CSE531: Computational Complexity Theory January 13, 2011

Homework 1

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Notes: You must work on the homework by yourself. Each problem is worth 10 points.

1. If $f, g \in \mathbf{NP}$, is $f \wedge g \in \mathbf{NP}$? What about $f \vee g$?

2. Let HALT be the halting function we defined in class (i.e. $\mathsf{HALT}(\alpha, x) = 1$ iff $M_\alpha(x)$ halts). Show that HALT is $\mathbf{NP}$-hard. Is it $\mathbf{NP}$-complete?

3. Give an example of a function that has polynomial sized circuits, but is not in $\mathbf{P}$.

4. A *formula* is a circuit where every gate has fan-out at most 1. Show that if $f : \{0,1\}^n \to \{0,1\}$ has a circuit of depth $O(\log n)$, then it has a formula of size poly(n).

5. Show that every function $f : \{0,1\}^n \to \{0,1\}$ can be computed by a circuit of size $O(2^n/n)$. HINT: Do the recursive argument as in class, but stop it when you reach functions on $\log n$ input bits, and compute all such functions by brute force.

6. (Hard.) Show that if a Turing Machine uses $o(\log \log n)$ space, then it must use $O(1)$ space.

   HINTS: Consider the shortest input $x_1, \ldots, x_n$ that requires space $S > 0$ (where S is chosen to be large enough). Let C_i denote the set of all possible configurations that are possible when the input head is over location i. Here a configuration is all information about the state of the machine *except* the location of the input head. Then prove

   Lemma 1. *For $i < j \leq n$, $C_i \neq C_j$.*

   To do this, assume that it is not the case, and consider the run of the machine on input $x_1, \ldots, x_i, x_{j+1}, \ldots, x_n$, and show that this run also uses space S, which contradicts the choice of $x_1, \ldots, x_n$.

   Finally, count the number of possible sets C_i, and use the pigeonhole principle to argue that if S is much smaller than $\log \log n$, then some i, j must give the same sets C_i, C_j and the same value $x_i = x_j$, which is a contradiction.