CSE531: Computational Complexity Theory	February 15, 2012

Lecture 11 Expander Graphs

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1 Combinatorial Definition

In order to simplify the presentation, we use numbers like 1/2, 5/4 which were chosen for no particular reason.

Recall that an undirected graph is d-regular if every vertex has exactly d edges. Throughout this lecture, think of d as a constant. Recall that $\Gamma(S)$ denotes the set of neighbors of a set of vertices S in a graph.

Definition 1. *We say that a d regular graph on n vertices is an expander if for every subset S of at most $n/2$ vertices, $|\Gamma(S)| \geq (5/4)|S|$.*

Next we use the probabilistic method to show that such graphs do exist:

Theorem 2. *There is a constant d such that for every n, there is a d-regular graph on n vertices.*

Proof We shall show that for d large enough, a random d-regular graph on n vertices is an expander with high probability.

Assume that n is even, and pick a d-regular graph by taking the union of d perfect matchings. In other words, we pick d sets of $n/2$ disjoint edges and then put all these edges together to obtain a d-regular graph.

For any subset S of $k \leq n/2$ vertices, and any subset T of $5k/4$ vertices, we shall bound the probability that $\Gamma(S) \subseteq T$ under one of the perfect matchings. Imagine sampling the matching by picking an arbitrary unmatched vertex of S and matching it to a random unmatched vertex of the graph, and repeating this process until all vertices of S are matched. Let E_i denote the event that the i'th vertex from S is matched to a vertex of T. Note that E_i is well defined for $i \leq k/2$ (after $k/2$ steps it may be that all vertices of S have been matched).

Then

$$\Pr[E_1 \wedge E_2 \wedge \ldots \wedge E_{\lceil k/2 \rceil}] = \prod_{i=1}^{\lceil k/2 \rceil} \Pr[E_i | E_{i-1} \ldots E_1] \leq \left(\frac{5k}{4n}\right)^{k/2}.$$

Thus,

$$\Pr[\Gamma(S) \subseteq T] \leq \left(\frac{5k}{4n}\right)^{dk/2}.$$

The number of sets S of size k is at most $\binom{n}{k}$, and the number of sets of size $5k/4$ is $\binom{n}{5k/4}$. Using the estimate $\binom{n}{k} \leq (en/k)^k$, we have that the probability that some set of size k does not expand is at most:

$$\begin{aligned}
&\binom{n}{k} \cdot \binom{n}{5k/4} \left(\frac{5k}{4n}\right)^{dk/2} \\
&\leq \left(\frac{en}{k}\right)^{k} \left(\frac{4en}{5k}\right)^{5k/4} \left(\frac{5k}{4n}\right)^{dk/2} \\
&\leq \left(\frac{en}{k}\right)^{9k/4} \left(\frac{5k}{4n}\right)^{dk/2} \\
&= \left(\frac{5e}{4}\right)^{9k/4} \left(\frac{5k}{4n}\right)^{dk/2-9k/4}
\end{aligned}$$

Note that since $\frac{5k}{4n} \leq 5/8 < 1$, choosing d to be a large enough constant makes this probability less than 2^{-100k}. By one more application of the union bound, we have that the probability that our random graph is not an expander is at most $\sum_{k=0}^{n/2} 2^{-100k} < 1$. ■

Fact 3. *The diameter of an expander is $O(\log n)$.*

Proof Consider any two vertices u, v in the graph. Then the set of vertices at distance t from u is $\Gamma^t(\{u\})$, which by the properties of the expander is a set of size at least $\min\{n/2, (5/4)^t\}$. Thus, at least half of all vertices are at distance $O(\log n)$ from u and at least half are at distance $O(\log n)$ from v. Thus, there must be some vertex z that is at distance $O(\log n)$ from both u and v, which means that there is a path of length $O(\log n)$ from u to v. ■

2 Algebraic Definition

One can also define expanders using the spectrum of the adjacency matrix. Suppose the adjacency matrix is A:

$$A_{i,j} = \begin{cases} 1 & \text{if } \{i, j\} \text{ is an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

The normalized adjacency matrix is $B = A/d$. B has a natural interpretation: it is the transition matrix for the stochastic process of taking a random step on the graph. In other words, if x is a column vector that corresponds to a probability distribution on the vertices (so that $\sum_i x_i = 1$ and $x_i \geq 0$), then Bx is the vector that is the distribution obtained by first sampling a vertex according to x and then picking a uniformly random neighbor of that vertex. Similarly, B^k is the matrix that corresponds to taking k random steps in the graph. Here are some other properties of B:

- B has exactly n real eigenvalues (possibly repeated): $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.
- The corresponding eigenvectors form an orthonormal basis.
- $\lambda_1 = 1$, and the first eigenvector is the vector that takes value $(1/\sqrt{n})$ everywhere.
- $\lambda_1 > \lambda_2$ if and only if the graph is connected.

- $\lambda_1 = -\lambda_n$ if and only if the graph is bipartite.

Let $\lambda = \max\{|\lambda_2|, \ldots, |\lambda_n|\}$.

We shall show that an equivalent way to define expanders is graphs for which $\lambda < 1 - \Omega(1)$, in other words λ bounded away from 1. We define the edge expansion of the graph:

$$h(G) = \min_{S, |S| \leq n/2} \frac{\#\text{ edges coming out of S}}{|S|}$$

Then one can show that edge expansion is closely tied to the value of λ:

Theorem 4.

$$d\left(\frac{1-\lambda}{2}\right) \leq h(G) \leq d\sqrt{2(1-\lambda)}$$

Thus, a good expander will have a constant eigenvalue gap $(1 - \lambda) = \Omega(1)$.

2.1 Analyzing Random Walks

Let x be the column vector for any distribution on the vertices of an expander graph. Then the distribution after k random steps is $B^k x$. By the properties of the eigenvalues, we know that there x can be expressed as a linear combination of eigenvectors $u, v_1, \ldots, v_n$ (where here u is the vector corresponding to the uniform distribution):

$$x = u + c_1 v_1 + c_2 v_2 + \ldots + c_n v_n.$$

This implies

$$B^k x = u + \sum_{i=2}^{n} \lambda_i^k c_i v_i$$

Thus

$$\|B^k x - u\| \leq \lambda \|x - u\|,$$

so the distribution rapidly converges to the uniform distribution.

In particular, this implies that not only is the diameter of an expander $O(\log n)$, but that after $O(\log n)$ random steps, the distribution of a random walk that began at any particular vertex is must have probability of roughly $1/n$ of being at any vertex.

Another very useful property of expanders is the following theorem (whose proof we do not discuss here).

Theorem 5. *Let $f_1, f_2, \ldots, f_t : [n] \to [0, 1]$ be a sequence of functions defined on the vertex set of an expander graph, each with mean $\mathbb{E}[f(v)] = \mu$. Let $X_1, \ldots, X_t$ be the vertices of a random walk starting at a uniformly random vertex in the graph. Then*

$$\Pr\left[\left|\sum_{i=1}^{t} f_i(X_i) - \mu t\right| \geq \epsilon t\right] < 2e^{-\frac{\epsilon^2(1-\lambda)t}{4}}.$$

The theorem is very useful, for example to reduce the error probability of algorithms without using too much additional randomness. One can use it to prove the following theorem, whose proof we leave as an exercise:

Theorem 6. *Let A be a randomized algorithm running in* $\mathrm{poly}(n)$ *time that uses r random bits to and computes a function correctly with probability* $2/3$*. Then there is another randomized algorithm running in* $\mathrm{poly}(n, t)$ *time that uses $r + t$ random bits and computes the function correctly with probability at least* $1 - 2^{-\Omega(t)}$.