CSE599s: Extremal Combinatorics	October 11, 2011

Homework 1

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1. Let G be a directed graph on $rs + 1$ vertices. Then show that one of the following must hold:
   - G contains a cycle.
   - G contains a path of length r.
   - G contains a set of vertices S, $|S| = s + 1$ such that there is no path that starts at S and ends in S.
2. In this problem we shall show that having access to randomness does not give polynomial sized circuits any additional power[1]. Suppose you are given a boolean circuit C of size s that takes in two n bit strings x, r as input and computes a function f, in the sense that for every x, $\Pr_r[C(x, r) = f(x)] \geq 2/3$, where the probability is taken over uniformly chosen r. Show that this implies that there is another circuit C' of size $\mathsf{poly}(s, n)$ such that $C'(x) = f(x)$ for every input x.
3. Prove that if an undirected graph has n vertices and $nk/2$ edges ($k \geq 1$), then there must be an independent set of size $n/(k + 1)$.
4. How many non-negative integer solutions are there to the equation $x_1 + x_2 + \cdots + x_n = 3k$ under the constraint that each $x_i \leq k$?
5. HARD: Prove or disprove: if $\mathcal{F}$ is a non-empty family of sets, $|\mathcal{F}| \geq 2$, such that for every $A, B \in \mathcal{F}$ we have $A \cup B \in \mathcal{F}$ (i.e. $\mathcal{F}$ is closed under union), then there is some element which occurs in at least $|\mathcal{F}|/2$ sets. EASIER: Prove or disprove that such an x exists if $\mathcal{F}$ is closed under both intersection and union.

[1] This is in contrast to Turing machines. As far as we know, polynomial time Turing machines with access to randomness could compute things that are not computable in polynomial time without randomness.