Homework 2

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1. Recall that a *star* of size k is a collection of k edges incident to the same vertex. Show that any graph with $2(k-1)^2+1$ edges contains a matching of size k or a star of size k.

2. Given a point in $\mathbb{R}^3$, its projection on the xy-plane is the point you obtain by dropping the z coordinate. Similarly define the projection to the yz, zx planes. Show that given a set of n points in $\mathbb{R}^3$, one of the projections of this set to the xy, yz or zx planes must have size at least $n^{2/3}$.

3. Given any set of vertices T in a graph, let $\Gamma(T)$ denote the set of neighbors of T. Show that the vertices of a graph can be partitioned such that each part is spanned by a single edge or a simple cycle if and only if for every independent set T, $|\Gamma(T)| \geq |T|$.

4. Let $\mathcal{F}$ be a family of sets, and A, B be sets such that every member of the family has a non-empty intersection with each of A, B. Further, suppose that no set that is smaller than A has this property. Let G be the bipartite graph with bipartition A, B such that (a, b) is an edge if and only if there is a set $S \in \mathcal{F}$ such that $a \in S, b \in S$. Show that G contains a matching that touches every element of A.

5. Suppose you have a family of sets $\mathcal{F}$ in an n element universe such that if $A, C \in \mathcal{F}$ and $A \subset B \subset C$, then $B \in \mathcal{F}$. Prove that $\sum_{A\in\mathcal{F}}(-1)^{|A|} \leq \binom{n}{\lfloor n/2 \rfloor}$.

6. Let $\mathcal{F}$ be a finite collection of disjoint subsets of $\mathbb{R}^d$, such that every set in $\mathcal{F}$ has volume at least 1. Show that there is a way to assign every set $A \in \mathcal{F}$ to a distinct point $x_A \in \mathbb{Z}^d$ (with integer coordinates), so that x_A is at most $\sqrt{d}$ away from A in Euclidean distance.