CSE599s: Extremal Combinatorics November 2, 2011

Lecture 10 Structured Intersections, and Partial Orders

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1 Families with Structured Intersection

We continue our study of families of sets that have a pairwise intersection. Recall that last time we showed that the largest intersecting family is the one that includes a single element.

Today, let us try to place some structure on the intersections. Let $\mathcal{G}$ be a family of graphs on the vertex set $[n]$. We say $\mathcal{G}$ is intersecting if for any two graphs $T, K \in \mathcal{G}$, $T \cap K$ has an edge. Then as before, $\mathcal{G}$ is of size at most $2^{\binom{n}{2}}/2$, which can be achieved with the family of all graphs that contain a designated edge.

Things get interesting if we ask for the intersections to have some structure. Say that $\mathcal{G}$ is $\bigtriangledown$-*intersecting* if for every $T, K \in \mathcal{G}$. $T \cap K$ contains a triangle. The trivial example gives a family of size $2^{\binom{n}{2}}/8$, but perhaps there is some clever way to get a $\bigtriangledown$-intersecting family that has size close to $2^{\binom{n}{2}}/2$, as in the examples above?

For the proof, we shall need to use the notion of *entropy*. The entropy of a random variable X is defined to be

$$H(X) = \sum_x \Pr[X = x] \log 1/\Pr[X = x].$$

The entropy is a generalization of the notion of size of a set. It satisfies the following inequalities:

Fact 1. $0 \leq H(X) \leq \log(\#\ \textit{elements in universe})$.

Given two random variables X, Y (possibly correlated), we define the conditional entropy:

$$H(X|Y) = \mathop{\mathbb{E}}_y [[H(X|Y = y)]].$$

Then we have

Fact 2 (Chain Rule)**.** $H(XY) = H(X) + H(Y|X) \leq H(X) + H(Y)$.

All of the above inequalities follow easily using the concavity of the logarithm function and Jensen's inequality.

Given random variables $X_1, \ldots, X_n$ and a set of coordinates $T = \{i_1, \cdots, i_k\} \subset [n]$, we write X_T to denote $X_{i_1}, \cdots, X_{i_k}$, the projection of X onto the coordinates in T, and we write $X_{<i}$ to denote X projected onto all coordinates less than i.

Lemma 3 (Shearer's Lemma)**.** *If S is any distribution on subsets of the coordinates $[n]$, such for every i, $\Pr[i \in S] \geq \mu$, then $\mathbb{E}[H(X_S)] \geq \mu \cdot H(X)$.*

We give a simple proof due to Jaikumar Radhakrishnan.

Proof For $T = \{i_1, \cdots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$, observe that

$$\begin{aligned} H(X_T) &= H(X_{i_1}) + H(X_{i_2}|X_{i_1}) + \cdots + H(X_{i_k}|X_{i_{k-1}}, \cdots, X_{i_1}) \\ &\geq H(X_{i_1}|X_{<i_1}) + H(X_{i_2}|X_{<i_2}) + \cdots + H(X_{i_k}|X_{<i_k}), \end{aligned}$$

where we used chain rule in the equality, and used the fact that entropy is only smaller if we condition on more variables, for the inequality.

Thus, we get that

$$\begin{aligned}
\mathop{\mathbb{E}}_{S}[H(X_S)] &\geq \mathop{\mathbb{E}}_{S}\left[\sum_{i\in S} H(X_i|X_{<i})\right] \\
&= \sum_{i\in[n]} \Pr[i\in S]\cdot H(X_i|X_{<i}) \qquad \text{whenever i is not in S, this term contributes 0} \\
&\geq \mu \sum_{i\in[n]} H(X_i|X_{<i}) \\
&= \mu\cdot H(X)
\end{aligned}$$

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Chung, Frankl, Graham and Shearer showed that no such example exists:

Theorem 4 ([1])**.** *If $\mathcal{G}$ is $\bigtriangledown$-intersecting, then $|\mathcal{G}| \leq 2^{\binom{n}{2}}/4$.*

Proof Let $X_1, \ldots, X_{\binom{n}{2}}$ be the indicator random variables for the edges of a uniformly random graph chosen from the family. Then we have

$$H\left(X_1,\ldots,X_{\binom{n}{2}}\right) = \log|\mathcal{G}|.$$

Now consider the experiment of picking a uniformly random subset $S \subset [n]$ of size $|S| = n/2$, and throwing out the coordinates that correspond to edges that go from S to outside S. This leaves $2\binom{n/2}{2}$ edges. Clearly, each edge is equally likely to be included with probability $2\binom{n/2}{2}/\binom{n}{2}$. Observe that for every triangle, some edge must lie entirely in S or the complement of S. Thus, after any such projection is applied to the whole family, we are left with an intersecting family. Thus, the entropy of the resulting variables is at most $2\binom{n/2}{2} - 1$.

By Shearer's lemma,

$$\begin{aligned}
\frac{2\binom{n/2}{2}}{\binom{n}{2}}\log|\mathcal{G}| &\leq 2\binom{n/2}{2} - 1 \\
\Rightarrow \log|\mathcal{G}| &\leq \binom{n}{2} - \frac{\binom{n}{2}}{2\binom{n/2}{2}} \\
&= \binom{n}{2} - \frac{n(n-1)}{2(n/2)(n/2-1)} \\
&= \binom{n}{2} - \frac{n-1}{n/2-1} \\
&\leq \binom{n}{2} - 2
\end{aligned}$$

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In fact, the obvious example is tight:

Theorem 5 ([2])**.** *If $\mathcal{G}$ is $\bigtriangledown$-intersecting, then $|\mathcal{G}| \leq 2^{\binom{n}{2}}/8$.*

The theorem is proved by spectral methods.

2 Partial Orders

Recall that given a partially ordered set $\mathcal{P}$, a chain is a sequence $x_1 < x_2 < \cdots < x_k$, and an antichain is a collection of incomparable elements.

Suppose there is a chain of length r. Then clearly, you cannot partition the set into less than r antichains, or two elements from the chain will end up in the same antichain. Similarly, if there is an antichain of length r, you cannot partition the set into less than r chains. In fact, these relationships are tight — the only reason you cannot partition the family in less than r chains (resp. s antichains) is because there is an antichain of size r (resp. a chain of length s).

Theorem 6. *If r is the length of the largest chain in $\mathcal{P}$, then you can partition $\mathcal{P}$ into r antichains*

Proof Let A_i be set of elements x such that longest chain ending at x has length i. Then each such A_i is clearly an antichain, and $A_1, \ldots, A_r$ is such a partition.
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Theorem 7. *If r is the size of the largest antichain, then you can partition $\mathcal{P}$ into r chains.*

Proof Observe that given any n chains, every antichain of size n must pick exactly one element from each chain. Suppose we have a collection of n chains and we know that there is an antichain of size n in the elements of these n chains. Let x_i be the maximal element of chain i that participates in some antichain. Then x_i, x_j are incomparable: if $x_i > x_j$, x_i cannot be part of an n element chain, since any such antichain must contain an element that is either equal to x_j or less than x_j. Thus $\{x_1, \ldots, x_n\}$ form an antichain.

We shall proceed by induction on the size of the poset $\mathcal{P}$. Let $a \in \mathcal{P}$ be a maximal element. Let n be the size of the biggest antichain in $\mathcal{P} - \{a\}$. Then by induction we can partition $\mathcal{P} - \{a\}$ into n chains. Let $x_1, \ldots, x_n$ be as above. Then there are two cases

Case 1: $\{a, x_1, \ldots, x_n\}$ form an antichain. In this case, the theorem is proved, since we add the chain containing only a to the partition to get a partition of $\mathcal{P}$ into $n+1$ chains.

Case 2: There is an $x_i < a$. In this case, suppose the chain containing x_i is $a_1 < a_2 < \cdots < a_k < x_i < b_1 < b_2 < \cdots < b_t$. We take out the chain $C = a_1 < \cdots < x_i < a$ from the poset. Then it must be that in the remaining elements, there is no antichain of size n. Thus by induction, we can partition the remaining elements into $n-1$ chains. We add the chain C to this collection to prove the theorem.

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Next we prove a few results about antichains in the partial order of sets.

Theorem 8 (Sperner)**.** *The largest antichain in the subsets of $[n]$ has size $\binom{n}{\lfloor n/2 \rfloor}$.*

We saw one proof using Hall's theorem. Here's another:

Claim 9. *If $\mathcal{F}$ is an antichain, then* $\sum_{A \in \mathcal{F}} \binom{n}{|A|}^{-1} \leq 1$.

The claim implies the theorem, since $\binom{n}{|A|}$ is maximized when $|A| = \lfloor n/2 \rfloor$. Thus the claim implies that $\sum_{A \in \mathcal{F}} \binom{n}{\lfloor n/2 \rfloor} \leq 1$, which proves the bound.

To prove the claim, say that a permutation $\pi : [n] \to [n]$ contains A if $\pi([|A|]) = A$. Then we see that the number of permutations that contain A is exactly $|A|! \cdot (n - |A|)!$. Further, now two elements of the antichain can be contained in the same permutation. Since the total number of permutations is $n!$, $\sum_{A \in \mathcal{F}} |A|!(n - |A|)! \leq n!$, which proves the claim.

References

[1] Fan R. K. Chung, Ronald L. Graham, Peter Frankl, and James B. Shearer. Some intersection theorems for ordered sets and graphs. *J. Comb. Theory, Ser. A*, 43(1):23–37, 1986.

[2] David Ellis, Yuval Filmus, and Ehud Friedgut. Triangle-intersecting families of graphs, November 07 2010.