Lecture 16: Ramsey and Hales-Jewett Theorem

Lecturer: Anup Rao

1 Ramsey Numbers

Suppose we color the edges of a complete graph with 2 colors, will there always be a monochromatic triangle? As we have seen, the maximum number of edges in a graph on $2n$ vertices without a triangle is n^2 (which is achieved by the complete bipartite graph). Since $(1/2)\binom{2n}{2} = 2n^2 - n$, each of the color classes may have less than n^2 edges, so just the number of edges is not enough to guarantee a monochromatic triangle. Nevertheless, it turns out that one can always find a monochromatic triangle, as long as n is large enough. With that in mind, we introduce the Ramsey numbers.

Definition 1. *$R_k(s,t)$ is the minimum number n such that any coloring of the k-sets of an $[n]$ with red and blue will have either a subset $S \subset [n]$ of size s such that every k-set supported on S is colored red, or a subset $T \subset [n]$ of size t such that k-set supported on T is colored blue.*

Observe that if $R_k(s,t)$ is well defined, then since any coloring of the k-sets of a universe $[n]$ with $n \geq R_k(s,t)$ is also a coloring of the k-sets supported on $[R_k(s,t)]$, this coloring must have either a subset of size s whose k-sets are all red or a subset of size t whose k-sets are all colored blue. $R_k(s,t)$ clearly always exists when $s = 1$ or $t = 1$, and we shall prove that $R_k(s,t)$ exists for any k, s, t. To prove this, we start by studying the case of $k = 2$ (namely undirected graphs).

Theorem 2. $R_2(s,t) \leq R_2(s,t-1) + R_2(s-1,t)$.

Proof Suppose $n = R_2(s,t-1) + R_2(s-1,t)$, and fix any coloring of the edges. Consider any vertex x. Let

$$A = \{u : \{x,u\} \text{ is colored red}\}$$

and

$$B = \{u : \{x,u\} \text{ is colored blue}\}$$

Since $|A| + |B| = R_2(s,t-1) + R_2(s-1,t) - 1$, either $|A| \geq R_2(s-1,t)$ or $|B| \geq R_2(s,t-1)$. Suppose $|A| \geq R_2(s-1,t)$, then A either has a blue clique of size t, in which case, so does the whole graph, or A has a red clique of size $s-1$, but then x together with this clique forms a red clique of size s. The case of $|B| \geq R_2(s,t-1)$ is similar. ■

The recurrence of 2 is satisfied by $\binom{s+t-2}{s-1}$, since $\binom{s+t-2}{s-1} = \binom{s+t-3}{s-1} + \binom{s+t-3}{s-2}$. Note that this is the smallest such expression that is well defined for $s,t \geq 1$. This gives the bound $R_2(s,t) \leq \binom{s+t-2}{s-1}$.

We have already seen that a random graph on n vertices (where each edge is included independently with probability 1/2) will not have any clique or independent set of size $2\log n$ with high probability. That argument can be used to show that $R_2(t,t) = \Omega(t2^{t/2})$. The upper bound from Theorem 2 gives $R_2(t,t) = \binom{2t-2}{t-1} = O(4^t/\sqrt{t})$.

A similar proof can be used to bound the Ramsey number for sets of larger size:

Theorem 3. $R_k(s,t) \le R_{k-1}(R_k(s-1,t), R_k(s,t-1)) + 1$.

Proof Consider any coloring χ of the k-sets of $[n = R_{k-1}(R_k(s-1,t), R_k(s,t-1)) + 1]$ with red and blue. Color every $k-1$ set $A \subset [n-1]$ with the same color that is given to $A \cup n$ in the original coloring, and call the induced coloring Γ. Then by the definition of n, in Γ there must be either a red subset of size $R_k(s-1,t)$ or a blue subset of size $R_k(s,t-1)$. If there is a red subset of size $R_k(s-1,t)$, then in χ, this subset either has a blue subset of size t in which case we are done, or it has a red subset of size $s-1$, in which case adding n gives us a red subset of size s by the definition of the coloring χ. The case of a blue subset of size $R_k(s,t-1)$ is the same. ■

One can generalize what we have seen so far to the case of $r > 2$ colors, as follows. We shall imagine coloring the k-sets of an n element universe with different colors. Given such a coloring, and a subset $T \subseteq [n]$, we shall say that T is monochromatic if every k-set of T is colored with the same color.

Definition 4. *$R_k(r;s)$ is the minimum n such that for any coloring of k-sets of $[n]$ with r colors there is a monochromatic subset $S \subset [n]$ of size s.*

Theorem 5. *$R_k(r;s)$ is finite.*

Proof We claim that it is enough to prove this for the case that $r = 2$. This is because $R_k(r;s) \le R_k(r-1, R_k(2;s))$. Indeed, given any coloring of the k sets of $[R_k(r-1; R_k(2;s))]$, let us recolor the sets by identifying the first two colors with each other: we view the coloring as using $r-1$ colors. Then, by definition there must be some monochromatic subset T of size $R_k(2;s) \ge s$. If the color corresponds to a single color in the original coloring, we are done. Otherwise, every subset of T must be colored with one of the first two colors, but since T is of size $R_k(2;s)$, T must have a monochromatic subset of size s in the original coloring. So all we need to do is show that $R_k(2;s)$ is finite, which is immediate from Theorem 3. ■

Next we see an interesting consequence of the finiteness of Ramsey numbers:

Theorem 6. *If n is large enough in terms of r, then any coloring of $\chi : [n] \to [r]$ will contain monochromatic $x, y, x+y$.*

Proof Set $n = R_2(r;3)$, and color every pair of elements $x, y \in [n]$ with the color $\chi(|x-y|)$. By the definition of $R_2(r;3)$, there is a monochromatic triangle with vertices $a < b < c$, and so $(a-b) + (b-c) = (a-c)$ forms the monochromatic triple that we were looking for. ■

2 The Hales-Jewett Theorem

A *combinatorial line* in $[n]^k$ is specified by a *root* in $\tau \in ([n] \cup \{*\})^k$. Given this root, and $a \in ([n] \cup \{*\})$, let $\tau(a)$ denote the element of $[n]^k$ obtained by replacing each $*$ in τ with a. The combinatorial line given by τ is the set $\{\tau(a) : a \in [n]\}$. Thus every combinatorial line either has one element (if there are no $*$'s), or n elements. If it has only one element, we shall say that the line (and the corresponding root) is *trivial*.

Theorem 7. *For every $r, n \ge 0$, there is k large enough so that in any coloring of $[n]^k$, there must be a nontrivial monochromatic combinatorial line.*

It is instructive to try and prove the theorem in the case of $n = 2$ as a warmup. In this case, we claim that $k = r$ suffices. Indeed, consider the $k+1$ points: $(11\dots1), (211\dots1), \dots, (22\dots2)$. Then by the pigeonhole principle, in any coloring we must have that there are $j \neq j'$ such that $\chi(1^j 2^{k-j}) = \chi(1^{j'} 2^{k-j'})$. But then these two points form a non-trivial monochromatic line. The proof of the general case will use this basic idea as a building block.

Proof We use induction on n. The case of $n = 1$ is trivial. For k that we shall set to be large enough during the proof, let $\chi : [n]^k \to [r]$ denote a coloring.

Given any sequence of roots $\tau = \tau_1, \dots, \tau_t$ where $\tau_i \in ([n] \cup \{*\})^{N_i}$ and $k = \sum_i N_i$, and given $a \in [n]^t$ given by $a = a_1, \dots, a_t$ we define $\tau(a) \in [n]^k$ to be $\tau(a) = \tau_1(a_1), \dots, \tau_t(a_t)$.

The proof will hinge on the following claim:

Claim 8. *There exists such a sequence of non-trivial roots $\tau = \tau_1, \dots, \tau_t$ as above, such that for any $i \in [t]$, and any $a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_t \in [n]$, we have*

$$\chi(\tau(a_1, \dots, a_{i-1}, n, a_{i+1}, \dots, a_t)) = \chi(\tau(a_1, \dots, a_{i-1}, n-1, a_{i+1}, \dots, a_t)).$$

Let us first show that the claim completes the proof. We shall set t to be large enough (by the induction hypothesis) to guarantee that any coloring of $[n-1]^t$ gives a non-trivial monochromatic line. Then we color each $a \in [n-1]^t$ with $\chi(\tau(a))$. By induction, this coloring has a monochromatic line $\{\sigma(b) : b \in [n-1]\}$ with some root $\sigma \in ([n-1] \cup \{*\})^t$. Thus we get that the set $\{\tau(\sigma(b)) : b \in [n-1]\}$ is monochromatic under χ. This is not quite a monochromatic line. However, by the property of τ guaranteed by the claim, we can take the point $\tau(\sigma(n-1))$, and iteratively replace each $n-1$ in a $*$ position with n and stay in the same color. Thus $\tau(\sigma)$ is actually the root of a non-trivial monochromatic line as required.

So all that is left to do is prove the claim. We find the roots of τ inductively. Assume we have found $\tau = \tau_i, \tau_{i+1}, \dots$ such that for every $z \in [n]^{\sum_{j=1}^{i-1} N_j}$, the root $z, \tau_i, \dots,$ has the neighbor property. Initially, this is true since any trivial root does satisfy the property of the claim.

Now for each $(z, y) \in [n]^{\sum_{j=1}^{j-1} N_j} \times [n]^{t-i}$, define the j'th color $\chi(z, n^j (n-1)^{N_i - j}, \tau(y))$. In this way, we gave defined $N_i + 1$ colorings of a set of size at most $n^{t + \sum_{j=1}^{i-1} N_j}$. We shall set $N_i \geq r^{n^{t + \sum_{j=1}^{i-1} N_j}}$. Then by the pigeonhole principle, there must be j, j' such that the j', j colorings coincide on all (z, y). These two colorings specify the i'th root τ_i.

We see that for the above argument to work, it is sufficient to have $N_i \geq r^{n^{t + \sum_{j=1}^{i-1} N_j}}$ for all i, which can easily be satisfied. This completes the proof. ■

Recall that an arithmetic progression is a sequence of numbers $a, a+d, a+2d, \dots, a+(t-1)d$.

Theorem 9. *For every r, n, there exists ℓ such that any coloring of $[\ell]$ with r colors has a monochromatic arithmetic progression of length n.*

Proof Let k be large enough to satisfy Theorem 7 with parameters r, n, and set $\ell = nk$. Then color each $(x_1, \dots, x_k)$ with the color of $\sum_{i=1}^k x_i$. Then the non-trivial monochromatic line induced by the coloring corresponds to a monochromatic arithmetic progression of length n. ■

One of the more spectacular applications of ergodic theory is the proof of the following stronger density version of the Hales-Jewett theorem, which still has no simple combinatorial proof (to my knowledge):

Theorem 10 (Density Hales-Jewett Theorem)**.** *For every ϵ, n, there is k large enough so that any subset of $[n]^k$ of size ϵn^k contains a non-trivial monochromatic combinatorial line.*

The above theorem has a couple of applications in computer science. In particular, it gives the only known proof of the *parallel repetition theorem* for multiple provers, but we shall not go into the details here.