# Lecture 17: The Borsuk-Ulam Theorem

*Lecturer: Anup Rao*

## 1 The Borsuk-Ulam Theorem

Today we discuss the Borsuk-Ulam Theorem. One of the reasons the theorem is so powerful is that it has many different convenient guises. We define $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ to be the $n$-dimensional sphere, and $\mathbb{B}^n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ to be the $n$-dimensional ball. The Borsuk-Ulam theorem says:

**Theorem 1.** *If $f : \mathbb{S}^n \to \mathbb{R}^n$ is continuous, then there exists $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$.*

It has many corollaries, most of which are actually equivalent to the theorem.

**Corollary 2.** *There is no continuous map $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$ satisfying $f(x) = -f(-x)$ for all $x$.*

**Proof** Suppose not, and such an $f$ does exist. Then by Theorem 1, there is a point $x$ such that $f(x) = f(-x)$ which contradicts the fact that $f(x) = -f(-x)$, since $f(x)$ is non-zero. ■

**Corollary 3.** *There is no continuous map $f : \mathbb{B}^n \to \mathbb{S}^{n-1}$ such that for any $x \in \mathbb{S}^{n-1}$, $f(x) = -f(-x)$.*

**Proof** Suppose not. Then consider the map $g : \mathbb{S}^n \to \mathbb{S}^{n-1}$ obtained by $g(x_1, \ldots, x_n) = f(x_2, \ldots, x_n)$. This map is continuous, and $g(x) = -g(-x)$, thus it contradicts Corollary 2. ■

**Corollary 4.** *If $\mathbb{S}^n = F_1 \cup \cdots \cup F_{n+1}$, where each $F_i$ is a closed or open set, then there must be some $i, x$ such that $F_i$ contains both $x$ and $-x$.*

**Proof** We prove the case of closed sets. Define the map $f : \mathbb{S}^n \to \mathbb{R}^n$, where the $i$'th coordinate $f(x)_i$ is the distance of $x$ to the set $F_i$. By Theorem 1, there is a point $x$ such that $f(x) = f(-x)$. If the $i$'th coordinate of $f(x)$ is 0 for some $i$, then $x, -x \in F_i$. Otherwise $x, -x \in F_{n+1}$. The case of open sets is only a little more involved, but we do not do it here. ■

**Corollary 5** (Brouwer's Fixed Point Theorem)**.** *If $f : \mathbb{B}^n \to \mathbb{B}^n$ is continuous, then there is an $x \in \mathbb{B}^n$ such that $f(x) = x$.*

**Proof** Suppose not. Then consider the map $g : \mathbb{B}^n \to \mathbb{S}^{n-1}$ that maps $x$ to the point on $\mathbb{S}^{n-1}$ that is reached by walking in a straight line from $f(x)$ to $x$ and continuing till you reach a point of $\mathbb{S}^{n-1}$. $g$ is continuous, and $g$ is the identity on $\mathbb{S}^{n-1}$. Thus $g$ contradicts Corollary 3. ■

A couple of proofs of Theorem 1 are known, including purely combinatorial ones. We shall not delve into a full proof. Instead, we give a direct combinatorial proof of Brouwer's Fixed Point theorem in the case that $n = 2$, to illustrate the kind of ideas that go into these things. Actually we shall prove that there is no continuous function $f : \mathbb{B}^2 \to \mathbb{B}^2$ for which $f$ is the identity on $\mathbb{S}^1$, which we saw is enough to prove Brouwer's theorem.

**Claim 6.** *There is no continuous map $f : \mathbb{B}^2 \to \mathbb{S}^1$ that is the identity on $\mathbb{S}^1$.*

**Proof** Suppose there is such a continuous map. Consider a triangle in the plane defined by the vertices $x, y, z$. Then the continuous map above can be deformed to get a continuous map $g$ that maps the triangle to its boundary in such a way that $g$ is the identity on the boundary. Now consider a triangulation of this big triangle into small triangles. Namely, we partition the big triangle into smaller triangles, in such a way that every edge is shared by exactly two internal triangles, or lies on the boundary of the big triangle $xyz$.

Next we label all the vertices of this triangulation as follows. Given any vertex $a$, we label $a$ with 1 if $g(a)$ is mapped to a point on the segment $[x, y)$ (namely $g(a)$ is on the segment but is not equal to $y$), we label $a$ with 2 if $g(a) \in [y, z)$ and 3 if $g(a) \in [z, x)$. Given this labeling, we claim that there must be some triangle in the triangulation whose vertices get all three labels $1, 2, 3$.

Since $g$ is continuous, we can pick a triangulation that is so fine that the value of $g$ barely changes within any one small triangle. Then the existence of a small triangle whose vertices are mapped very far from each other gives a contradiction.

To see why there must be a small triangle getting all labels, consider the graph whose vertices are the regions in the space, namely all the triangles and the exterior of the triangle. Put an edge between two vertices of the graph if the corresponding edge of the triangulation is labelled by $1, 2$. Thus we connect two regions if and only if the edge between them is labeled by 1 and 2.

Observe that since $g$ is the identity on the boundary, the region that is outside the big triangle has degree exactly 1 in this graph. On the other hand, if a triangle is *not* labeled by $1, 2, 3$, then its degree in the graph must be even! Thus there must be some triangle that gets all three labels, or we would have constructed a graph such that the sum of the degrees of the vertices is odd. ■

# 2 Applications

## 2.1 Knesser Graphs

Given a family of sets $\mathcal{F}$, its Knesser graph is the graph where every vertex is a set from the family, and there is an edge if and only the corresponding sets are disjoint. It is easy to show that *every* graph is the Knesser graph of some family.

We define the graph $\mathsf{KG}_{n,k}$ to be the Knesser graph of the family $\binom{[n]}{k}$ (subsets of $[n]$ of size $k$). Here we shall study how to bound the Chromatic number of this graph. Recall that the chromatic number is the minimum number of colors needed to color the vertices so that each edge gets two distinct colors.

**Claim 7.** *The chromatic number of $\mathsf{KG}_{n,k}$ is at most $n - 2k + 2$.*

**Proof** Given a set $A$, let $x$ be its largest element. Color $A$ with $\max\{x, 2k - 1\}$. The number of colors is at most $n - 2k + 2$. Now if two sets $A, B$ are disjoint, they cannot be both colored by a number $x \geq 2k - 1$, or they must both contain $x$. On the other hand, if they are both colored $2k - 1$, then they must both be contained in $[2k - 1]$, which again implies that they intersect, since both sets are of size $k$. ■

Here we prove that this is tight:

**Theorem 8.** *If $n \geq 2k - 1$, then the chromatic number of $\mathsf{KG}_{n,k}$ is $n - 2k + 2$.*

**Proof** Let $d = n - 2k + 1$. For each element of $i \in [n]$ , we identify a point $v_i \in \mathbb{S}^d$ in such a way that no hyperplane that passes through the origin can pass through $d + 1$ of the points we have defined. Such an embedding can be found greedily, namely we can always pick each new point in such a way that it avoids all the hyperplanes defined by the earlier points. The number of such hyperplanes is only finite, so there are many choices for where to put the next point.

Suppose $\mathsf{KG}_{n,k}$ can be colored with $d$ colors. Now we define a collection of sets $A_1, \ldots, A_d$ as follows.

$$A_i = \left\{ x \in \mathbb{S}^d : \exists T \in \binom{[n]}{k} \text{ colored } i \text{ such that for all } j \in T,\ \langle x, v_j \rangle > 0 \right\}$$

In words, $x \in A_i$ if and only if there is a set of $k$ points colored $i$ in the hemisphere defined by $x$.

Let $A_{d+1} = \mathbb{S}^d - \bigcup_{i=1}^d A_i$.

By Corollary 4, there must exist $i, x$ such that $x, -x \in A_i$. If $i < d + 1$, then this means that there is a $k$-set in the open hemisphere centered at $x$ colored $i$, and another $k$-set in the open hemisphere centered at $-x$ colored $i$. These two sets are disjoint, and so we have a found an edge of the Knesser graph that is not properly colored. On the other hand, if $i = d + 1$, then there is *no* $k$ set in open hemispheres centered at $x, -x$, so there must be $n - 2(k - 1) = n - 2k + 2 = d + 1$ points on the equator. This gives a hyperplane that passes through the origin and contains $d + 1$ points, contradicting our placement of the points. ■

### 2.2 The Ham Sandwich Theorem

Another beautiful consequence of the Borsuk-Ulam theorem is the so called Ham-Sandwich Theorem.

**Theorem 9.** *Let $W_1, \ldots, W_d$ be compact subsets of $\mathbb{R}^d$ such that the volume of each subset is non-zero. Then there is a hyperplane that simultaneously bisects all sets $W_i$.*

**Proof** The hyperplane corresponding to a point $h \in \mathbb{S}^{d-1}$ is the set $\{x : \langle h, x \rangle = b\}$. Given any such $h$, there is always a choice for $b$ such that the hyperplane bisects $W_d$. Let us denote this choice $b_h$.

Now consider the map $f : \mathbb{S}^{d-1} \to \mathbb{R}^{d-1}$ such that the $i$'th coordinate is the volume of $W_i$ on the positive side of the hyperplane bisecting $W_d$:

$$f(h)_i = \mathsf{volume}(\{x \in W_i : \langle h, x \rangle > b_h\}).$$

This map is continuous, and so by Theorem 1, we must have that there exists $h$ such that $f(h) = f(-h)$, which means that $h, b_h$ define a hyperplane that bisects all $W_i$. ■

### 2.3 The Necklace Theorem

Given a necklace consisting of $d$ types of gems, two thieves would like to cut the necklace in a few locations in such a way that each type of gem can be equally divided. Formally, assume that the necklace is a string, and the thieves want to find a few locations where the string can be cut.

**Theorem 10.** *The necklace can be divided with $d$ cuts.*

**Proof** Imagine that the necklace corresponds to the unit interval $[0, 1]$, and place the necklace in $\mathbb{R}^d$ in such a way that the point $x \in [0, 1]$ is placed at $(x, x^2, x^3, \ldots, x^d)$. Each type of gem corresponds to a set $W_i$, so we can apply Theorem 9 to conclude that there is a hyperplane that bisects each type of gem simultaneously. Let $\{\langle h, x \rangle = b\}$ be this hyperplane.

We claim that this hyperplane can intersect the necklace in at most $d$ locations. Indeed, this hyperplane corresponds to the polynomial

$$p(X) = -b + h_1 X + h_2 X^2 + \ldots + h_d X^d,$$

and every point of intersection is a root of $p(X)$. Since $p(X)$ has degree $d$, there can be at most $d$ roots. ■