CSE599s: Extremal Combinatorics October 10, 2011

Lecture 4 Inclusion Exclusion (contd.) and the Pigeonhole principle

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1 More examples of Inclusion-Exclusion

Suppose we are given a family of sets $\{A_1, \ldots, A_n\}$. Can we estimate the size of their union in terms of their intersection sizes? For every non-empty subset $I \subseteq [n]$, define $A_I = \bigcap_{i \in I} A_i$, with the convention that $A_\emptyset$ is the set of all elements in the universe.

Last time we showed:

Proposition 1. *The number of elements not in* any *of the sets is* $\sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$.

Proof If the universe is of size t, then

$$\sum_{I \subseteq [n]} (-1)^{|I|} |A_I| = t - \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} |A_I| = t - |\bigcup_{i=1}^{n} A_i|.$$

■

1.1 Counting Surjections

Question: How many onto functions are there from $[m]$ to $[n]$?

There are n^m functions in total. Let A_i denote the set of functions that do not map anything to i. Then $|A_I| = (n - |I|)^m$. The number of functions that are not in any of these A_i (and hence are onto) is

$$\sum_{I \subseteq [n]} (-1)^{|I|} |A_I| = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m$$

1.2 Computing the Permanent

Given an $n \times n$ boolean matrix M (i.e. a matrix with 0 or 1 entries), its permanent is

$$\mathsf{perm}(M) = \sum_{\pi \in S_n} \prod_{i=1}^{n} M_{i,\pi(i)}.$$

The naive way to compute the permanent is just by computing each term, which would take $n! \cdot n$ arithmetic operations. Note that computing the permanent is $\#P$-complete, which means that if one can compute the permanent quickly, then we would be able to count the number of satisfying inputs to a circuit quickly or solve SAT.

A faster way to compute the permanent is via Ryser's formula:

$$\mathsf{perm}(M) = (-1)^n \sum_{I\subseteq[n]} (-1)^{|I|} \prod_{i=1}^{n} \sum_{j\in I}^{n} M_{i,j}$$

This takes time $n^2 2^n$.

Given the matrix M, define the family of functions $S = \{f : [n] \to [n] | \forall i, M_{i,f(i)} = 1\}$. The permanent counts the number of these that are permutations!

So let us define the set $A_i \subseteq S$ to be the set of all functions in S that do not map onto i. Then we are interested in exactly the number of functions that avoid all the A_i's (and hence are permutations). $|A_I|$ is exactly the number of functions in S that avoid mapping into I and yet pick out the 1 entries of M. For each i, such a function can take exactly $\sum_{j\notin I} M_{i,j}$ values, so $|A_I| = \prod_{i=1}^{n} \sum_{j\notin I} M_{i,j}$.

By inclusion-exclusion, we have that

$$\sum_{I\subseteq[n]} (-1)^{|I|}|A_I| = \sum_{I\subseteq[n]} (-1)^{|I|} \prod_{i=1}^{n} \sum_{j\notin I} M_{i,j} = \sum_{I\subseteq[n]} (-1)^{n-|I|} \prod_{i=1}^{n} \sum_{j\in I}^{n} M_{i,j} = (-1)^n \sum_{I\subseteq[n]} (-1)^{|I|} \prod_{i=1}^{n} \sum_{j\in I} M_{i,j}$$

as required.

2 What do the first k terms of the sum say?

The *Bonferroni inequalities* say that the partial sums alternate between being above the target and below the target:

Lemma 2. *For any even number $2k$,*

$$\sum_{j=1}^{2k} \sum_{I\subseteq[n],|I|=j} |A_I| \le |\bigcup_{i=1}^{n} A_i| \le \sum_{j=1}^{2k+1} \sum_{I\subseteq[n],|I|=j} |A_I|$$

For the proof, we again count the contribution of a particular element x. If x occurs t times, then the sets of size t are enough to estimate the contribution of x to the union. If we are only going up to sets of size $2k < t$, then the contribution of x is $1 - \sum_{j=1}^{2k} (-1)^j \binom{t}{j}$.

We have the following simple lemma:

Lemma 3. *Suppose $a_0, \ldots, a_t$ is a sequence of numbers such that $\sum_j (-1)^j a_j = 0$ and the a_i's are increasing until a_s and decreasing after, then for any odd k, $\sum_{j=1}^{k} (-1)^j a_j$ is negative, and it is positive for even k.*

The proof is easiest when one draws a picture. Lemma 3 easily gives Lemma 2.

In general, the values of the terms in the sum upto size k are not enough to estimate the size of the union. However, Linial and Nisan proved that once you get to the size of intersections of size $\sqrt{n}$ sets, then that information is enough to give good estimates on the size of the union:

Theorem 4. *Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be two collections of sets such that for all $I \subset [n], |I| < k$, $|A_I| = |B_I|$. Then,*

1. *For $k \geq \Omega(\sqrt{n})$,*

$$\frac{|\bigcup_{i=1}^n A_i|}{|\bigcup_{i=1}^n B_i|} = 1 + O(e^{-2k/\sqrt{n}}).$$

2. *For $k \leq O(\sqrt{n})$,*

$$\frac{|\bigcup_{i=1}^n A_i|}{|\bigcup_{i=1}^n B_i|} = O(n/k^2).$$

3 The Pigeonhole Principle

The principle: n pigeons cannot fit in $n-1$ holes.

Here are some clever applications.

Proposition 5. *Every graph must have two vertices of the same degree.*

Proof If the graph has n vertices, then the degree of every vertex x satisfies $0 \leq d(x) \leq n-1$. Let $T = \{d(x) : x \text{ is a vertex }\}$ be the set of degrees of the graph. If $|T| < n-1$, we are done by the pigeonhole principle. Otherwise, some vertex must have degree 0 and some other vertex must have degree $n-1$. That is impossible! ■

3.1 Erdős-Szekeres and Dilworth Theorems

Let $a_1, a_2, \ldots, a_n$ be a sequence of numbers. A *subsequence* $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$ is a sequence such that $i_1 < i_2 < \cdots < i_t$. We say that the subsequence is *increasing* if $a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_t}$ and *decreasing* if $a_{i_1} \geq a_{i_2} \geq a_{i_t}$.

Theorem 6 (Erdős-Szekeres)**.** *If $n > rs$, then there is either an increasing subsequence of length $r+1$ or a decreasing subsequence of length $s+1$.*

Proof Suppose not. For each $i \in [n]$, let x_i be the length of the longest increasing subsequence that ends at a_i, and y_i be the length of the longest decreasing subsequence that starts at a_i. Since there is no increasing subsequence of length $r+1$, $x_i \in [r]$ for every i. Similarly, $y_i \in [s]$ for all i. Thus the number of such tuples (x_i, y_i) is at most rs. By the pigeonhole principle, there must be $i < j$ such that $x_i = x_j, y_i = y_j$. If $a_i \leq a_j$, this is impossible, since we can extend the longest increasing subsequence that ends at a_i by a_j to get a longer one. If $a_i \leq a_j$, this is again impossible since we can extend the longest decreasing sequence that starts at a_j by prefixing a_i to it. ■

A *partial* order on a set S is a subset of $S \times S$. We write $x < y$ to indicate that (x, y) is in the partial order. We require that $x < y$ and $y < z$ implies that $x < z$. A useful example to keep in mind is partial order on $2^{[n]}$ where $I < J$ if and only if $I \subseteq J$.

A *chain* in the partial order is an increasing sequence in it. An *antichain* is a set of incomparable elements.

Using the same idea as above, we can prove:

Theorem 7 (Dilworth)**.** *If $|S| > rs$, then there is either a chain of size $r+1$ or an antichain of size $s+1$.*

Proof Suppose every chain is of length at most r. Then for each element x, let a_x denote the length of longest chain that ends at x. By averaging, there must be $|S|/r > s$ elements for which this value is the same. We claim that this set forms a set of incomparable elements. Indeed, if two of the elements of this set are comparable, i.e. $y \neq x, a_y = a_x$, then if $y < x$, we can extend the longest chain ending at y by x to get a chain that ends at x with $r + 1$ elements. ■