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Lecture 6 Hall's Theorem

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1 Hall's Theorem

In an undirected graph, a *matching* is a set of disjoint edges.

Given a bipartite graph with bipartition A, B, every matching is obviously of size at most $|A|$. Hall's Theorem gives a nice characterization of when such a matching exists.

Theorem 1. *There is a matching of size A if and only if every set $S \subseteq A$ of vertices is connected to at least $|S|$ vertices in B.*

Proof If such a matching exists, then clearly S must have at least $|S|$ neighbors just by the edges of the matching.

We shall prove the other direction by induction on the size of $|A|$. When $|A| = 1$, the result is trivial.

For the general case, pick an arbitrary $x \in A$. x must have at least one neighbor $y \in B$. We can try to match x to y and find a matching of size $|A| - 1$ in the graph induced on $A - \{x\}$, $B - \{y\}$ by induction. The only case in which this would not work is when there is some set $S \subseteq A - \{x\}$ that has less than $|S|$ neighbors in $B - \{y\}$. Then S must have *exactly* $|S|$ neighbors in B.

Let T denote the neighbors of S. By induction, there is a matching that matches all vertices of S to a vertex of T. For every set $S' \subseteq A - S$, since $S' \cup S$ has at least $|S'| + |S| = |S'| + |T|$ neighbors in B, S' must have at least $|S'|$ neighbors in $B - T$. Thus there is a matching that matches all the vertices of S' to $B - T$. ■

What about when A, B are countable but not necessarily finite? If A, B are the positive integers, and $1 \in A$ is connected to every vertex of B, and every other vertex $a \in A$ is connected only to $a - 1$, then you can check that there is no matching that matches all of A. On the other hand, the following is a theorem:

Theorem 2. *If A, B are countable, every vertex of A has a finite number of neighbors, and for every $S \subseteq A$ of finite size, S has at least $|S|$ neighbors, then there is a matching touching every vertex of A.*

1.1 Characterizing Doubly Stochastic Matrices

Given an $n \times n$ matrix with non-negative entries, we say that the matrix is *stochastic* if for each row, the sum of the entries in the row add up to exactly 1. The matrix is said to be *doubly stochastic* if the sum of the entries in each column also add up to 1. A stochastic matrix corresponds to the transition function of a Markov chain on n states. A doubly stochastic matrix corresponds to the transition function of a chain which the uniform distribution is a stationary distribution.

The simplest examples of doubly stochastic matrices are permutation matrices: given a permutation π of $[n]$, construct the matrix H^π where

$$H^\pi_{i,j} = \begin{cases} 1 & \text{if } \pi(i) = j \\ 0 & \text{else.} \end{cases}$$

Clearly, every row and column of a permutation matrix contains exactly one 1.

Theorem 3. *Every doubly stochastic matrix is a convex combination of permutation matrices — i.e. the doubly stochastic matrix is a weighted average of permutation matrices.*

In other words, if we have a doubly stochastic matrix, then we can always describe the corresponding Markov chain as: sample a permutation from some distribution, and then apply the permutation.

Proof Suppose we have a matrix M whose column and row sums are exactly λ. We shall proceed by induction on the number of non-zero entries in the matrix. The number of non-zeroes is at least n (or else some row will add up to 0). In this case, by pigeonhole, every row and column must contain exactly 1, which means that the matrix is a permutation matrix.

For the general case, consider the bipartite graph where A is the set of rows, B is the set of columns, and we connect (i, j) exactly when $A_{i,j} > 0$. If $S \subseteq A$ has less than $|S|$ neighbors, that means that the total sum of entries in the rows of S is less than $\lambda|S|$, since each column can contribute at most λ. This cannot happen, since each row must contribute exactly λ. Thus Hall's condition is met, and we can find a matching in the graph of size n, which corresponds to some permutation matrix P.

Let μ be equal to the smallest non-zero entry of the matrix, then in $M - \mu P$, all entries are non-negative, the number of non-negative entries is one less than in M, and the row and column sums are exactly $\lambda - \mu$. The inductive hypothesis completes the proof.

The sum of all the weights in this combination must add to 1 if the initial λ was 1.

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1.2 Embedding Small Sets to Larger Sets

Theorem 4. *Let $k \le (n-1)/2$. Then there is an injective function $f : \binom{[n]}{k} \to \binom{[n]}{k+1}$.*

Proof On the left, sets of size k. On the right, sets of size $k + 1$, and an edge if the left vertex is included in the right vertex. Note that every vertex on the left has $n - k$ neighbors, and every vertex on the right has $k + 1$ neighbors.

Let S be any family of k-sets. The number of edges coming out of S is $|S|(n - k)$, therefore, the number of neighbors of S is at least $|S|(n - k)/(k + 1) \ge |S|$. Hall's theorem then completes the proof. ■

Corollary 5. *Let $\mathcal{F}$ be an antichain of sets of size at most $t \le (n-1)/2$. Let $\mathcal{F}_t$ denote all sets of size t that contain a set of $\mathcal{F}$. Then $|\mathcal{F}_t| \ge |\mathcal{F}|$.*

Proof Use Theorem 4 to find a function that maps sets of size 1 into sets of size 2 injectively. Apply this theorem to the sets of size 1 in $\mathcal{F}$ to find a new family where every set is a superset of some set of $\mathcal{F}$ and there are exactly $\mathcal{F}$ sets, and no sets of size 1. Repeat this to get rid of sets of size $2, 3, \ldots, k$. ■

1.3 Vertex Covers in Bipartite Graphs

First we discuss the case of general undirected graphs (not necessarily bipartite). A *vertex cover* in the graph is a set of vertices in a graph that touches every edge. Let $v(G)$ denote the size of the smallest vertex cover in the graph.

Suppose there is some maximal matching with k edges.

Claim 6. *$k \leq v(G) \leq 2k$.*

Proof Include both endpoints of each edge in the matching to get a vertex cover of size $2k$. Every other edge must touch one of the edges of the matching (or else the matching is not maximal). On the other hand, every vertex cover must cover the edges of the matching, for which k vertices are required. ■

Remark 7. *Observe that the claim implies that there cannot be a maximal matching those size is more than a factor of 2 bigger than the size of another maximal matching.*

Although a trivial algorithm can find maximal matchings in undirected graphs, it is NP-hard to solve the decision version of the vertex cover problem: given an undirected graph G and a number k, answer whether or not there is a vertex cover of size k in the graph. So any polynomial time algorithm for the vertex cover problem would show that P=NP.

In fact, we know how to find a matching of largest size in polynomial time. The next theorem shows that this gives an algorithm for vertex cover in bipartite graphs:

Theorem 8. *In a bipartite graph, the size of the smallest vertex cover is equal to the size of the biggest matching.*

Proof By Claim 6, we only need to show that there is a matching which is as large as the smallest vertex cover. Let A, B be the bipartition, and let $S \subseteq A, T \subseteq B$ be such that $S \cup T$ is a smallest vertex cover.

There are no edges from $A - S$ to $B - T$, since $S \cup T$ is a vertex cover. We shall find a matching of size $|S|$ from S to $B - T$, and a matching of size $|T|$ from T to $A - S$. The union will give us a matching of size $|S| + |T|$. Consider any subset $Q \subseteq S$. Note that every edge in the graph that touches Q touches a neighbor of Q. Thus, if Q has less than $|Q|$ neighbors in $B - T$, then we can remove Q from the vertex cover and add its neighbors to get a smaller vertex cover. This will cover all edges from Q into B. Thus Q must have at least $|Q|$ neighbors in $B - T$. By Hall's theorem, there is a matching from S to $B - T$ of size $|S|$. Proceeding in the same way, we find a matching from T to $A - S$ of size $|T|$.

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