CSE599s: Extremal Combinatorics	October 26, 2011
Lecture 9 Intersecting Families	
Lecturer: Anup Rao	

We started this lecture by finishing the proof of the monotone circuit lowerbound (see lecture 8).

1 Intersecting Families

An *intersecting* family of sets is a collection of sets such that every two sets intersect. One can obtain a large family on the universe $[n]$ be forcing all sets to include 1. In fact, you cannot do any better:

Fact 1. *Every intersecting family $\mathcal{F}$ satisfies $|\mathcal{F}| \geq 2^{\binom{n}{2}}/2$.*

Proof For every set A, the family can contain either A or its complement, but not both. ■

What if we restrict the family to have sets of size k? If $k \geq n/2$ then every two k-sets intersect, so the result is trivial. If $k \leq n$, including a fixed element gives a family of size $\binom{n-1}{k-1}$. Indeed, this is optimal:

Theorem 2 (Erdös, Ko, Rado)**.** *Every intersecting family of k-sets has at most $\binom{n-1}{k-1}$ elements.*

Proof (Due to Katona)

We start by understanding the special case when all the sets are intervals. For each $s \in [n]$, define the cyclic interval sets $B_s = \{s, s+1, \ldots, s+k-1\}$, where the numbers are viewed mod n.

Claim 3. *At most k of the sets B_s can belong to $\mathcal{F}$.*

Note that B_i is disjoint from B_{i+k}. Thus if B_0 is included, then $-(k-1) \leq s \leq (k-1)$ for all other sets of $\mathcal{F}$, but only half of these remaining ones can be included since only one of each pair $(B_{-(k-1)}, B_1), \ldots, (B_{-1}, B_{k-1})$ can be included. That proves the claim.

Now apply a random permutation to the universe. Any fixed k-set can get mapped to $\binom{n}{k}$ positions by the permutation, of which n correspond to intervals. Thus the probability that the set becomes an interval is exactly $n/\binom{n}{k}$. Thus the expected number of intervals from the family is $|\mathcal{F}| \cdot n \cdot \binom{n}{k} \leq k \Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$. ■

One can also investigate the size of general families when they are forced to have larger intersections (say intersections of size t). In this case it turns out that the optimum is achieved when you take all sets of size larger than $n/2 + t/2$.

Theorem 4. *If $n+t$ is even, the size of the largest t-intersecting family is at most $\sum_{i=(n+t)/2}^{n} \binom{n}{i}$.*

Proof We shall massage the family into structure.

Given a t-intersecting family $\mathcal{F}$, and $A \in \mathcal{F}$ let $C_{ij}(A)$ be obtained as follows. For each set $A \in \mathcal{F}$, if replacing j with i gives a set A' that is not in $\mathcal{F}$, do so and replace A with the new version. Let $C_{ij}(\mathcal{F})$ denote the family obtained by applying this transformation to all sets in the family.

Claim 5. *$C_{ij}(\mathcal{F})$ is still t-intersecting.*

Given any two sets A, B, if the operation changes both sets, or neither of the sets, then the size of $A \cap B$ remains the same. If one of A, B contain both i, j, then also the intersection size remains as large as it was before. Otherwise, it must be that before the transformation A contains j and not i and B contains j and not i, but A is changed and B is not. In this case, there must be a set B' that is the same as B except that it contains i and not j. Then $|A \cap B'| \geq t$ before the transformation, so $|A \cap B| \geq t$ after the transformation.

We repeatedly apply C_{ij} for all $i < j$, until the family is no longer affected. The family is now *left compressed*. We continue the proof by induction on n, t. When $t = 1$, the claim is easily seen to be true.

Otherwise, we have families $\mathcal{F}_1 = \{A - \{1\} : 1 \in A \in \mathcal{F}\}$. By induction we get a $t-1$ intersecting family on $n-1$ elements so

$$|\mathcal{F}| \leq \sum_{i=(n+t)/2-1} \binom{n-1}{i}.$$

Next, let $\mathcal{F}_{1^c} = \{B : 1 \notin B \in \mathcal{F}\}$. If $|B_1 \cap B_2| = t$, $x \in B_1 \cap B_2$, $|C_{1x}(B_1) \cap B_2| < t$. So again by induction, we have $t+1$ intersecting family on a universe $|\mathcal{F}_{1^c}| \leq \sum_{i=(n+t)/2} \binom{n-1}{i}$. Together $|\mathcal{F}| \leq \binom{n-1}{n-1} + \sum_{i=(n+t)/2}^{n-1} \binom{n-1}{i} + \binom{n-1}{i-1}$ which is what we want. ■