CSE431: Complexity Theory April 26, 2024

Midterm

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For each of the following assertions:

- (3 points) State whether they are True, False, or Unknown to the best of your knowledge of complexity theory.
- (2 points) Briefly justify your answer.

1. (25 points, 5 each)
   (a) $\mathbf{P} \neq \mathbf{NP}$.
   (b) There is a function $f : \{0,1\}^* \to \{0,1\}$ that cannot be computed by any Turing machine.
   (c) For every n, there is a function $f : \{0,1\}^n \to \{0,1\}$ that cannot be computed by any boolean circuit.
   (d) For n large enough, it holds that there is a function $f : \{0,1\}^n \to \{0,1\}$ that can be computed by a circuit of size $2^{\sqrt{n}}$, but cannot be computed by a circuit of size n^2.
   (e) If $\mathbf{P} \neq \mathbf{NP}$, and n is large enough, then any algorithm for solving the independent set problem must take time at least 2^n on graphs with n vertices.

2. (25 points, 5 each)
   (a) Given the code of a Turing machine α and an input x, there is a machine $M(\alpha, x)$ that outputs 1 if the machine corresponding to α halts on input x within $2^{|x|}$ steps, and outputs 0 otherwise.
   (b) Every function $f : \{0,1\}^n \to \{0,1\}$ can be computed by a circuit of size $O(2^n/n^2)$.
   (c) If there is a deterministic logspace algorithm to check whether or not two vertices in a directed graph are connected, then $\mathbf{L} = \mathbf{NL}$.
   (d) There is a non-deterministic logspace algorithm that takes as input directed graph, a vertex s and a number k, and either aborts, or outputs all the vertices at distance k from s in the graph.
   (e) There is an algorithm that can take an undirected graph and two vertices s, t as input and output whether or not there is a path between s and t in $O(\log^2 n)$ space.

3. (25 points, 5 each)
   (a) $\mathbf{PSPACE} \subseteq \mathbf{EXP}$.
   (b) $co\mathbf{NL} \neq \mathbf{PSPACE}$.
   (c) The problem of determining whether or not a graph can be colored with 5 colors is $\mathbf{NP}$-complete.
   (d) There is an algorithm for $\mathsf{3SAT}$ that takes n^2 time.

(e) Every function $f : \{0,1\}^* \rightarrow \{0,1\}$ that is computed by a Turing machine can also be computed by a polynomial sized family of circuits.

4. (20 points) A 2CNF formula is a formula of the type $(x \vee y) \wedge (\neg x \vee z) \wedge \ldots$. In other words, each clause contains 2 variables. The function 2SAT(ϕ) takes a 2CNF as input, and outputs 1 if ϕ is satisfiable. Otherwise it outputs 0. Every clause of the form $(x \vee y)$ in ϕ can be thought of as asserting the statement $\neg x \Rightarrow y$) (i.e. if x is not true then y is true). Thus the formula defines a directed graph, where the vertices are variables or their negations, and every clause in the formula gives an edge of the graph.

(a) Show that the formula is satisfiable if and only if there is no variable x such that there is a path from x to $\neg x$ and a path from $\neg x$ to x in the graph.

(b) Prove that 2SAT is in NL.