Another counting argument

September 1, 2026

1 Idea

I want to try and come up with a proof for the $G(n,p)$ case that avoids the coupling argument altogether.

Suppose G is a $G(n,p)$ graph with p very close to 1, but here assume that the graph has exactly $p\binom{n}{2}$ edges. Suppose Z is a random clique of size at most t (vertices), with

$$p(S \subseteq Z) \leq r^{-\tau(S)},$$

where here S is another graph that is a partition into cliques, $\tau(S)$ is the usual measure (sum of size of each clique minus 1), and r is a parameter that we leave free for now.

For each Z as above, there is the minimal fragment A, namely

$$A = Z' \setminus G,$$

where $Z' \in \mathsf{supp}(Z)$ is the lexicographically first graph that minimizes $|A|$ subject to the constraint that $Z' \subseteq G \cup Z$.

Then we can count the number of pairs (G, A) with $|A| = a$ as follows (here a is the number of edges in A). The number of choices for $G \cup A$ is at most

$$\begin{aligned}
\binom{\binom{n}{2}}{p\binom{n}{2}+a} &= \frac{\binom{n}{2}!}{\left(p\binom{n}{2}\right)!\,\left((1-p)\binom{n}{2}\right)!} \cdot \frac{\left(p\binom{n}{2}\right)!}{\left(p\binom{n}{2}+a\right)!} \cdot \frac{\left((1-p)\binom{n}{2}\right)!}{\left((1-p)\binom{n}{2}-a\right)!} \\
&= \binom{\binom{n}{2}}{p\binom{n}{2}} \prod_{i=1}^{a} \frac{(1-p)\binom{n}{2}-a+i}{p\binom{n}{2}+i} \\
&\leq \binom{\binom{n}{2}}{p\binom{n}{2}} \cdot \left(\frac{(1-p)}{p}\right)^{a}.
\end{aligned}$$

Given $G \cup A$, find the lexicographically first graph Z'' in the support of Z that is contained in it. Then, the choice of A implies that $A \subseteq Z''$. We can specifiy A by describing the partition of vertices that corresponds to its connected components, and then by describing its edges. If A has ℓ connnected components, and the i'th connected component has v_i vertices and a_i edges, then the number of choices for A given $G \cup A$ is at most

$$\prod_{i=1}^{\ell} \binom{t}{v_i} \cdot \binom{\binom{v_i}{2}}{a_i},$$

so the expected number of fragments of size a is at most:

$$\sum_{\ell=1}^{t} \sum_{\substack{v_1+\cdots+v_\ell \le t-\ell \\ a_1+\cdots+a_\ell = a}} \prod_{i=1}^{\ell} \Big(\frac{1-p}{p}\Big)^{a_i} \cdot \binom{t}{v_i} \cdot \binom{\binom{v_i}{2}}{a_i} \cdot r^{-(v_i-1)}.$$

When $(1-p)/p \le \frac{v'}{\binom{v'}{2}-v'+1}$ for $v' \ge 2$, the ratio between consecutive terms satisfies:

$$\begin{aligned} \frac{((1-p)/p)^{a'+1}\binom{t}{v'}\binom{\binom{v'}{2}}{a'+1}}{((1-p)/p)^{a'}\binom{t}{v'}\binom{\binom{v'}{2}}{a'}} &= \Big(\frac{1-p}{p}\Big)\frac{\binom{v'}{2}-a'}{a'+1} \\ &\le \Big(\frac{1-p}{p}\Big)\frac{\binom{v'}{2}-(v'-1)}{v'} =: \delta \le \frac{1}{2} \end{aligned}$$

for all $a' \ge v'-1$. We get

$$\Big(\frac{1-p}{p}\Big)^{a'}\binom{t}{v'}\binom{\binom{v'}{2}}{a'} \le \Big(\frac{1-p}{p}\Big)^{v'-1}\binom{t}{v'}\binom{\binom{v'}{2}}{v'-1} \cdot \delta^{a'-(v'-1)}.$$

Multiplying over all $i \in \{1,\ldots,\ell\}$ and using the exact identity $\sum_{i=1}^{\ell} a_i = a$, the product of the decay factors factors out of the sum:

$$\prod_{i=1}^{\ell} \delta^{a_i-(v_i-1)} = \delta^{\sum_{i=1}^{\ell} a_i - \sum_{i=1}^{\ell}(v_i-1)} = \delta^a \prod_{i=1}^{\ell} \delta^{-(v_i-1)}.$$

Substituting this back in, bounding the number of compositions of a into ℓ parts ($a_1+\cdots+a_\ell = a$ with $a_i \ge v_i - 1 \ge 1$) by $\binom{a-1}{\ell-1}$, and using $\binom{t}{v'}\binom{\binom{v'}{2}}{v'-1} \le \frac{et}{v'}(\frac{e^2t}{2})^{v'-1} \le \frac{e^2t}{2}(\frac{e^2t}{2})^{v'-1}$ (via $v' \le e^{v'-1}$):

$$\begin{aligned} \mathbb{E}[\text{fragments of size } a] &\le \sum_{\substack{1\le\ell\le\min(t,a) \\ a_1+\cdots+a_\ell=a \\ v_1+\cdots+v_\ell\le t-\ell}} \prod_{i=1}^{\ell}\Bigg[\Big(\frac{1-p}{p}\Big)^{v_i-1}\binom{t}{v_i}\binom{\binom{v_i}{2}}{v_i-1} r^{-(v_i-1)}\delta^{a_i-(v_i-1)}\Bigg] \\ &= \delta^a \sum_{\substack{1\le\ell\le\min(t,a) \\ v_1+\cdots+v_\ell\le t-\ell}} \binom{a-1}{\ell-1}\prod_{i=1}^{\ell}\Bigg[\binom{t}{v_i}\binom{\binom{v_i}{2}}{v_i-1}\Big(\frac{1-p}{\delta p r}\Big)^{v_i-1}\Bigg] \\ &\le \delta^a \sum_{\ell=1}^{a}\binom{a-1}{\ell-1}\Bigg(\sum_{v'=2}^{\infty}\frac{e^2t}{2}\Big(\frac{e^2t(1-p)}{2\delta p r}\Big)^{v'-1}\Bigg)^{\ell}. \end{aligned}$$

Let $\varepsilon := \sum_{v'=2}^{\infty} \frac{e^2 t}{2} \left(\frac{e^2 t(1-p)}{2\delta pr}\right)^{v'-1} = \frac{e^2 t}{2} \cdot \frac{\frac{e^2 t(1-p)}{2\delta pr}}{1-\frac{e^2 t(1-p)}{2\delta pr}}$. Under our assumption that $\frac{t^2(1-p)}{pr} \ll \frac{1}{100}$, we have $\varepsilon \leq \frac{1}{100}$.

We then have

$$
\begin{aligned}
\mathbb{E}[\text{fragments of size } a] &\leq \delta^a \sum_{\ell=1}^{a} \binom{a-1}{\ell-1} \varepsilon^{\ell} \\
&= \varepsilon\, \delta^a (1+\varepsilon)^{a-1} \\
&= \frac{\varepsilon}{1+\varepsilon} \big(\delta(1+\varepsilon)\big)^a .
\end{aligned}
$$

Since $\delta \leq \frac{1}{2}$ and $\varepsilon \leq \frac{1}{100}$, $\delta(1+\varepsilon) \leq \frac{1}{2} \cdot \frac{101}{100} = \frac{101}{200} < 1$, which gives :

$$
\mathbb{E}[\text{fragments of size } a] \leq \frac{1}{100} \left(\frac{101}{200}\right)^a = 2^{-\Omega(a)}.
$$

2 Getting exponentially small error

I do not know how to do this yet, but here is a vague idea. Instead of encoding one fragment A, we should encode $G, A_1, \ldots A_k$ where A_i correspond to independent fragments. The point is that if any one of these fragments is non-empty, then all of them are. So, if we can recover all of them efficiently, we can do a union bound over the size of all fragments, which will look like

$$
\sum_{a_1, \ldots, a_k > 0} \cdots ,
$$

an expression that will look like a product of geometric series, each of which is bounded by 1/2.