Chaotic Iterations, Termination, Optimality, Correctness, JOP vs. LFP

cs264
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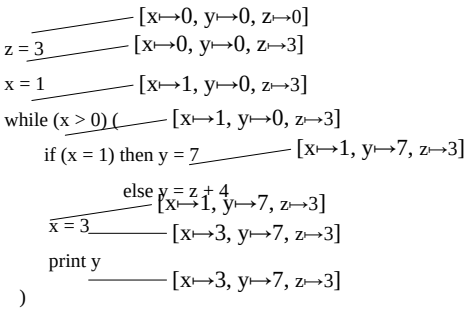
slides by Mooly Sagiv

Outline

- familiar constant propagation
- mathematical background: posets and lattices
- chaotic iterations
- termination
- precision

A Simple Example Program



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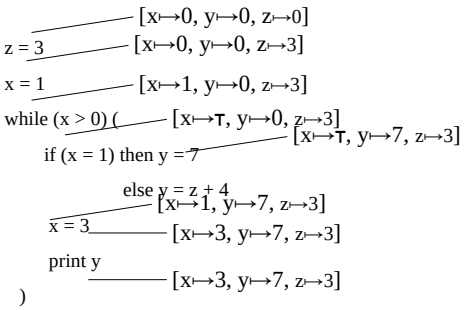

A Simple Example Program

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$[x\mapsto 0, y\mapsto 0, z\mapsto 0]$

z = 3 $[x\mapsto 0, y\mapsto 0, z\mapsto 3]$

x = 1 $[x\mapsto 1, y\mapsto 0, z\mapsto 3]$

while (x > 0) ($[x\mapsto \top, y\mapsto 0, z\mapsto 3]$

if (x = 1) then y = 7 $[x\mapsto 1, y\mapsto 7, z\mapsto 3]$

else y =z + 4 $[x\mapsto \top, y\mapsto 7, z\mapsto 3]$ $[x\mapsto \top, y\mapsto 7, z\mapsto 3]$

x = 3 $[x\mapsto 1, y\mapsto 7, z\mapsto 3]$

$[x\mapsto 3, y\mapsto 7, z\mapsto 3]$

print y

$[x\mapsto 3, y\mapsto 7, z\mapsto 3]$

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Computing Constants

- Construct a control flow graph (CFG)
- Associate transfer functions with control flow graph edges
- Iterate until a solution is found
- The solution is unique
  - But order of evaluation may affect the number of iterations

Constructing CFG

z = 3

x = 1

while (x > 0) (

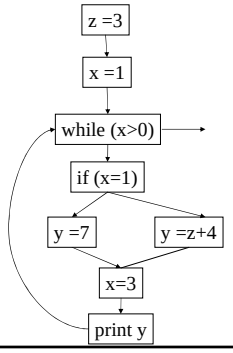
if (x = 1) then y = 7

else y =z + 4

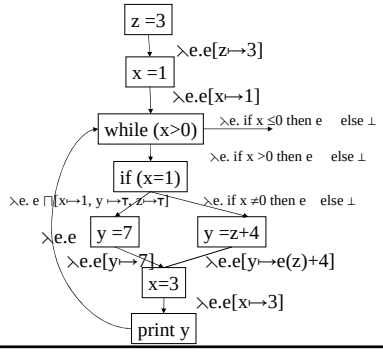
x = 3

print y

)



Associating Transfer Functions



Iterative Computation

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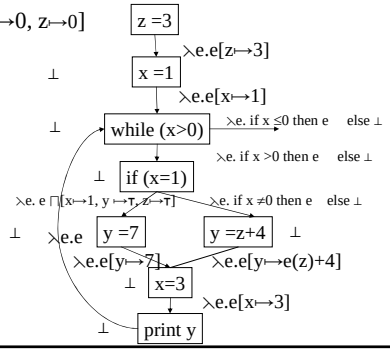

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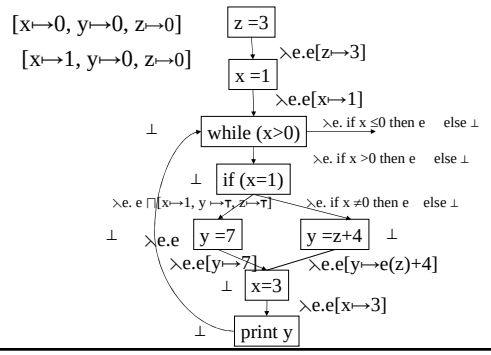

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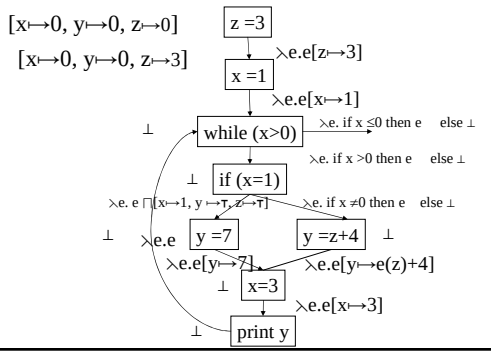
Iterative Computation



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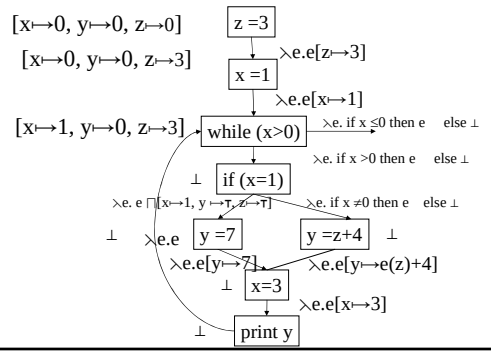
Iterative Computation



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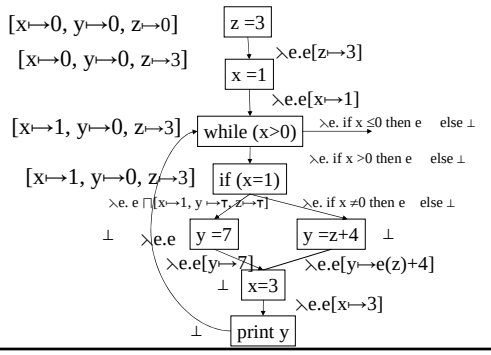
Iterative Computation



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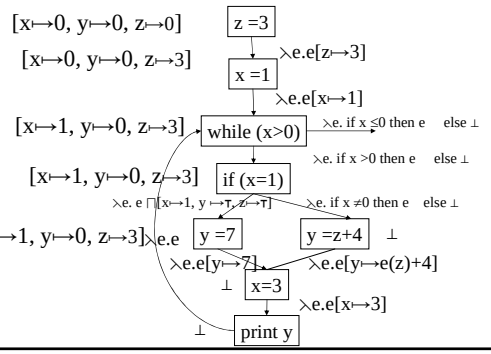
Iterative Computation



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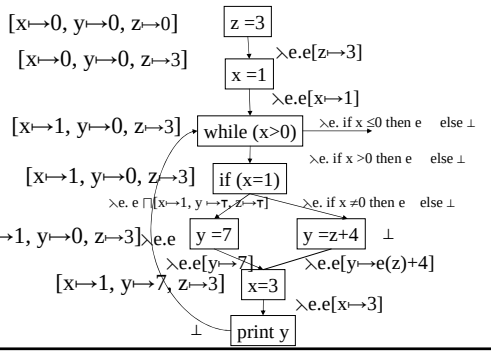
Iterative Computation



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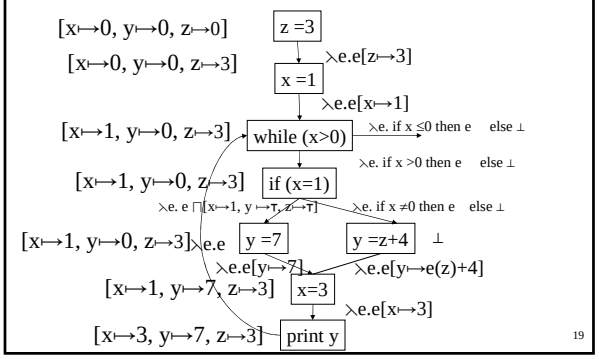
Iterative Computation



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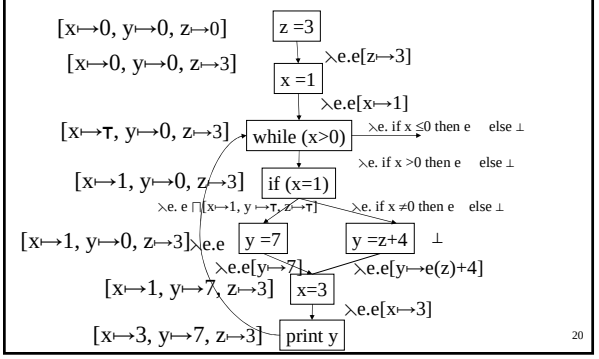
Iterative Computation



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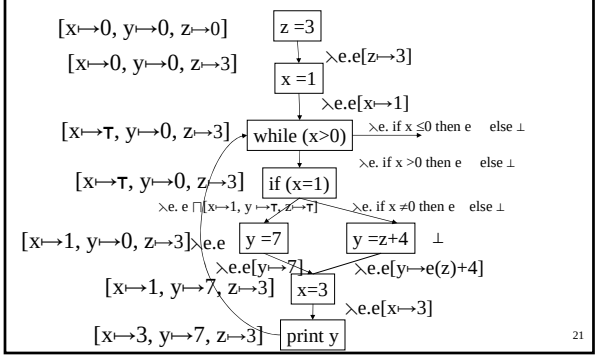
Iterative Computation



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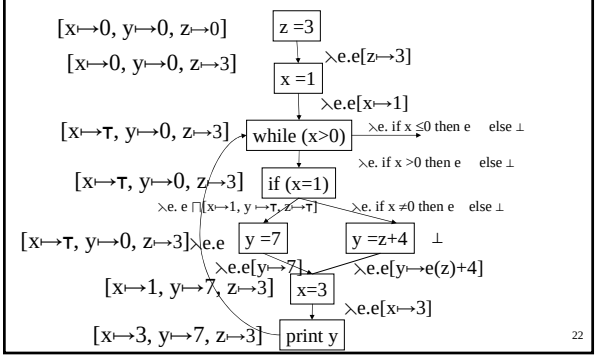
Iterative Computation



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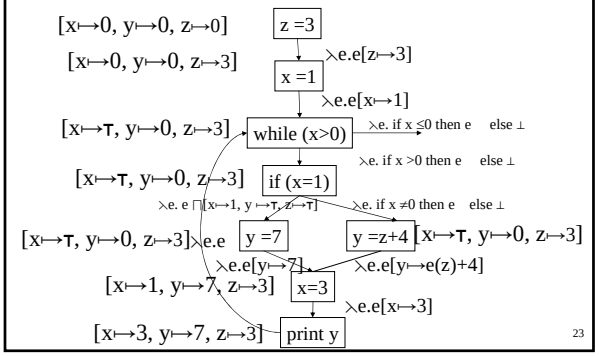
Iterative Computation



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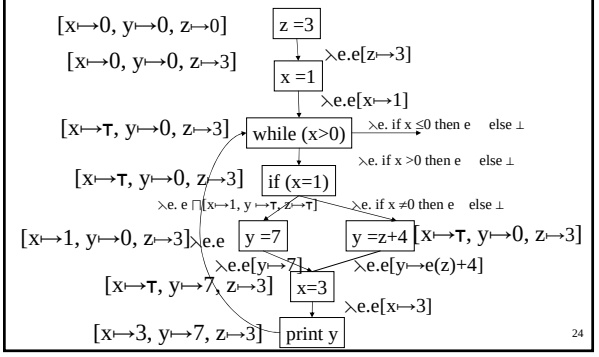
Iterative Computation



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Mathematical Background

- Declaratively define
  - The result of the analysis
  - The exact solution
  - Allow comparison

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Posets

- A partial ordering is a binary relation
  $\sqsubseteq : L \times L \rightarrow \{false, true\}$
  - For all $l \in L : l \sqsubseteq l$ (Reflexive)
  - For all $l_1, l_2, l_3 \in L : l_1 \sqsubseteq l_2, l_2 \sqsubseteq l_3 \Rightarrow l_1 \sqsubseteq l_3$ (Transitive)
  - For all $l_1, l_2 \in L : l_1 \sqsubseteq l_2, l_2 \sqsubseteq l_1 \Rightarrow l_1 = l_2$ (Anti-Symmetric)

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Posets

- Denoted by $(L, \sqsubseteq)$
- In program analysis:
  $l_1 \sqsubseteq l_2 \Leftrightarrow$
  l_1 is more precise than $l_2 \Leftrightarrow$
  l_1 represents fewer concrete states than l_2
- Examples
  - Total orders $(N, \leq)$
  - Powersets $(P(S), \subseteq)$
  - Powersets $(P(S), \supseteq)$
  - Constant propagation

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Upper and Lower Bounds

- Consider a poset $(L, \sqsubseteq)$
- A subset $L' \subseteq L$ has a lower bound $l \in L$ if for all $l' \in L' : l \sqsubseteq l'$
- A subset $L' \subseteq L$ has an upper bound $u \in L$ if for all $l' \in L' : l' \sqsubseteq u$

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Bounds

- A greatest lower bound of a subset $L' \subseteq L$ is a lower bound $l_0 \in L$ such that $l \sqsubseteq l_0$ for any lower bound l of L'
- A lowest upper bound of a subset $L' \subseteq L$ is an upper bound $u_0 \in L$ such that $u_0 \sqsubseteq u$ for any upper bound u of L'
- For every subset $L' \subseteq L$:
  - The greatest lower bound of L' is unique if at all exists
    - $\sqcap L'$ (meet) $a \sqcap b$
  - The lowest upper bound of L' is unique if at all exists
    - $\sqcup L'$ (join) $a \sqcup b$

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Complete Lattices

- A poset $(L, \sqsubseteq)$ is a complete lattice if every subset has least and upper bounds
- $L = (L, \sqsubseteq) = (L, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$
  - $\bot = \sqcup \emptyset = \sqcap L$
  - $\top = \sqcup L = \sqcap \emptyset$
- Examples
  - Total orders $(N, \leq)$
  - Powersets $(P(S), \subseteq)$
  - Powersets $(P(S), \supseteq)$
  - Constant propagation

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Complete Lattices

- Lemma For every poset $(L, \sqsubseteq)$ the following conditions are equivalent
  - L is a complete lattice
  - Every subset of L has a least upper bound
  - Every subset of L has a greatest lower bound

Cartesian Products

- A complete lattice $(L_1, \sqsubseteq_1) = (L_1, \sqsubseteq, \sqcup_1, \sqcap_1, \bot_1, \top_1)$
- A complete lattice $(L_2, \sqsubseteq_2) = (, \sqsubseteq, \sqcup_2, \sqcap_2, \bot_2, \top_2)$
- Define a Poset $L = (L_1 \times L_2, \sqsubseteq)$ where
  - $(x_1, x_2) \sqsubseteq (y_1, y_2)$ if
    - $x_1 \sqsubseteq y_1$ and
    - $x_2 \sqsubseteq y_2$
- L is a complete lattice

Finite Maps

- A complete lattice $(L_1, \sqsubseteq_1) = (L_1, \sqsubseteq, \sqcup_1, \sqcap_1, \bot_1, \top_1)$
- A finite set V
- Define a Poset $L = (V \to L_1, \sqsubseteq)$ where
  - $e_1 \sqsubseteq e_2$ if for all $v \in V$
    - $e_1 v \sqsubseteq e_2 v$
- L is a complete lattice

Chains

- A subset $Y \subseteq L$ in a poset $(L, \sqsubseteq)$ is a chain if every two elements in Y are ordered
  - For all $l_1, l_2 \in Y$: $l_1 \sqsubseteq l_2$ or $l_2 \sqsubseteq l_1$
- An ascending chain is a sequence of values
  - $l_1 \sqsubseteq l_2 \sqsubseteq l_3 \sqsubseteq \ldots$
- A strictly ascending chain is a sequence of values
  - $l_1 \sqsubset l_2 \sqsubset l_3 \sqsubset \ldots$
- A descending chain is a sequence of values
  - $l_1 \sqsupseteq l_2 \sqsupseteq l_3 \sqsupseteq \ldots$
- A strictly descending chain is a sequence of values
  - $l_1 \sqsupset l_2 \sqsupset l_3 \sqsupset \ldots$
- L has a finite height if every chain in L is finite
- Lemma A poset $(L, \sqsubseteq)$ has finite height if and only if every strictly decreasing and strictly increasing chains are finite

Monotone Functions

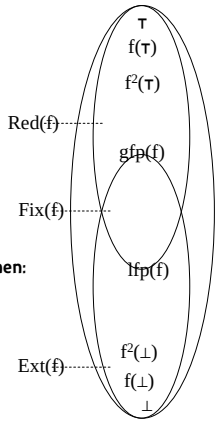
- A poset $(L, \sqsubseteq)$
- A function $f: L \to L$ is monotone if for every $l_1, l_2 \in L$:
  - $l_1 \sqsubseteq l_2 \Rightarrow f(l_1) \sqsubseteq f(l_2)$

Galois Connection

- Lattices C and A and functions $\alpha: C \to A$ and $\gamma: A \to C$
- The pair of functions (α, γ) form Galois connection if
  - α and γ are monotone
  - $\forall a \in A: \alpha(\gamma(a)) \sqsubseteq a$
  - $\forall c \in C: c \sqsubseteq \gamma(\alpha(C))$
- Alternatively if:
  $\forall c \in C . \forall a \in A$
  $\alpha(c) \sqsubseteq a$ iff $c \sqsubseteq \gamma(a)$
- α and γ uniquely determine each other

Fixed Points

- A monotone function f: L → L where (L, ⊑, ⊔, ⊓, ⊥, ⊤) is a complete lattice
- Fix(f) = { l: l ∈ L, f(l) = l}
- Red(f) = {l: l ∈ L, f(l) ⊑ l}
- Ext(f) = {l: l ∈ L, l ⊑ f(l)}
  - $l_1 \sqsubseteq l_2 \Rightarrow f(l_1) \sqsubseteq f(l_2)$
- Tarski's Theorem 1955: if f is monotone then:
  - lfp(f) = ⊓ Fix(f) = ⊓ Red(f) ∈ Fix(f)
  - gfp(f) = ⊔ Fix(f) = ⊔ Ext(f) ∈ Fix(f)



Chaotic Iterations

- A lattice L = (L, ⊑, ⊔, ⊓, ⊥, ⊤) with finite strictly increasing chains
- $L^n = L \times L \times \ldots \times L$
- A monotone function $\underline{f}: L^n \rightarrow L^n$
- Compute lfp($\underline{f}$)
- The simultaneous least fixed of the system $\{x[i] = \underline{f}_i(x) : 1 \leq i \leq n\}$

$\underline{x} := (\bot, \bot, \ldots, \bot)$

while ($\underline{f}(x) \neq \underline{x}$) do

$\underline{x} := \underline{f}(\underline{x})$

Chaotic Iterations

```
for i :=1  to n do
    x[i] = ⊥
WL = {1, 2, …, n}
while (WL ≠  Ø ) do
   select and remove an element i ∈ WL
   new := f_i(x)
    if (new ≠ x[i]) then
        x[i] := new;
        Add all the indexes that directly depends on i to WL
```

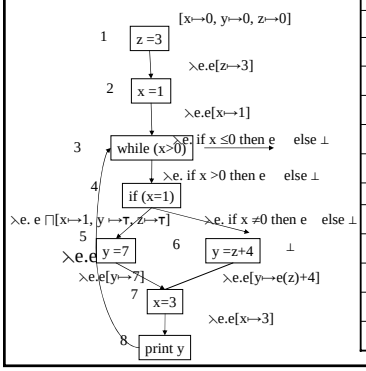

Precision

Least fixed point gives us

- Minimum number of non-constant
- Maximum number of ⊥

Specialized Chaotic Iterations

```
Chaotic(G(V, E): Graph, s: Node, L: Lattice, ι: L, f: E →(L →L) ){
   for each v in V to n do df_entry[v] := ⊥
   df[v] = ι
   WL = {s}
   while (WL ≠  Ø ) do
      select and remove an element u ∈ WL
      for each v, such that. (u, v) ∈E do
            temp = f(e)(df_entry[u])
            new := df_entry(v)⊔ temp
            if (new ≠ df_entry[v]) then
                  df_entry[v] := new;
                  WL := WL ∪{v}
```

WL	$df_{entry}[v]$
{1}	
{2}	df[2]:=[x↦0, y↦0, z↦3]
{3}	df[3]:=[x↦1, y↦0, z↦3]
{4}	df[4]:=[x↦1, y↦0, z↦3]
{5}	df[5]:=[x↦1, y↦0, z↦3]
{7}	df[7]:=[x↦1, y↦7, z↦3]
{8}	df[8]:=[x↦3, y↦7, z↦3]
{3}	df[3]:=[x↦⊤, y↦⊤, z↦3]
{4}	df[4]:=[x↦⊤, y↦⊤, z↦3]
{5,6}	df[5]:=[x↦1, y↦⊤, z↦3]
{6,7}	df[6]:=[x↦⊤, y↦⊤, z↦3]
{7}	df[7]:=[x↦⊤, y↦7, z↦3]

Specialized Chaotic Iterations: System of Equations

$$S = \left\{ \begin{array}{l} df_{entry}[s] = \iota \\ df_{entry}[v] = \sqcup\{f(u, v)\,(df_{entry}[u]) \mid (u, v) \in E\} \end{array} \right\}$$

$F_S: L^n \rightarrow L^n$

$F_S(X)[s] = \iota$

$F_S(X)[v] = \sqcup\{f(u, v)(X[u]) \mid (u, v) \in E\}$

$$lfp(S) = lfp(F_S)$$

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Complexity of Chaotic Iterations

- Parameters:
  - n the number of CFG nodes
  - k is the maximum outdegree of edges
  - A lattice of height h
  - c is the maximum cost of
    - applying $f_{(e)}$
    - $\sqcup$
    - L comparisons
- Complexity: O(n * h * c * k)

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Soundness, Completeness

- Soundness:
  - Every detected constant is indeed such
  - Every error will be detected
  - The least fixed point contains all occurring runtime states
- Completeness:
  - Every constant is indeed detected as such
  - Every detected error is real
  - Every state contained in the least fixed point is reachable for some input

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The Abstract Interpretation Technique

- The foundation of program analysis
- Goals
  - Establish soundness of (find faults in) a given program analysis algorithm
  - Design new program analysis algorithms
- The main ideas:
  - Relate each step in the algorithm to a step in a structural operational semantics
  - Establish global correctness using a general theorem
- Not limited to a particular form of analysis

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Soundness in Constant Propagation

- Every detected constant is indeed such
- May include fewer constants
- May miss $\perp$
- At every CFG node *l*
  All constants in $df_{entry}(l)$ are indeed constants
- At every CFG node l $df_{entry}(l)$ "represents" all the possible concrete states arising when the structural operational semantics reaches *l*

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Proof of Soundness

- Define an "appropriate" structural operational semantics
- Define "collecting" structural operational semantics
- Establish a Galois connection between collecting states and reaching definitions
- (Local correctness) Show that the abstract interpretation of every atomic statement is sound
  w.r.t. the collecting semantics
- (Global correctness) Conclude that the analysis is sound CC1976

$$\gamma(lfp(S)[v, entry]) \supseteq REACH[v, entry]$$

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Structural Semantics for While

axioms

$[ass_{sos}]$ $\langle x := a, s\rangle \Rightarrow s[x \mapsto \mathbf{A}[\![a]\!]s]$

$[skip_{sos}]$ $\langle \mathbf{skip}, s\rangle \Rightarrow s$

rules

$$[comp^1_{sos}]\ \frac{\langle S_1, s\rangle \Rightarrow \langle S'_1, s'\rangle}{\langle S_1; S_2, s\rangle \Rightarrow \langle S'_1; S_2, s'\rangle}$$

$$[comp^2_{sos}]\ \frac{\langle S_1, s\rangle \Rightarrow s'}{\langle S_1; S_2, s\rangle \Rightarrow \langle S_2, s'\rangle}$$

Structural Semantics for While if construct

$[if^{tt}_{sos}]$ $\langle$if b then S_1 else S_2, s$\rangle \Rightarrow \langle S_1, s\rangle$ if $\mathbf{B}[\![b]\!]s=tt$

$[if^{ff}_{os}]$ $\langle$if b then S_1 else S_2, s$\rangle \Rightarrow \langle S_2, s\rangle$ if $\mathbf{B}[\![b]\!]s=ff$

Structural Semantics for While while construct

$[while_{sos}]$ $\langle$while b do S, s$\rangle \Rightarrow$
$\langle$if b then (S; while b do S) else skip, s$\rangle$

Collecting Semantics

- **The input state is not known at compile-time**
- **"Collect" all the states for all possible inputs to the program**
- **No lost of precision**

A Simple Example Program

$\{[x\mapsto 0, y\mapsto 0, z\mapsto 0]\}$

z = 3 $\{[x\mapsto 0, y\mapsto 0, z\mapsto 3]\}$

x = 1 $\{[x\mapsto 1, y\mapsto 0, z\mapsto 3]\}$

while (x > 0) ($\{[x\mapsto 1, y\mapsto 0, z\mapsto 3], [x\mapsto 3, y\mapsto 0, z\mapsto 3],\}$

if (x = 1) then y = 7 $\{[x\mapsto 1, y\mapsto 7, z\mapsto 3], [x\mapsto 3, y\mapsto 7, z\mapsto 3]\}$

else y = z + 4

x = 3 $\{[x\mapsto 1, y\mapsto 7, z\mapsto 3], [x\mapsto 3, y\mapsto 7, z\mapsto 3]\}$

print y $\{[x\mapsto 3, y\mapsto 7, z\mapsto 3]\}$

) $\{[x\mapsto 3, y\mapsto 7, z\mapsto 3]\}$

Another Example

x= 0

while (true) do

x = x +1

An "Iterative" Definition

- Generate a system of monotone equations
- The least solution is well-defined
- The least solution is the collecting interpretation
- But may not be computable

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Equations Generated for Collecting Interpretation

- Equations for elementary statements
  - [skip]
    $CS_{exit}(1) = CS_{entry}(l)$
  - [b]
    $CS_{exit}(1) = \{\sigma: \sigma \in CS_{entry}(l), [\![b]\!]\sigma = tt\}$
  - [x := a]
    $CS_{exit}(1) = \{(s[x \mapsto \mathbf{A}[\![a]\!]s]) \mid s \in CS_{entry}(l)\}$
- Equations for control flow constructs
  $CS_{entry}(l) = \cup CS_{exit}(l')$ *l'* immediately precedes *l* in the control flow graph
- An equation for the entry
  $CS_{entry}(1) = \{\sigma \mid \sigma \in Var_* \to Z\}$

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Specialized Chaotic Iterations System of Equations (Collecting Semantics)

S =

$$\left\{\begin{array}{l} CS_{entry}[s] = \{\sigma_0\} \\ CS_{entry}[v] = \cup\{f(e)(CS_{entry}[u]) \mid (u, v) \in E\} \\ \text{where } f(e) = \lambda X.\ \{[\![st(e)]\!]\sigma \mid \sigma \in X\} \text{ for atomic statements} \\ \qquad f(e) = \lambda X.\{\sigma \mid [\![b(e)]\!]\sigma = tt\} \end{array}\right\}$$

$F_S: L^n \to L^n$

$F_s(X)[v] = \cup\{f(e)[u] \mid (u, v) \in E\}$

$lfp(S) = lfp(F_S)$

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The Least Solution

- 2n sets of equations
  $CS_{entry}(1), \ldots, CS_{entry}(n), CS_{exit}(1), \ldots, CS_{exit}(n)$
- Can be written in vectorial form
  $$\overrightarrow{CS} = F_{cs}(\overrightarrow{CS})$$
- The least solution $lfp(F_{cs})$ is well-defined
- Every component is minimal
- Since F_{cs} is monotone such a solution always exists
- $CS_{entry}(v) = \{s \mid \exists s_0 \mid \langle P, s_0\rangle \Rightarrow^* (S', s)),$
  $init(S') = v\}$
- Simplify the soundness criteria

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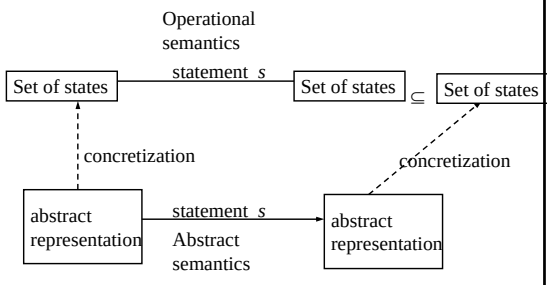
Abstract (Conservative) interpretation



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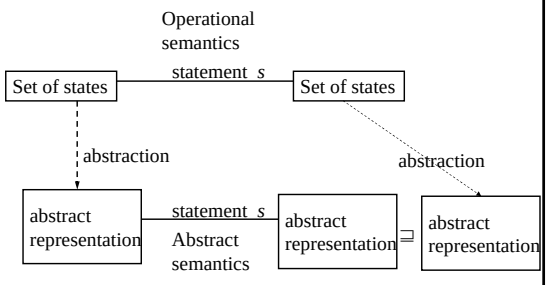
Abstract (Conservative) interpretation



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The Abstraction Function

- Map collecting states into constants
- The abstraction of an individual state
  β_{CP}:[Var$_\star \to$Z] $\to$ [Var$_\star \to$Z$\cup\{\bot, \top\}$]
  $\beta_{CP}(\sigma) = \sigma$
- The abstraction of set of states
  α_{CP}:P([Var$_\star \to$Z]) $\to$ [Var$_\star \to$Z$\cup\{\bot, \top\}$]
  α_{CP} (CS) = $\sqcup \{ \beta_{CP}(\sigma) \mid \sigma \in CS\} = \sqcup\{\sigma \mid \sigma \in CS\}$
- Soundness
  α_{CP} (CS_{entry} (v)) $\sqsubseteq$ df_{entry}(v)
- Completeness

The Concretization Function

- Map constants into collecting states
- The formal meaning of constants
- The concretization
  γ_{CP}: [Var$_\star \to$Z$\cup\{\bot, \top\}$] $\to$P([Var$_\star \to$Z])
  γ_{CP} (df) = $\{\sigma \mid \beta_{CP}(\sigma) \sqsubseteq df\} = \{\sigma \mid \sigma \sqsubseteq df\}$
- Soundness
  CS_{entry} (v) $\subseteq \gamma_{CP}$ (df_{entry}(v))
- Optimality

Galois Connection

- α_{CP} is monotone
- γ_{CP} is monotone
- $\forall$ df $\in$ [Var$_\star \to$Z$\cup\{\bot, \top\}$]
  - $\alpha_{CP}(\gamma_{CP}(df)) \sqsubseteq df$
- $\forall$ c $\in$ P([Var$_\star \to$Z])
  - $c_{CP} \sqsubseteq \gamma_{CP}(\alpha_{CP}(C))$

Local Concrete Semantics

- For every atomic statement S
  - $[\![S]\!]$: [Var$_\star \to$Z] $\to$[Var$_\star \to$Z]
  - $[\![x := a]\!]s = s[x \mapsto \mathbf{A}[\![a]\!]s]$
  - $[\![skip]\!]s = s$
- For Boolean conditions …

Local Abstract Semantics

- For every atomic statement S
  - $[\![S]\!]^\#$:Var$_\star \to$L $\to$ Var$_\star \to$L
  - $[\![x := a]\!]^\#(e) = e[x \mapsto [\![a]\!]^\#(e)]$
  - $[\![skip]\!]^\#(e) = e$
- For Booleans …

Local Soundness

- For every atomic statement S show one of the following
  - $\alpha_{CP}(\{[\![S]\!]\sigma \mid \sigma \in CS\} \sqsubseteq [\![S]\!]^\#(\alpha_{CP}(CS))$
  - $\{[\![S]\!]\sigma \mid \sigma \in \gamma_{CP}(df)\} \subseteq \gamma_{CP}([\![S]\!]^\#(df))$
  - $\alpha(\{[\![S]\!]\sigma \mid \sigma \in \gamma_{CP}(df)\}) \sqsubseteq [\![S]\!]^\#(df)$
- The above condition implies global soundness [Cousot & Cousot 1976]
  $\alpha(CS_{entry}(l)) \sqsubseteq df_{entry}(l)$
  $CS_{entry}(l) \subseteq \gamma(df_{entry}(l))$

Lemma 1

Consider a lattice L.

f: $L \to L$ is monotone iff

for all $X \subseteq L$: $\sqcup\{f(z) \mid z \in X\} \sqsubseteq f(\sqcup\{z \mid z \in X\})$

Assignments in constant propagation

- **Monotone**
  - $df_1 \sqsubseteq df_2 \to [\![x := e]\!]^{\#}(df_1) \sqsubseteq [\![x := e]\!]^{\#}(df_2)$
- Local Soundness
  - $\alpha(\{[\![x := e]\!]\sigma \mid \sigma \in \mathbf{CS}\}) \sqsubseteq [\![x := e]\!]^{*}(\alpha(\mathbf{CS}))$

Soundness Theorem(1)

1. Let (α, γ) form Galois connection from C to A
2. f: $C \to C$ be a monotone function
3. $f^{\#}: A \to A$ be a monotone function
4. $\forall a \in A$: $f(\gamma(a)) \sqsubseteq \gamma(f^{\#}(a))$

$lfp(f) \sqsubseteq \gamma(lfp(f^{\#}))$

$\alpha(lfp(f)) \sqsubseteq lfp(f^{\#})$

Soundness Theorem(2)

1. Let (α, γ) form Galois connection from C to A
2. f: $C \to C$ be a monotone function
3. $f^{\#}: A \to A$ be a monotone function
4. $\forall c \in C$: $\alpha(f(c)) \sqsubseteq f^{\#}(\alpha(c))$

$\alpha(lfp(f)) \sqsubseteq lfp(f^{\#})$

$lfp(f) \sqsubseteq \gamma(lfp(f^{\#}))$

Soundness Theorem(3)

1. Let (α, γ) form Galois connection from C to A
2. f: $C \to C$ be a monotone function
3. $f^{\#}: A \to A$ be a monotone function
4. $\forall a \in A$: $\alpha(f(\gamma(a))) \sqsubseteq f^{\#}(a)$

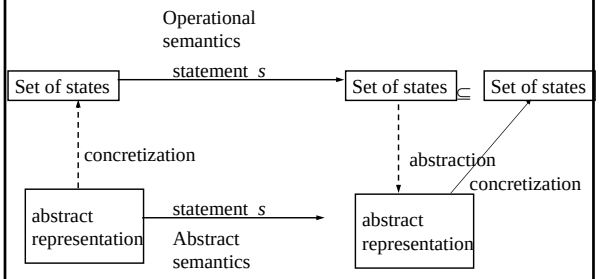
$\alpha(lfp(f)) \sqsubseteq lfp(f^{\#})$

$lfp(f) \sqsubseteq \gamma(lfp(f^{\#}))$

Proof of Soundness (Summary)

- **Define an "appropriate" structural operational semantics**
- **Define "collecting" structural operational semantics**
- **Establish a Galois connection between collecting states and reaching definitions**
- **(Local correctness) Show that the abstract interpretation of every atomic statement is sound w.r.t. the collecting semantics**
- **(Global correctness) Conclude that the analysis is sound**

Best (Conservative) interpretation



Induced Analysis (Relatively Optimal)

- It is sometimes possible to show that a given analysis is not only sound but optimal w.r.t. the chosen abstraction
  - but not necessarily optimal!
- Define $[\![S]\!]^{\#}(df) = \alpha(\{[\![S]\!]\sigma \mid \sigma \in \gamma(df)\})$
- But this $[\![S]\!]^{\#}$ may not be computable
- Derive (at compiler-generation time) an alternative form for $[\![S]\!]^{\#}$
- A useful measure to decide if the abstraction must lead to overly imprecise results

Example Dataflow Problem

- Formal available expression analysis
- Find out which expressions are available at a given program point
- Example program

```
x = y + t
z = y + r
while (…) {
    t = t + (y + r)
        }
```

- Lattice
- Galois connection
- Basic statements
- Soundness

Example: May-Be-Garbage

- A variable x may-be-garbage at a program point v if there exists a execution path leading to v in which x's value is unpredictable:
  - Was not assigned
  - Was assigned using an unpredictable expression
- Lattice
- Galois connection
- Basic statements
- Soundness

Points-To Analysis

- Determine if a pointer variable p may point to q on some path leading to a program point
- "Adapt" other optimizations
  - Constant propagation
    x = 5; *p = 7 ; … x …
- Pointer aliases
  - Variables p and q are may-aliases at v if the points-to set at v contains entries (p, x) and (q, x)
- Side-effect analysis
  *p = *q + * * t

The **PWhile** Programming Language Abstract Syntax

$a := x \mid *x \mid \&x \mid n \mid a_1\ op_a\ a_2$

$b :=$ true $\mid$ false $\mid$ not $b \mid\ b_1\ op_b\ b_2 \mid a_1\ op_r\ a_2$

$S := x := a \mid *x := a \mid$ skip $\mid S_1 ; S_2 \mid$
if b **then** S_1 **else** $S_2 \mid$ **while** b do S

Concrete Semantics 1 for PWhile

State1= [Loc→Loc∪Z]

For every atomic statement S

⟦S ⟧ : States1 →States1

⟦x := a ⟧(σ)=σ[loc(x) ↦**A**⟦a⟧ σ]

⟦x := &y ⟧(σ)

⟦x := *y ⟧(σ)

⟦x := y ⟧(σ)

⟦*x := y ⟧(σ)

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Points-To Analysis

- **Lattice L_{pt} =**
- **Galois connection**

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Abstract Semantics for PWhile

- For every atomic statement S

⟦S ⟧ #: P(Var*× Var*)→ P(Var*× Var*)

⟦x := &y ⟧#

⟦x := *y ⟧#

⟦x := y ⟧#

⟦*x := y ⟧#

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Example

t := &a;
y := &b;

z := &c;

if x> 0;
then p:= &y;
else p:= &z;

*p := t;

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Example

/* ∅ */ t := &a; /* {(t, a)}*/
/* {(t, a)}*/ y := &b; /* {(t, a), (y, b) }*/

/* {(t, a), (y, b)}*/ z := &c; /* {(t, a), (y, b), (z, c) }*/

if x> 0;
then p:= &y; /* {(t, a), (y, b), (z, c), (p, y)}*/

else p:= &z; /* {(t, a), (y, b), (z, c), (p, z)}*/
/* {(t, a), (y, b), (z, c), (p, y), (p, z)}*/

*p := t;

/* {(t, a), (y, b), (y, c), (p, y), (p, z), (y, a), (z, a)}*/

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Flow-insensitive points-to-analysis Steengard '96

- **Ignore control flow**
- **One set of points-to per program**
- **Can be represented as a directed graph**
- **Conservative approximation**
  - **Accumulate pointers**
- **Can be computed in almost linear time**

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Example

```
t := &a;
y := &b;
z := &c;
if x> 0;
      then p:= &y;
      else p:= &z;
*p := t;
```

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Precision

- We cannot usually have
  - α(CS) = DF on all programs
- But can we say something about precision in all programs?

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The Join-Over-All-Paths (JOP)

- Let paths(v) denote the potentially infinite set paths from start to v (written as sequences of labels)
- For a sequence of edges $[e_1, e_2, \ldots, e_n]$ define $f[e_1, e_2, \ldots, e_n]: L \rightarrow L$ by composing the effects of basic blocks
  $f[e_1, e_2, \ldots, e_n](l) = f(e_n)(\ldots(f(e_2)(f(e_1)(l)))\ldots)$
- $JOP[v] = \sqcup\{f[e_1, e_2, \ldots, e_n](\iota)$
  $[e_1, e_2, \ldots, e_n] \in paths(v)\}$

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JOP vs. Least Solution

- The DF solution obtained by Chaotic iteration satisfies for every *l*:
  - $JOP[v] \sqsubseteq DF_{entry}(v)$
- A function f is additive (distributive) if
  - $f(\sqcup\{x \mid x \in X\}) = \sqcup\{f(x) \mid \in X\}$
- If every f_l is additive (distributive) for all the nodes v
  - $JOP[v] = DF_{entry}(v)$

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Conclusions

- Chaotic iterations is a powerful technique
- Easy to implement
- Rather precise
- But expensive
  - More efficient methods exist for structured programs
- Abstract interpretation relates runtime semantics and static information
- The concrete semantics serves as a tool in designing abstractions
  - More intuition will be given in the sequel

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