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A relational approach to the compilation of sparse matrix programs

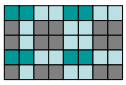
The Problem

- Matrix Algorithms tend to be relatively simple
  - Matrix Vector multiplication
  - Matrix-Matrix multiplication
- Coding them for sparse matrices is hard
  - sparse matrix formats can be hard to work with
    - Involve a lot of indirection
    - hence iteration space is very complex
  - there are lots of matrix formats
    - Makes library-based approaches infeasible

Example: Matrix Vector Multiply

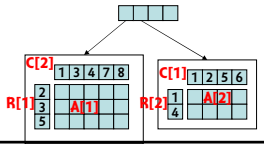
- In Dense Form

```
Do i = 1, N
  Do j = 1, N
        Y[i] = Y[i] + A[i j] * X[j];
```



- In Sparse form

```
Do t = 1, K
Do i = 1, M[t]
  Do j = 1, N[t]
        Y[R[t,][i]] = Y[R[t,][i] ]
        + A[t,][ i, j] * X[C[t][y] ];
```

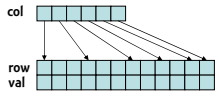


The Solution

- Describe the sparse matrix format
  - opt for a declarative specification
    - we want to avoid pointer analysis
- Define the algorithm on a dense matrix
  - imperative specification (like in SKETCH)
  - but on a restricted language
    - we want to avoid dependence analysis
  - Convert it to a relational query
- Generate the sparse implementation
  - must iterate over the matrices in an efficient way
  - relational calculus helps us with this

An Example

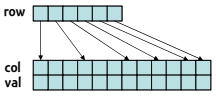
- CRS Format
  - Compressed Row Storage



- What we know
  - we can do random access of rows
  - within a row we can iterate over the columns

Another Example

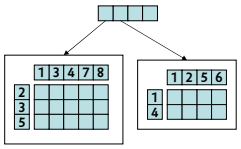
- CCS Format
  - Compressed Column Storage



- What we know
  - we can do random access of columns
  - within a column we can iterate over the rows

Another Example

- INODE Format



- What we know
  - we can do random access of inodes
  - within an inode I can iterate over row and column id
- What should we convey to the compiler?

Description of Sparse Format

- What the compiler needs to know
  - what is the hierarchy of indices
  - what can I do at each level of the hierarchy
    - can I do random access?
    - do I have to search sequentially?
  - how do I access the next level of the hierarchy

Specifying the hierarchy of indices

T:= V | F | F>$_{op}$T | F x F x F x … > T | (F, F, F, …) > T | T ∪ T

- The > operator indicates the nesting of fields
  - Example: in CCS, we have i>j because we have to get the column before we can access an element in the row

Specifying the hierarchy of indices

T:= V | F | F>$_{op}$T | F x F x F x … > T | (F, F, F, …) > T | T ∪ T

- The (F_1 x … x F_n) operator indicates that the indices corresponding to the F_i can be enumerated independently
  - Example: Dense storage

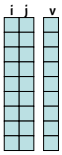
(I x J) > V

Specifying the hierarchy of indices

T:= V | F | F>$_{op}$T | F x F x F x … > T | (F, F, F, …) > T | T ∪ T

- The (F_1 , … , F_n) operator indicates that the indices corresponding to the F_i are stored together as a list
  - Example: Coordinate representation

(I , J) > V

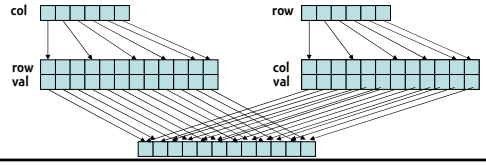


Specifying the hierarchy of indices

T:= V | F | F>$_{op}$T | F x F x F x … > T | (F, F, F, …) > T | T ∪ T

- The ∪ operator indicates alternative index hierarchies
  - Example: A combined CSS/CRS format

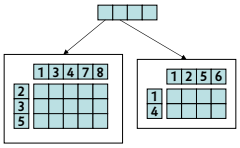
((i>j) ∪ (j>i))> V



Hidden fields

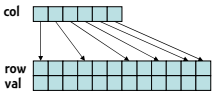
- Sometimes hidden fields are needed to specify additional structure
  - Example INODE:

    INODE > (i x j) > V
- Note: can't handle unbounded recursion



An Example

- CRS Format
  - Compressed Row Storage

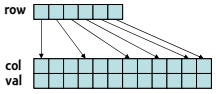


- What we know
  - we can do random access of columns
  - within a row we can iterate over the rows

j > i > v

Another Example

- CCS Format
  - Compressed Column Storage

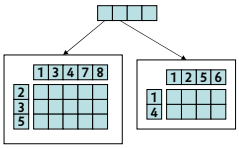


- What we know
  - we can do random access of rows
  - within a column we can iterate over the columns

i > j > v

Another Example

- INODE Format



- What we know
  - we can do random access of inodes
  - within an inode I can iterate over row and column id

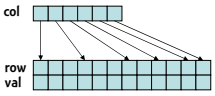
INODE > (i x j) > v

Access methods

- Must tell the compiler how to get to the actual entries in the matrix
- Three basic methods
  - Search(x_k)
    - return a pointer to entry containing x_k
  - Enum()
    - enumerate each value together with a pointer to it
  - Deref(p)
    - get the value referenced by p
- The details vary depending on the level of the hierarchy

Access Method example

- For CRS, for the column
  - Search(col) = <true, col>
    - Because col storage is dense
  - Enum() = {<1,1>, <2,2>, …, <n,n>}
  - Deref(col) = rowstart



- For the row on a given column
  - Search(row) = <b, i>
    - b says whether that row exists, and I is it's position in row
  - Enum()= the row array
  - Deref(i) = val

What about performance

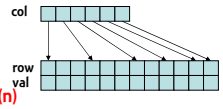
- So far we haven't said anything about performance
  - the following two formats are represented by the same index hierarchy (i x j) > V
  - one requires searching to find entries, the other one does not!

Performance information

- Each level of the hierarchy we need to provide the following info
  - Cost of searching
  - Ordering of indices
  - Range of indices
  - Type and range of handles
  - Arity of dereference
  - Kind of dereference

Access Method example

- For CRS, for the column
  - Search(col) = <true, col> O(1)
    - Because col storage is dense
  - Enum() = {<1,1>, <2,2>, …, <n,n>} O(n)
  - Deref(col) = rowstart O(1)
- For the row on a given column
  - Search(row) = <b, i> O(n)
    - b says whether that row exists, and I is it's position in row
  - Enum()= the row array O(1)
  - Deref(row) = val O(1)



The algorithm as relational query

- Matrices can be seen as relations (i, j, val)
- The traversal performed by the algorithm is just a join over different relations
  - The $\blacklozenge_P$ makes Iter.i = Y.i=A.i and Iter.j = A.j = X.j
  - Database people could solve this problem

```
Do i = 1, N
  Do j = 1, N
      Y[i] = Y[i] + A[i j] * X[j];
```

$\blacklozenge_P$ (Iter x Y(I, v) x A(i, j, v) x X(j))

- Only do-all loops are allowed in the spec
  - loop iteration order is arbitrary
  - eliminates need for dependence analysis

The compiler's view

- The algorithm as relational query
  - how do we get equijoins?
  - how do we order them?
  - how do we implement joins?
- Implemented in a two level plan
  - high-level planning decides how to order the joins
  - Low-level planning decides how the joins are to be implemented

High-level planning: Ordering of joins

- Key idea:
  - ordering of joins should respect hierarchy of indices when possible
  - equijoins preferred whenever possible

Ordering of loops, Example

```
DO i = 1 , n
   DO j = 1, n
      Y(i) = Y(i) + A(i-j, j)*X(j)
```

Relational Query

- We can define an affine equation to describe the selection

$a = H\,i$ $\qquad H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$a = \langle i, j, s, t, i_y, j_x \rangle$

- vector a corresponds to actual indices to the matrices

Ordering of loops, Example

```
DO i = 1 , n
   DO j = 1, n
      Y(i) = Y(i) + A(i-j, j)*X(j)
```

Relational Query

- We can rearrange the Matrix using Column operations

$a = H'\,i$ $\qquad H' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$

$a = \langle s, i, j, i_y, j_x, t \rangle$

- Now we can do equijoins easily

Low-Level planning

- For this, we can use standard techniques from databases
- Tradeoffs between space and time
  - For example, when you do scatter/gather
- Also guided by the complexity information

Conclusion

- Key Ideas
  - We need to provide the compiler with all the necessary information about the matrix format
  - The problem becomes tractable when we push it to a higher level
    - Relational queries in this case
    - Just like last week with garbage collection
- Why is nobody using it?