Microsoft Research

2012 Summer School on Concurrency

August 22–29, 2012 | St. Petersburg, Russia

Introduction to Parallel Programming

Section 3.

Parallel Methods for Matrix Multiplication

Victor Gergel, Professor, D.Sc.
Lobachevsky State University of Nizhni Novgorod (UNN)

Contents

Problem Statement

Matrix multiplication:

$$C = A \cdot B$$

or

$$\begin{pmatrix} c_{0,0}, & c_{0,1}, & \ldots, & c_{0,l-1} \\ & & \ldots & \\ c_{m-1,0}, & c_{m-1,1}, & \ldots, & c_{m-1,l-1} \end{pmatrix} = \begin{pmatrix} a_{0,0}, & a_{0,1}, & \ldots, & a_{0,n-1} \\ & & \ldots & \\ a_{m-1,0}, & a_{m-1,1}, & \ldots, & a_{m-1,n-1} \end{pmatrix} \begin{pmatrix} b_{0,0}, & b_{0,1}, & \ldots, & a_{0,l-1} \\ & & \ldots & \\ b_{n-1,0}, & b_{n-1,1}, & \ldots, & b_{n-1,l-1} \end{pmatrix}$$

➩ The matrix multiplication problem can be reduced to the execution of *m·l* independent operations of matrix *A* rows and matrix *B* columns inner product calculation

$$c_{ij} = \left(a_i, b_j^T\right) = \sum_{k=0}^{n-1} a_{ik} \cdot b_{kj}, \; 0 \le i < m, \; 0 \le j < l$$

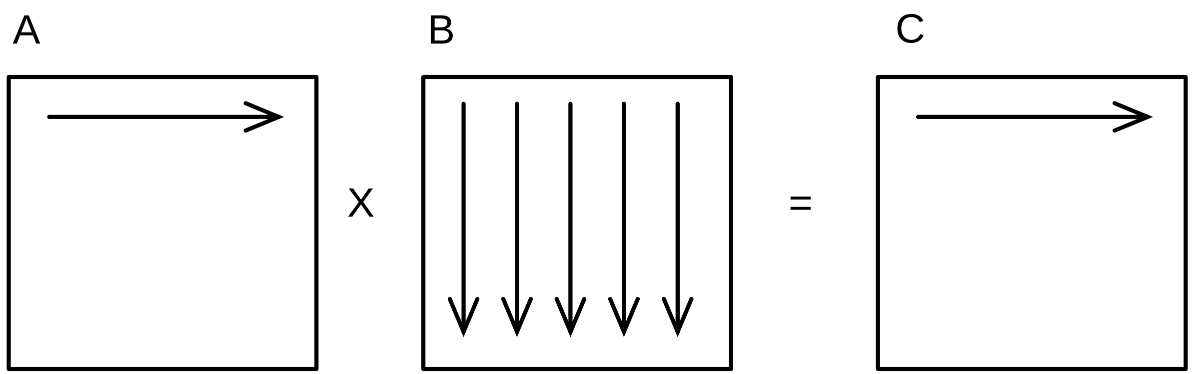
Data parallelism can be exploited to design parallel computations

Sequential Algorithm…

```
// Algorithm 8.1
// Sequential algorithm of matrix multiplication
double MatrixA[Size][Size];
double MatrixB[Size][Size];
double MatrixC[Size][Size];
int i,j,k;
...
for (i=0; i<Size; i++){
  for (j=0; j<Size; j++){
    MatrixC[i][j] = 0;
    for (k=0; k<Size; k++){
      MatrixC[i][j] = MatrixC[i][j] + MatrixA[i][k]*MatrixB[k][j];
    }
  }
}
```

Sequential Algorithm

- Algorithm performs the matrix ***C*** rows calculation sequentially
- At every iteration of the outer loop on ***i*** variable a single row of matrix ***A*** and all columns of matrix ***B*** are processed



- ***m·l*** inner products are calculated to perform the matrix multiplication
- The complexity of the matrix multiplication is *O(mnl)*.

Algorithm 1: Block-Striped Decomposition…

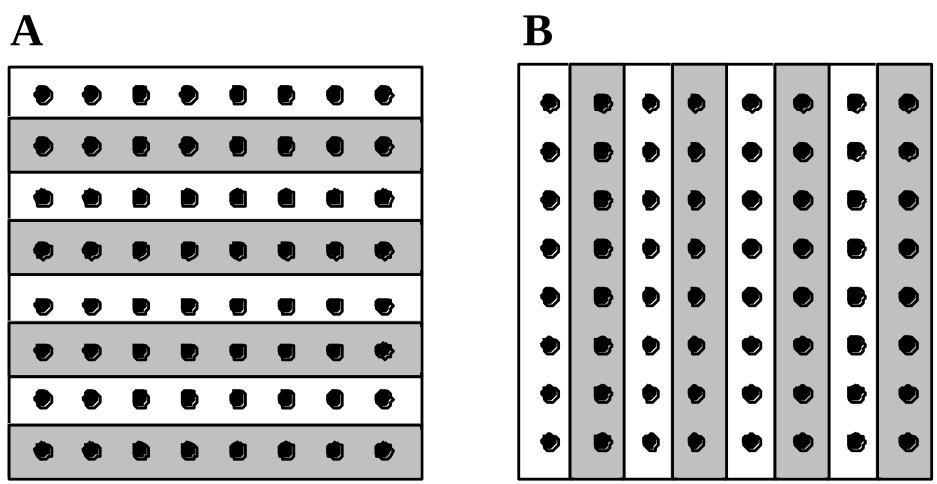
- **A fine-grained approach** – *the basic subtask* is calculation of one element of matrix ***C***

$$c_{ij} = \left(a_i, b_j^T\right),\ a_i = \left(a_{i0}, a_{i1}, \ldots, a_{in-1}\right),\ b_j^T = \left(b_{0j}, b_{1j}, \ldots, b_{n-1j}\right)^T$$

- Number of basic subtasks is equal to n^2. Achieved parallelism level is redundant!
- As a rule, the number of available processors is less then n^2 ($p<n^2$), so it will be necessary to perform the subtask scaling

Algorithm 1: Block-Striped Decomposition…

- **The aggregated subtask** – the calculation of one row of matrix ***C*** (the number of subtasks is ***n***)
- **Data distribution** – *rowwise block-striped* decomposition for matrix ***A*** and *columnwise block-striped* decomposition for matrix ***B***



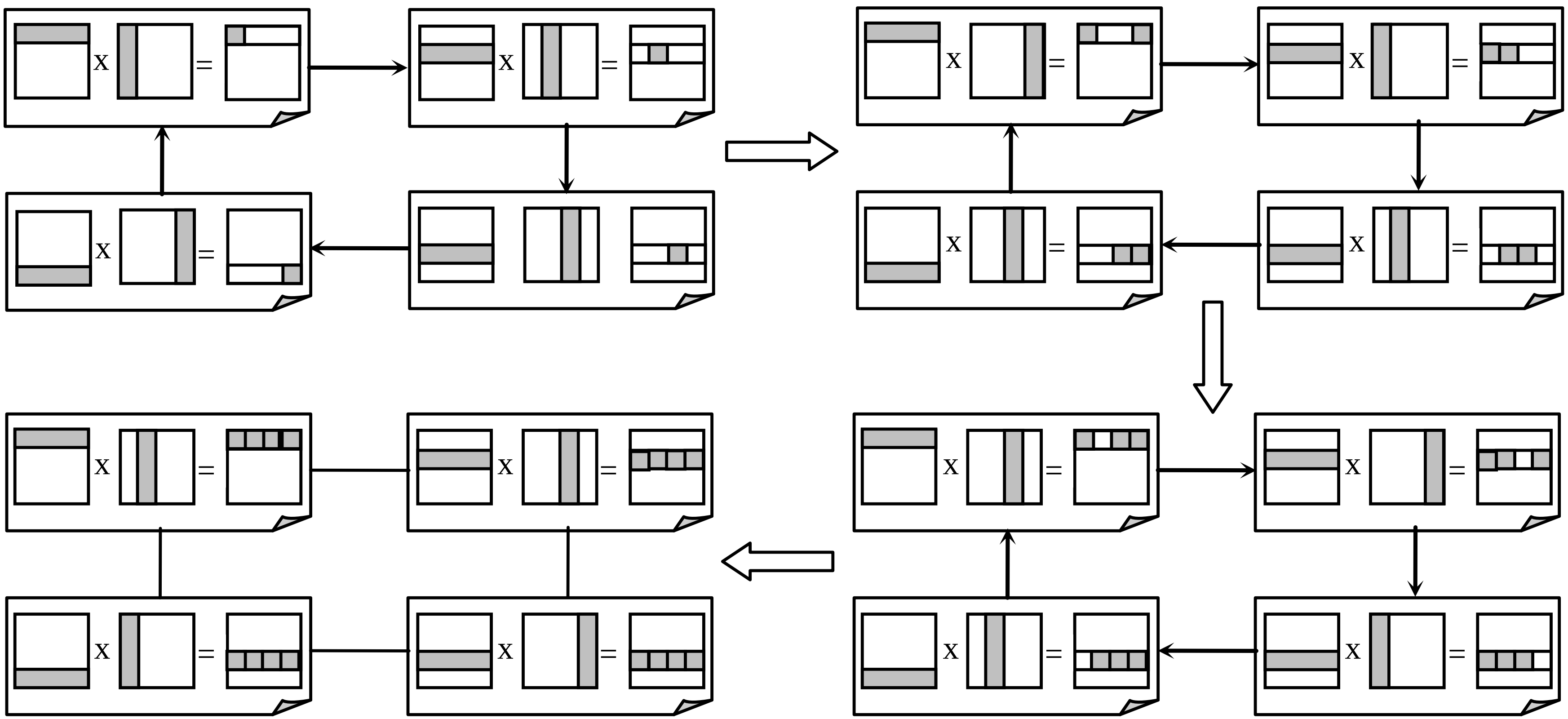
Algorithm 1: Block-Striped Decomposition…

- **Analysis of Information Dependencies…**
  - Each subtask hold one row of matrix ***A*** and one column of matrix ***B***,
  - At every iteration each subtask performs the inner product calculation of its row and column, as a result the corresponding element of matrix ***C*** is obtained
  - Then every subtask i, $0 \le i < n$, transmits its column of matrix ***B*** for the subtask with the number $(i+1)\ mod\ n$.

*After all algorithm iterations all the columns of matrix **B** were come within each subtask one after another*

<u>Algorithm 1</u>: based on block-striped decomposition

❑ Scheme of Information Dependences



Algorithm 1: Block-Striped Decomposition…

- **Aggregating and Distributing the Subtasks among the Processors:**
  - In case when the number of processors ***p*** is less than the number of basic subtasks ***n***, calculations can be aggregated in such a way that each processor would execute several inner products of matrix ***A*** rows and matrix ***B*** columns. In this case after the completion of computation, each aggregated basic subtask determines several rows of the result matrix ***C***,
  - Under such conditions the initial matrix ***A*** is decomposed into ***p*** horizontal stripes and matrix ***B*** is decomposed into ***p*** vertical stripes,
  - Subtasks distribution among the processors have to meet the requirements of effective representation of the ring structure of subtask information dependencies

Algorithm 1: Block-Striped Decomposition…

❑ Efficiency Analysis…

– Speed-up and Efficiency generalized estimates

$$S_p = \frac{n^3}{(n^3/p)} = p \qquad E_p = \frac{n^3}{p\cdot(n^3/p)} = 1$$

Developed method of parallel computations allows to achieve ideal speed-up and efficiency characteristics

Algorithm 1: Block-Striped Decomposition…

❑ Efficiency Analysis (detailed estimates):

- Time of parallel algorithm execution, that corresponds to the processor calculations:

$$T_p(calc) = (n^2 / p) \cdot (2n - 1) \cdot \tau$$

- Time of the data transmission operations can be obtained by means of the Hockney model:

$$T_p(comm) = (p - 1) \cdot (\alpha + w \cdot n \cdot (n/p)/\beta)$$

(we assume that all the data transmission operations between the processors during one iteration can be executed in parallel)

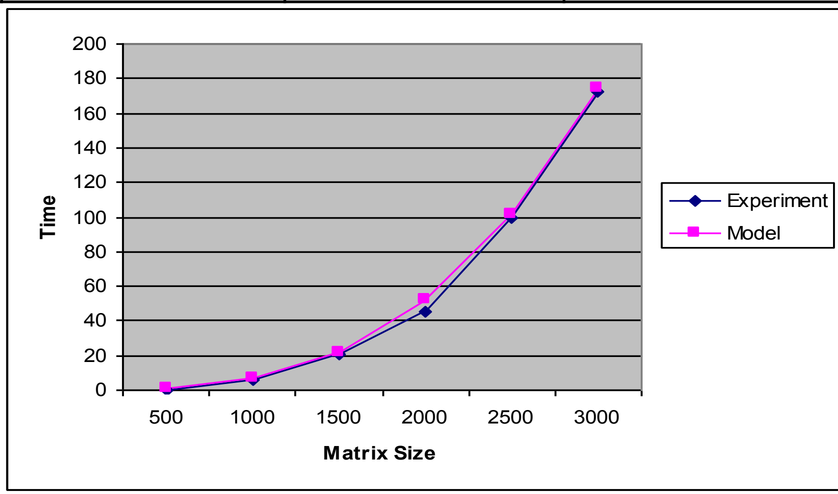
Total time of parallel algorithm execution is:

$$T_p = (n^2 / p)(2n - 1) \cdot \tau + (p - 1) \cdot (\alpha + w \cdot n \cdot (n/p)/\beta)$$

Algorithm 1: Block-Striped Decomposition…

- **Results of computational experiments…**
  - Comparison of theoretical estimations and results of computational experiments

Matrix Size	2 processors		4 processors		8 processors	
	Model	Experiment	Model	Experiment	Model	Experiment
500	0,8243	0,3758	0,4313	0,1535	0,2353	0,0968
1000	6,51822	5,4427	3,3349	2,2628	1,7436	0,6998
1500	21,9137	20,9503	11,1270	11,0804	5,7340	5,1766
2000	51,8429	45,7436	26,2236	21,6001	13,4144	9,4127
2500	101,1377	99,5097	51,0408	56,9203	25,9928	18,3303
3000	174,6301	171,9232	87,9946	111,9642	44,6772	45,5482

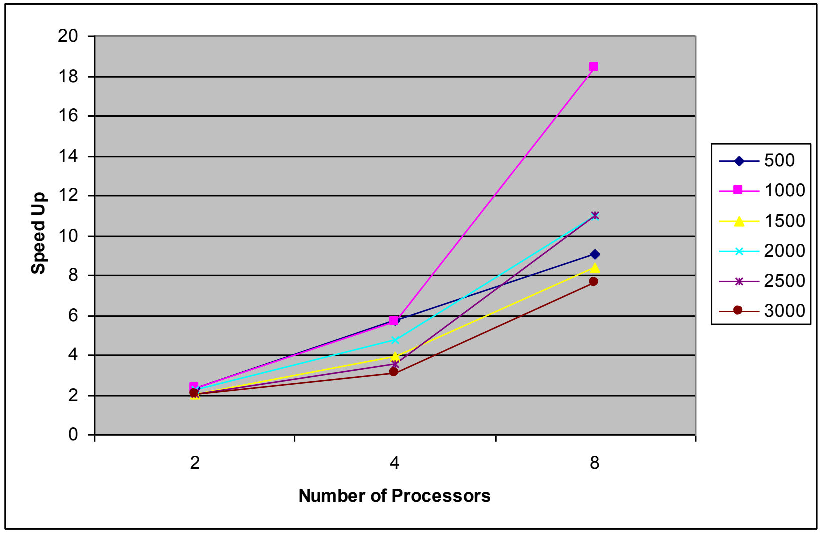


Algorithm 1: Block-Striped Decomposition…

❑ Results of computational experiments:

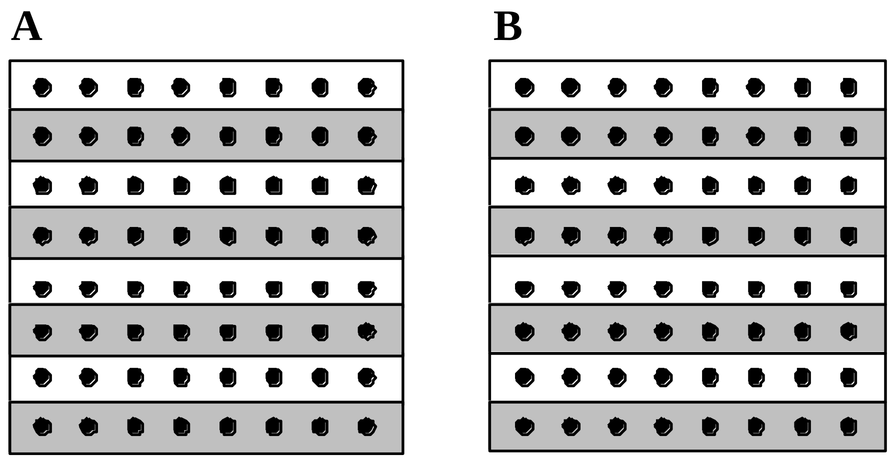
– Speedup

Matrix Size	Serial Algorithm	2 processors		4 processors		8 processors	
		Time	Speed Up	Time	Speed Up	Time	Speed Up
500	0,8752	0,3758	2,3287	0,1535	5,6982	0,0968	9,0371
1000	12,8787	5,4427	2,3662	2,2628	5,6912	0,6998	18,4014
1500	43,4731	20,9503	2,0750	11,0804	3,9234	5,1766	8,3978
2000	103,0561	45,7436	2,2529	21,6001	4,7710	9,4127	10,9485
2500	201,2915	99,5097	2,0228	56,9203	3,5363	18,3303	10,9813
3000	347,8434	171,9232	2,0232	111,9642	3,1067	45,5482	7,6368



Algorithm 1': Block-Striped Decomposition…

- Another possible approach for the data distribution is *the rowwise block-striped decomposition* for matrices ***A*** and ***B***



Algorithm 1': Block-Striped Decomposition…

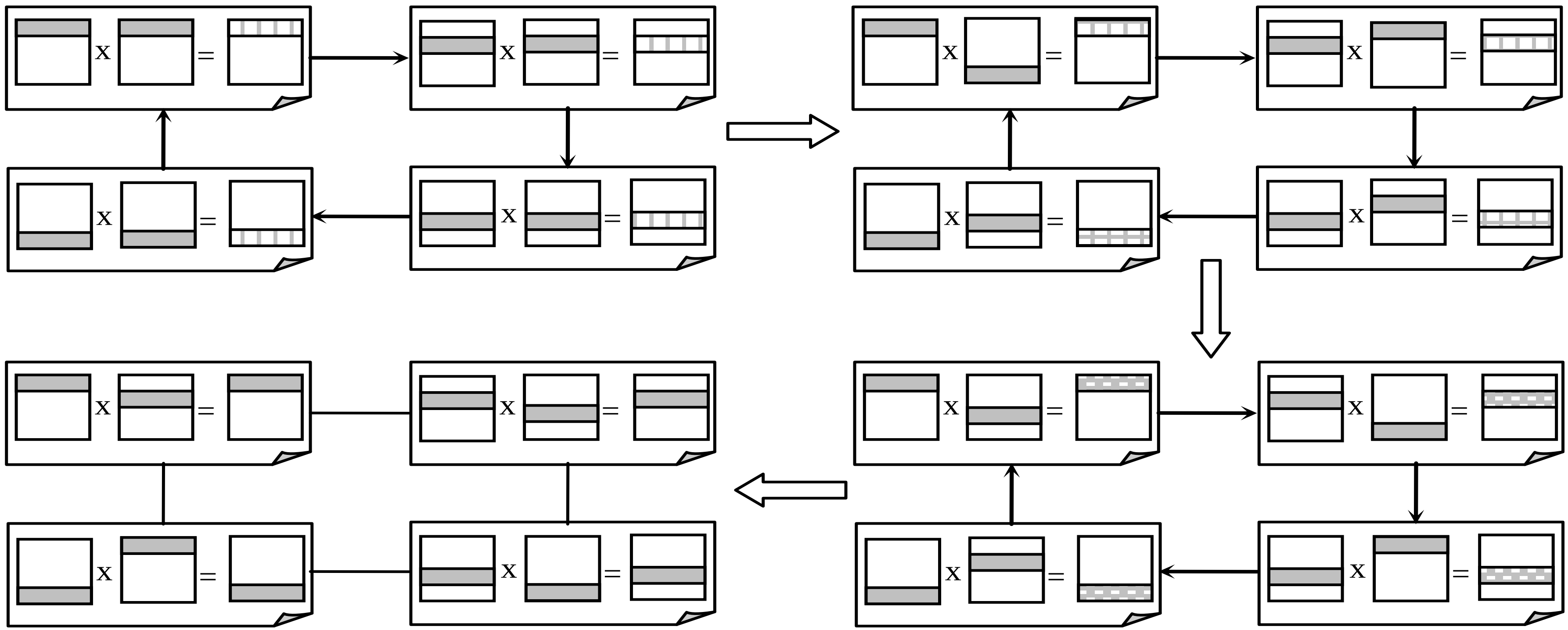
Analysis of Information Dependencies

- Each subtask hold one row of matrix ***A*** and one row of matrix ***B***,
- At every iteration the subtasks perform the element-to-element multiplications of the rows; as a result the row of partial results for matrix ***C*** is obtained,
- Then every subtask i, $0 \le i < n$, transmits its row of matrix ***B*** for the subtask with the number $(i+1) \bmod n$.

*After all algorithm iterations all rows of matrix **B** were come within every subtask one after another*

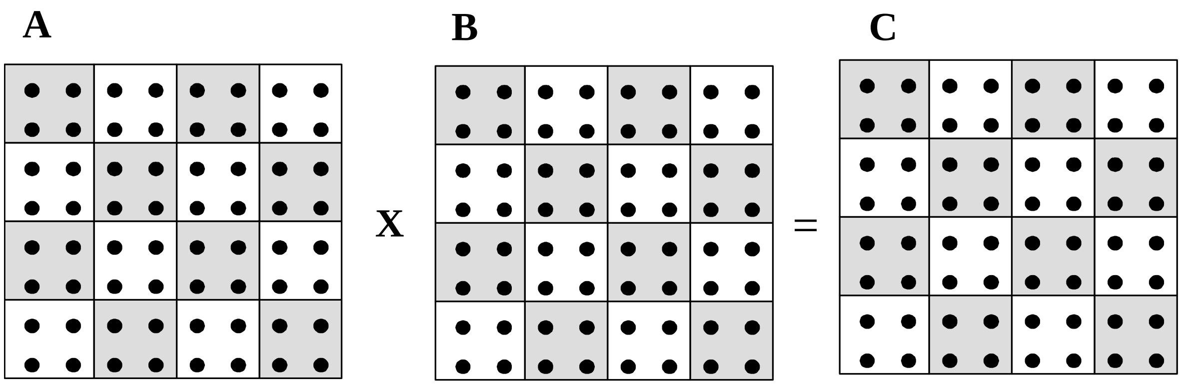
Algorithm 1': Block-Striped Decomposition

❑ Scheme of Information Dependences



Algorithm 2: Fox's method…

- **Data distribution** – *checkerboard scheme*



- **Basic subtask** is a procedure, that calculates all elements of one block of matrix ***C***

$$\begin{pmatrix} A_{00}A_{01}\ldots A_{0q-1} \\ \cdots \\ A_{q-10}A_{q-11}\ldots A_{q-1q-1} \end{pmatrix} \times \begin{pmatrix} B_{00}B_{01}\ldots B_{0q-1} \\ \cdots \\ B_{q-10}B_{q-11}\ldots B_{q-1q-1} \end{pmatrix} = \begin{pmatrix} C_{00}C_{01}\ldots C_{0q-1} \\ \cdots \\ c_{q-10}C_{q-11}\ldots C_{q-1q-1} \end{pmatrix}, \quad C_{ij} = \sum_{s=0}^{q-1} A_{is}B_{sj}$$

Algorithm 2: Fox's method…

❑ Analysis of Information Dependencies

– Subtask with *(i,j)* number calculates the block C_{ij}, of the result matrix ***C***. As a result, the subtasks form the $q \times q$ two-dimensional grid,

– Each subtask holds 4 matrix blocks:

- block C_{ij} of the result matrix ***C***, which is calculated in the subtask,
- block A_{ij} of matrix ***A***, which was placed in the subtask before the calculation starts,
- blocks A_{ij}' and B_{ij}' of matrix ***A*** and matrix ***B***, that are received by the subtask during calculations.

Algorithm 2: Fox's method…

- **Analysis of Information Dependencies** – during iteration *l*, $0 \le l < q$, algorithm performs:
  - The subtask *(i,j)* transmits its block $\boldsymbol{A_{ij}}$ of matrix ***A*** to all subtasks of the same horizontal row *i* of the grid; the *j* index, which detemines the position of the subtask in the row, can be obtained using equation:

    $$j = (\,i + l\,)\ mod\ q,$$

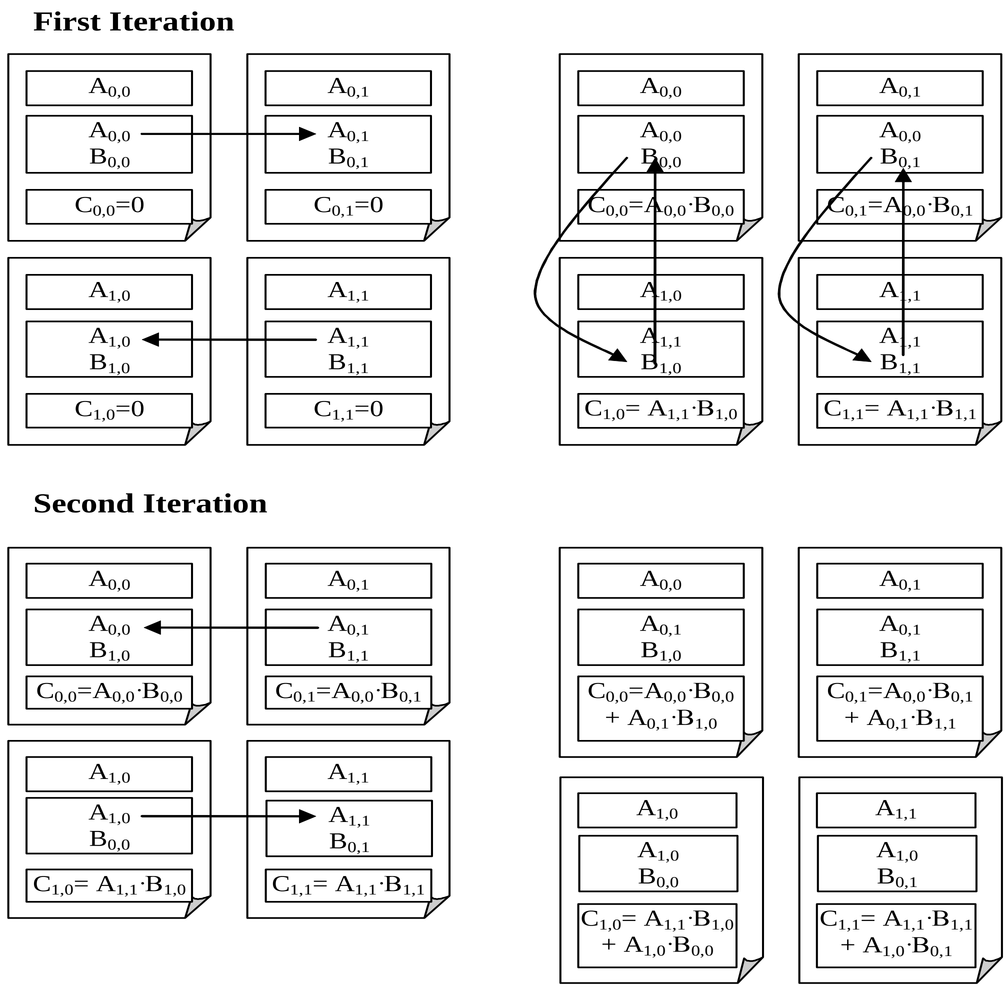
    where *mod* operation is the procedure of calculating the remainder of integer-valued division,
  - Every subtask performs the multiplication of received blocks $\boldsymbol{A_{ij}'}$ and $\boldsymbol{B_{ij}'}$ and adds the result to the block $\boldsymbol{C_{ij}}$

    $$C_{ij} = C_{ij} + A'_{ij} \times B'_{ij}$$

  - Every subtask *(i,j)* transmits its block $\boldsymbol{B_{ij}'}$ to the neighbor, which is previous in the same vertical line (the blocks of subtasks of the first row are transmitted to the subtasks of the last row of the grid).

Algorithm 2: Fox's method…

Scheme of Information Dependences



Algorithm 2: Fox's method..

- **Scaling and Distributing the Subtasks among the Processors**
  – The sizes of the matrices blocks can be selected so that the number of subtasks will coincides the number of available processors ***p***,
  – The most efficient execution of the parallel the Fox's algorithm can be provided when the communication network topology is a two-dimensional grid,
  – In this case the subtasks can be distributed among the processors in a natural way: the subtask (*i,j*) has to be placed to the $\boldsymbol{p_{i,j}}$ processor

Algorithm 2: Fox's method…

❑ Efficiency Analysis…

– Speed-up and Efficiency generalized estimates

$$S_p = \frac{n^2}{n^2 / p} = p \qquad E_p = \frac{n^2}{p \cdot (n^2 / p)} = 1$$

Developed method of parallel computations allows to achieve ideal speed-up and efficiency characteristics

Algorithm 2: Fox's method

- **Efficiency Analysis** (detailed estimates):

-Time of parallel algorithm execution, that corresponds to the processor calculations:

$$T_p\left(calc\right) = q[(n^2 / p) \cdot \left(2n / q - 1\right) + (n^2 / p)] \cdot \tau$$

- At every iteration one of the processors in each row of the processors grid transmits its block of matrix ***A*** to the rest processors in the row:

$$T_p^1\left(comm\right) = \log_2 q\,(\alpha + w(n^2 / p) / \beta)$$

- After the matrix block multiplication each processor transmits its blocks of matrix ***B*** to its neighbor in the vertical column:

$$T_p^2\left(comm\right) = \alpha + w \cdot (n^2 / p) / \beta$$

Total time of parallel algorithm execution is:

$$T_p = q[(n^2 / p) \cdot \left(2n / q - 1\right) + (n^2 / p)] \cdot \tau + (q \log_2 q + (q - 1))(\alpha + w(n^2 / p) / \beta)$$

Algorithm 2: Fox's method…

- Description of the parallel program sample…
  - The main function
    - implements the computational method scheme by sequential calling out the necessary subprograms

Code

Algorithm 2: Fox's method…

- **Description of the parallel program sample…**
  - **The function *CreateGridCommunicators*:**
    - creates a communicator as a two-dimensional square grid,
    - determines the coordinates of each process in the grid,
    - creates communicators for each row and each column separately.

Code

Algorithm 2: Fox's method…

- **Description of the parallel program sample…**
  - **The function *ProcessInitialization*:**
    - sets the matrix sizes,
    - allocates memory for storing the initial matrices and their blocks,
    - initializes all the original problem data (in order to determine the elements of the initial matrices we will use the functions *DummyDataInitialization* and *RandomDataInitialization*)

Code

Algorithm 2: Fox's method…

- **Description of the parallel program sample…**
  - **The function *ParallelResultCalculation*:**
    - executes the parallel Fox algorithm of matrix multiplication (the matrix blocks and their sizes must be given to the function as its arguments)

Code

Algorithm 2: Fox's method…

- **Description of the parallel program sample…**
  - Iteration execution: transmission of matrix ***A*** blocks to the processors in the same row of the processors grid (the function ***AblockCommunication***):
    - The number of sending processor *Pivot* is determined, then the data transmission is performed,
    - For the transmission the block *pAblock* of matrix ***A*** is used. This block was placed on the processor before the calculation started,
    - The block transmission is performed by means the function *MPI_Bcast* (the communicator *RowComm* is used).

Code

Algorithm 2: Fox's method…

- **Description of the parallel program sample…**
  - Iteration execution: Matrix block multiplication is executed (the function ***BlockMultiplication***):
    - This function multiplies the block of matrix ***A*** (*pAblock*) and the block of matrix ***B*** (*pBblock*) and adds the result to the block of matrix ***C***

Code

Algorithm 2: Fox's method…

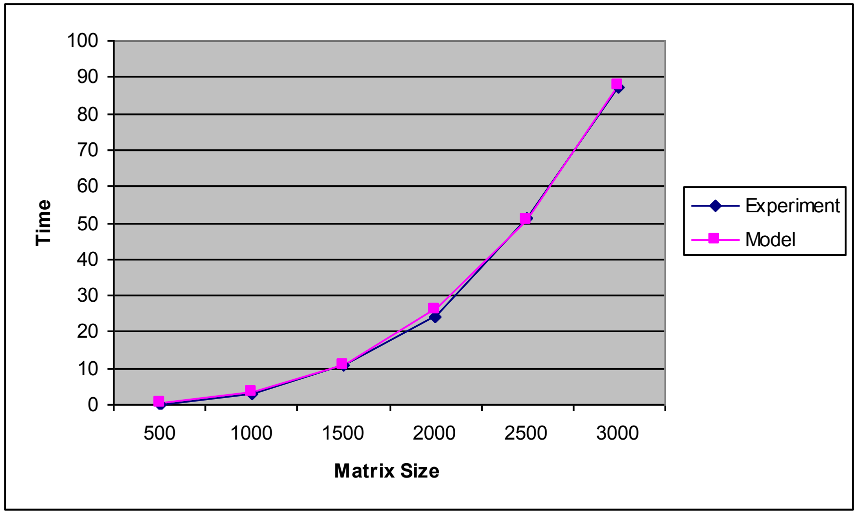
- **Description of the parallel program sample**
  - Iteration execution: Cycle shift of the matrix B blocks along the columns of the processor grid is implemented (the function ***BblockCommunication***):
    - Every processor transmits its block to the processor with the number *NextProc* in the grid column,
    - Every processor receives the block, which was sent by the processor with the number *PrevProc* from the same grid column,
    - To perform such actions the fucntion *MPI_SendRecv_replace* is used, which provides all necessary block transmissions and uses only single memory buffer *pBblock*.

Code

Algorithm 2: Fox's method…

- **Results of computational experiments…**
  - Comparison of theoretical estimations and results of computational experiments

Matrix Size	4 processors		9 processors	
	Model	Experiment	Model	Experiment
500	0,4217	0,2190	0,2200	0,1468
1000	3,2970	3,0910	1,5924	2,1565
1500	11,0419	10,8678	5,1920	7,2502
2000	26,0726	24,1421	12,0927	21,4157
2500	50,8049	51,4735	23,3682	41,2159
3000	87,6548	87,0538	40,0923	58,2022

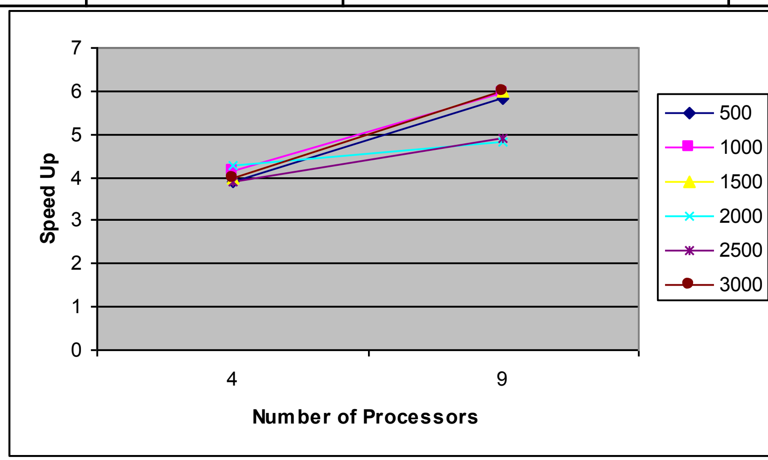


Algorithm 2: Fox's method

❑ Results of computational experiments:

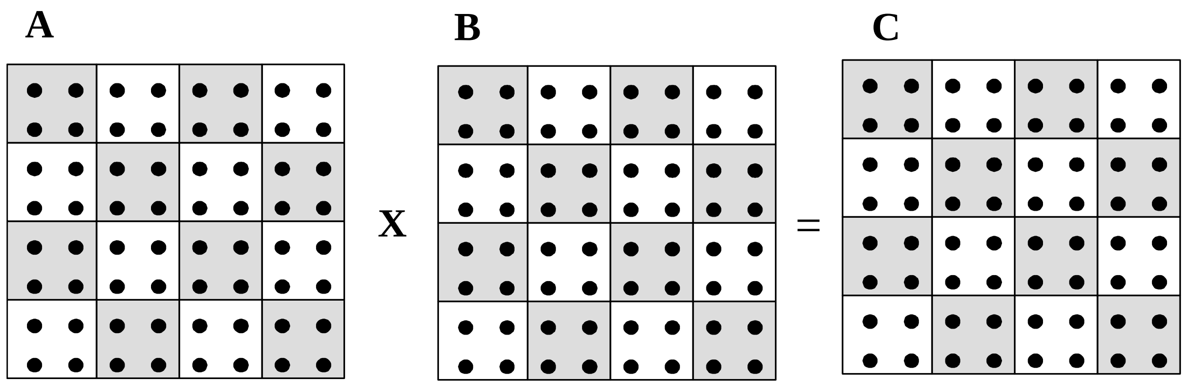
– Speedup

Matrix Size	Serial Algorithm	Parallel Algorithm			
		4 processors		9 processors	
		Time	Speed Up	Time	Speed Up
500	0,8527	0,2190	3,8925	0,1468	5,8079
1000	12,8787	3,0910	4,1664	2,1565	5,9719
1500	43,4731	10,8678	4,0001	7,2502	5,9960
2000	103,0561	24,1421	4,2687	21,4157	4,8121
2500	201,2915	51,4735	3,9105	41,2159	4,8838
3000	347,8434	87,0538	3,9957	58,2022	5,9764



Algorithm 3: Cannon's Method…

- **Data distribution** – *Checkerboard scheme*



- **Basic subtask** is a procedure, that calculates all elements of one block of matrix ***C***

$$\begin{pmatrix} A_{00}A_{01}\ldots A_{0q-1} \\ \cdots \\ A_{q-10}A_{q-11}\ldots A_{q-1q-1} \end{pmatrix} \times \begin{pmatrix} B_{00}B_{01}\ldots B_{0q-1} \\ \cdots \\ B_{q-10}B_{q-11}\ldots B_{q-1q-1} \end{pmatrix} = \begin{pmatrix} C_{00}C_{01}\ldots C_{0q-1} \\ \cdots \\ c_{q-10}C_{q-11}\ldots C_{q-1q-1} \end{pmatrix}, \quad C_{ij} = \sum_{s=0}^{q-1} A_{is}B_{sj}$$

Algorithm 3: Cannon's Method…

❑ Analysis of Information Dependencies:

– The subtask with the number *(i,j)* calculates the block C_{ij}, of the result matrix ***C***. As a result, the subtasks form the $q \times q$ two-dimensional grid,

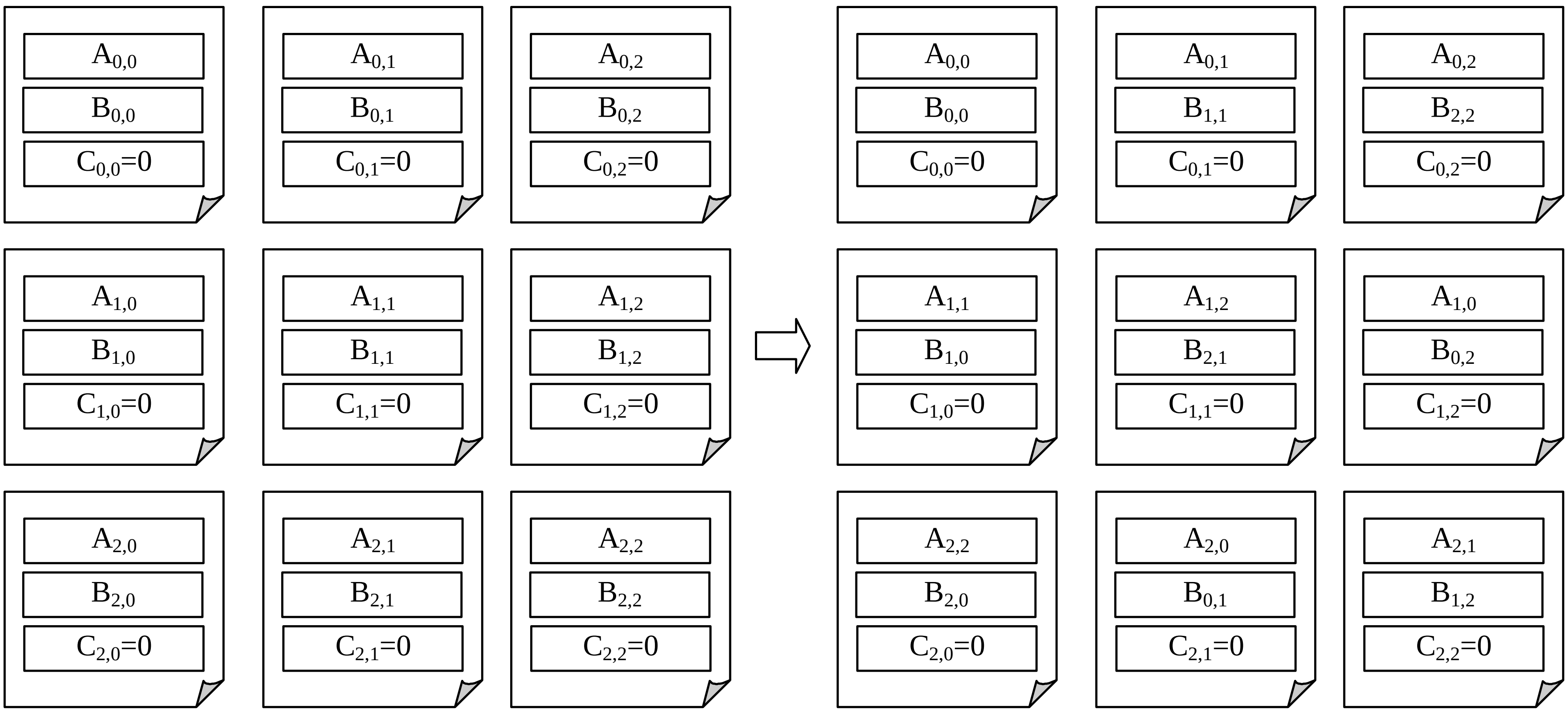
– The initial distribution of matrix blocks in Cannon's algorithm is selected in such a way that the first block multiplication can be performed without additional data transmission:

- At the beginning each subtask (*i,j*) holds the blocks A_{ij} and B_{ij},
- For the *i*-th row of the subtasks grid the matrix ***A*** blocks are shifted for *(i-1)* positions to the left,
- For the *j* –th column of the subtasks grid the matrix ***B*** blocks are shifted for *(j-1)* positions upward,

– Data transmission operations are the example of the *circular shift* communication

Algorithm 3: Cannon's Method…

- Redistribution of matrix blocks on the first stage of the algorithm



Algorithm 3: Cannon's Method…

❑ Analysis of Information Dependencies:

- After the redistribution, which was performed at the first stage, the matrix blocks can be multiplied without additional data transmission operations,
- To obtain all of the rest blocks after the operation of blocks multiplication:
  - Matrix ***A*** blocks are shifted for one position left along the grid row,
  - Matrix ***B*** blocks are shifted for one position upward along the grid column.

Algorithm 3: Cannon's Method…

- **Scaling and Distributing the Subtasks among the Processors:**
  - The sizes of the matrices blocks can be selected so that the number of subtasks will coincides the number of available processors ***p***,
  - The most efficient execution of the parallel Canon's algorithm can be provided when the communication network topology is a two-dimensional grid,
  - In this case the subtasks can be distributed among the processors in a natural way: the subtask (*i,j*) has to be placed to the $p_{i,j}$ processor

Algorithm 3: Cannon's Method…

- **Efficiency Analysis…**
  - Speed-up and Efficiency generalized estimates

$$S_p = \frac{n^2}{n^2/p} = p \qquad E_p = \frac{n^2}{p \cdot (n^2/p)} = 1$$

Developed method of parallel computations allows to achieve ideal speed-up and efficiency characteristics

Algorithm 3: Cannon's Method…

❑ **Efficiency Analysis** (detailed estimates):

- The Cannon's algorithm differs from the Fox's algorithm only in the types of communication operations, that is why:

$$T_p(calc) = (n^2/p)\cdot(2n-1)\cdot\tau$$

- Time of the initial redistribution of matrices blocks:

$$T_p^1(comm) = 2\cdot\left(\alpha + w\cdot(n^2/p)/\beta\right)$$

- After every multiplication operation matrix blocks are shifted:

$$T_p^2(comm) = 2\cdot\left(\alpha + w\cdot(n^2/p)/\beta\right)$$

Total time of parallel algorithm execution is:

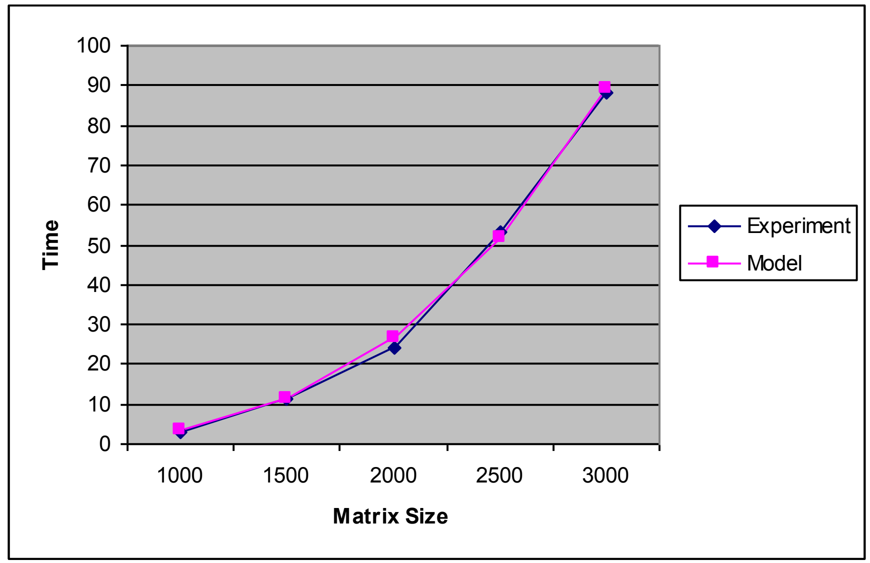
$$T_p = q[(n^2/p)\cdot(2n/q-1)+(n^2/p)]\cdot\tau + (2q+2)(\alpha + w(n^2/p)/\beta)$$

Algorithm 3: Cannon's Method…

❑ Results of computational experiments…

– Comparison of theoretical estimations and results of computational experiments

Matrix Size	4 processors		9 processors	
	Model	Experiment	Model	Experiment
1000	3,4485	3,0806	1,5669	1,1889
1500	11,3821	11,1716	5,1348	4,6310
2000	26,6769	24,0502	11,9912	14,4759
2500	51,7488	53,1444	23,2098	23,5398
3000	89,0138	88,2979	39,8643	36,3688

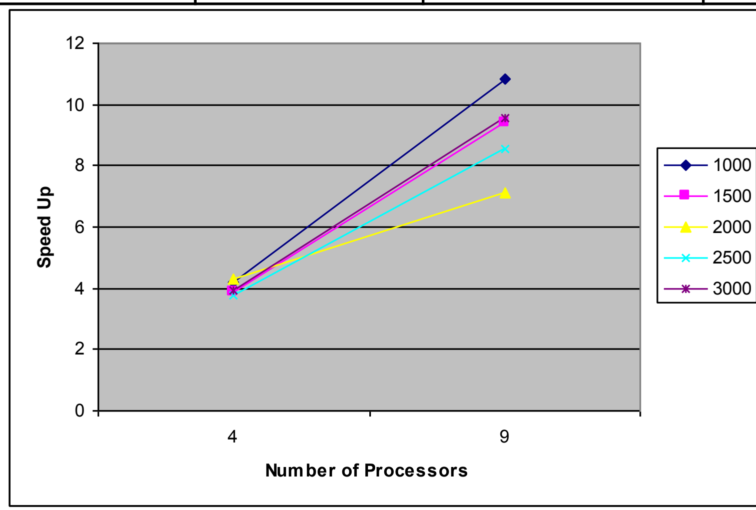


Algorithm 3: Cannon's Method

❑ Results of computational experiments:

– Speedup

Matrix Size	Serial Algorithm	Parallel Algorithm			
		4 processors		9 processors	
		Time	Speed Up	Time	Speed Up
1000	12,8787	3,0806	4,1805	1,1889	10,8324
1500	43,4731	11,1716	3,8913	4,6310	9,3872
2000	103,0561	24,0502	4,2850	14,4759	7,1191
2500	201,2915	53,1444	3,7876	23,5398	8,5511
3000	347,8434	88,2979	3,9394	36,3688	9,5643

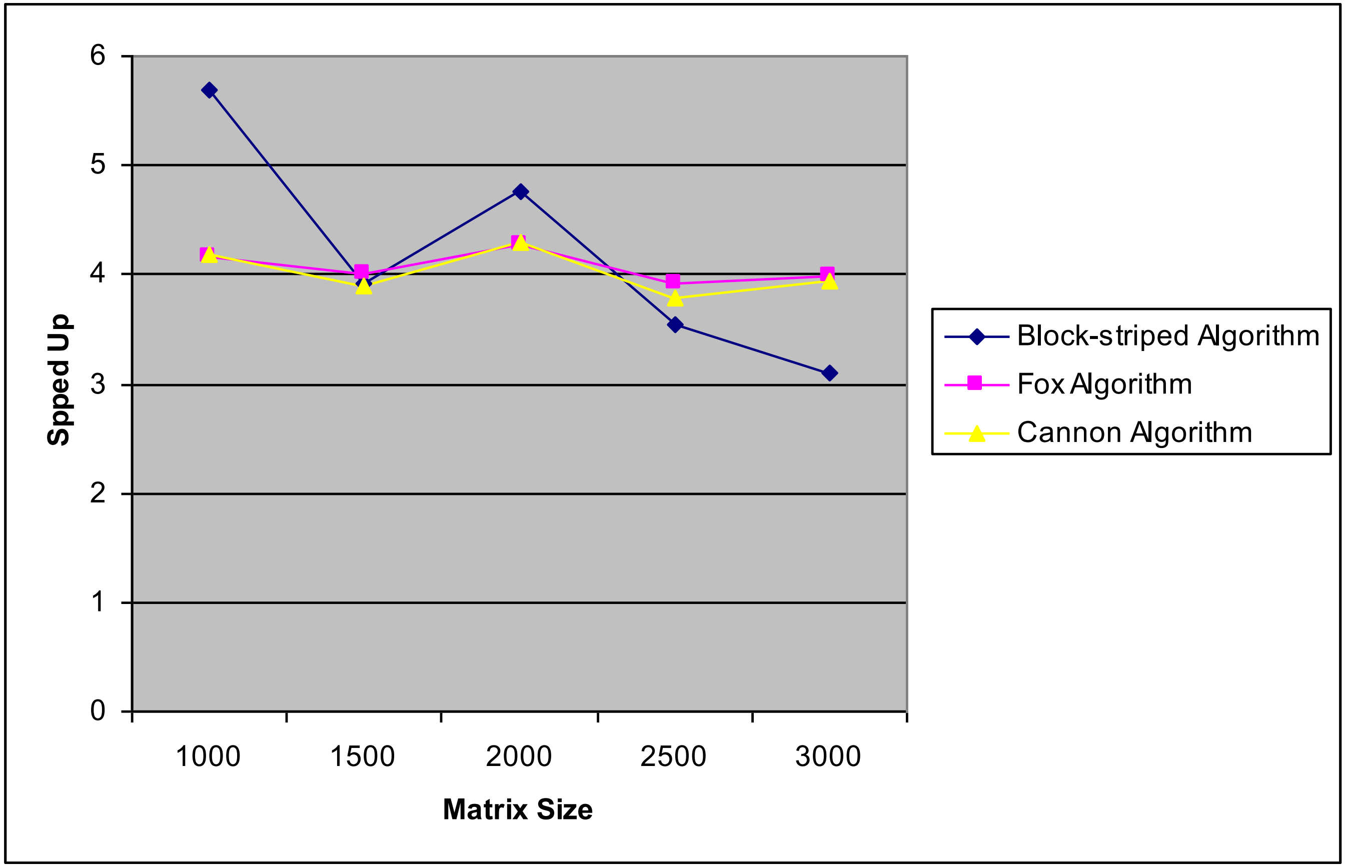


Summary…

- Three parallel algorithms for matrix multiplication are discussed:
  – Algorithm 1 - Block-Striped Decomposition
  – Algorithm 2 – Fox's method (checkerboard decomposition),
  – Algorithm 3 – Cannon's method (checkerboard decomposition),
- The parallel program sample for the Fox's algorithm is described
- Theoretical analysis allows to predict the speed-up and efficiency characteristics of parallel computations with sufficiently high accuracy

Summary

- All presented algorithms have nearly the same theoretical estimations for speed-up and efficiency characteristics



Discussions

- What sequential algorithms for matrix multiplication are known? What is the complexity of these algorithms?
- What basic approaches can be used for developing the parallel algorithms for matrix multiplication?
- What algorithm possesses the best speedup and efficiency characteristics?
- Which one of the described algorithms requires the maximal and minimal size of memory?
- What data communication operations are necessary for the parallel algorithms of matrix multiplication?

Exercises

- Develop the parallel programs for two algorithms of matrix multiplication based on block-striped decomposition. Compare the time of their execution.
- Develop the parallel program for the Cannon's algorithm.
- Formulate the theoretical estimations for the execution time of these algorithms
- Execute programs. Compare the time of computational experiments and the theoretical estimations being obtained

References

- **Gergel, V.P.** Theory and Practice of Parallel Computations. - Moscow: INTUIT.RU, 2007. - 424 p. (In Russian)
- **Gergel, V.P.** High Performance Computations for Multiprocessor Multicore Systems. - Moscow: MSU, 2010. – 544 c. (In Russian)
- **Kumar V., Grama, A., Gupta, A., Karypis, G.** (1994). Introduction to Parallel Computing. - The Benjamin/Cummings Publishing Company, Inc. (2nd edn., 2003)
- **Quinn, M. J.** (2004). Parallel Programming in C with MPI and OpenMP. – New York, NY: McGraw-Hill.
- **Fox, G.C., Otto, S.W. and Hey, A.J.G.** (1987) Matrix Algorithms on a Hypercube I: Matrix Multiplication. Parallel Computing. 4 H. 17-31.

Next Section

- Parallel Methods for Solving Linear Systems

Author’s Team

Gergel V.P., Professor, Doctor of Science in Engineering, Course Author

Grishagin V.A., Associate Professor, Candidate of Science in Mathematics

Abrosimova O.N., Assistant Professor (chapter 10)

Kurylev A.L., Assistant Professor (learning labs 4,5)

Labutin D.Y., Assistant Professor (ParaLab system)

Sysoev A.V., Assistant Professor (chapter 1)

Gergel A.V., Post-Graduate Student (chapter 12, learning lab 6)

Labutina A.A., Post-Graduate Student (chapters 7,8,9, learning labs 1,2,3, ParaLab system)

Senin A.V., Post-Graduate Student (chapter 11, learning labs on Microsoft Compute Cluster)

Liverko S.V., Student (ParaLab system)

About the project

The purpose of the project is to develop the set of educational materials for the teaching course “Multiprocessor computational systems and parallel programming”. This course is designed for the consideration of the parallel computation problems, which are stipulated in the recommendations of IEEE-CS and ACM Computing Curricula 2001. The educational materials can be used for teaching/training specialists in the fields of informatics, computer engineering and information technologies. The curriculum consists of **the training course “Introduction in the methods of parallel programming”** and **the computer laboratory training “The methods and technologies of parallel program development”**. Such educational materials makes possible to seamlessly combine both the fundamental education in computer science and the practical training in the methods of developing the software for solving complicated time-consuming computational problems using the high performance computational systems.

The project was carried out in Nizhny Novgorod State University, the Software Department of the Computing Mathematics and Cybernetics Faculty (http://www.software.unn.ac.ru). The project was implemented with the support of Microsoft.