

CS-XXX: Graduate Programming Languages

Lecture 7 — Lambda Calculus

Dan Grossman
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Where we are

- ▶ Done: Syntax, semantics, and equivalence
 - ▶ For a language with little more than loops and global variables
- ▶ Now: Didn't IMP leave some things out?
 - ▶ In particular: scope, functions, and data structures
 - ▶ (Not to mention threads, I/O, exceptions, strings, ...)

Time for a new model...

Data + Code

Higher-order functions work well for scope *and* data structures

- ▶ Scope: not all memory available to all code

```
let x = 1
let add3 y =
  let z = 2 in
  x + y + z
let seven = add3 4
```

- ▶ Data: Function closures store data. Example: Association “list”

```
let empty = (fun k -> raise Empty)
let cons k v lst = (fun k' -> if k'=k then v else lst k')
let lookup k lst = lst k
```

(Later: Objects do both too)

Adding data structures

Extending IMP with data structures is not too hard:

$$\begin{aligned}e & ::= c \mid x \mid e + e \mid e * e \mid (e, e) \mid e.1 \mid e.2 \\v & ::= c \mid (v, v) \\H & ::= \cdot \mid H, x \mapsto v\end{aligned}$$

$H ; e \Downarrow v$ all old rules plus:

$$\frac{H ; e_1 \Downarrow v_1 \quad H ; e_2 \Downarrow v_2}{H ; (e_1, e_2) \Downarrow (v_1, v_2)} \quad \frac{H ; e \Downarrow (v_1, v_2)}{H ; e.1 \Downarrow v_1} \quad \frac{H ; e \Downarrow (v_1, v_2)}{H ; e.2 \Downarrow v_2}$$

Notice:

- ▶ We allow pairs of values, not just pairs of integers
- ▶ We now have *stuck* programs (e.g., $c.1$)
 - ▶ What would C++ do? Scheme? ML? Java? Perl?
 - ▶ Division also causes stuckness

What about functions

But adding functions (or objects) does not work well:

$$\begin{array}{l} e ::= \dots \mid \text{fun } x \rightarrow s \\ v ::= \dots \mid \text{fun } x \rightarrow s \\ s ::= \dots \mid e(e) \end{array}$$

$$\boxed{H ; e \Downarrow v}$$

$$\boxed{H ; s \rightarrow H' ; s'}$$

Additions:

$$\frac{}{H ; \text{fun } x \rightarrow s \Downarrow \text{fun } x \rightarrow s} \quad \frac{H ; e_1 \Downarrow \text{fun } x \rightarrow s \quad H ; e_2 \Downarrow v}{H ; e_1(e_2) \rightarrow H ; x := v ; s}$$

Does this match “the semantics we want” for function calls?

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NO: Consider $x := 1 ; (\text{fun } x \rightarrow y := x)(2) ; \text{ans} := x$.

Scope matters; variable name does not. That is:

- ▶ Local variables should “be local”
- ▶ Choice of local-variable names should have only local ramifications

Another try

$$\frac{H ; e_1 \Downarrow \text{fun } x \rightarrow s \quad H ; e_2 \Downarrow v \quad y \text{ "fresh"}}{H ; e_1(e_2) \rightarrow H ; y := x; x := v; s; x := y}$$

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- ▶ “fresh” is not very IMP-like but okay (think malloc)
- ▶ not a good match to how functions are implemented
- ▶ yuck: the way we want to think about something as fundamental as a call?
- ▶ **NO: wrong model for most functional and OO languages**
 - ▶ (Even wrong for C if s calls another function that accesses the global variable x)

The wrong model

$$\frac{H ; e_1 \Downarrow \text{fun } x \rightarrow s \quad H ; e_2 \Downarrow v \quad y \text{ "fresh"}}{H ; e_1(e_2) \rightarrow H ; y := x; x := v; s; x := y}$$

```
f1 := (fun x -> f2 := (fun z -> ans := x + z));  
f1(2);  
x := 3;  
f2(4)
```

“Should” set ans to 6:

- ▶ f₁(2) should assign to f₂ a function that adds 2 to its argument and stores result in ans

“Actually” sets ans to 7:

- ▶ f₂(2) assigns to f₂ a function that adds *the current value of* x to its argument

Punch line

Cannot properly model local scope via a global heap of integers.

- ▶ Functions are not syntactic sugar for assignments to globals

So let's build a new model that focuses on this essential concept

- ▶ (can add back IMP features later)

Or just borrow a model from Alonzo Church

And drop mutation, conditionals, integers (!), and loops (!)

The Lambda Calculus

The Lambda Calculus:

$$\begin{aligned} e & ::= \lambda x. e \mid x \mid e e \\ v & ::= \lambda x. e \end{aligned}$$

You *apply* a function by *substituting* the argument for the *bound variable*

- ▶ (There is an equivalent *environment* definition not unlike heap-copying; see future homework)

Example Substitutions

$$\begin{aligned} e & ::= \lambda x. e \mid x \mid e e \\ v & ::= \lambda x. e \end{aligned}$$

Substitution is the key operation we were missing:

$$(\lambda x. x)(\lambda y. y) \rightarrow (\lambda y. y)$$

$$(\lambda x. \lambda y. y x)(\lambda z. z) \rightarrow (\lambda y. y \lambda z. z)$$

$$(\lambda x. x x)(\lambda x. x x) \rightarrow (\lambda x. x x)(\lambda x. x x)$$

After substitution, the bound variable is gone, so its “name” was irrelevant. (Good!)

A Programming Language

Given substitution ($e_1[e_2/x] = e_3$), we can give a semantics:

$$\boxed{e \rightarrow e'}$$
$$\frac{e[v/x] = e'}{(\lambda x. e) v \rightarrow e'} \quad \frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2} \quad \frac{e_2 \rightarrow e'_2}{v e_2 \rightarrow v e'_2}$$

A small-step, *call-by-value* (CBV), left-to-right semantics

- ▶ Terminates when the “whole program” is some $\lambda x. e$

But (also) gets stuck when there's a *free variable* “at top-level”

- ▶ Won't “cheat” like we did with $H(x)$ in IMP because scope is what we are interested in

This is the “heart” of functional languages like OCaml

- ▶ But “real” implementations do not substitute; they do something *equivalent*

Roadmap

- ▶ Motivation for a new model (done)
- ▶ CBV lambda calculus using substitution (done)
- ▶ Notes on concrete syntax
- ▶ Simple Lambda encodings (it is Turing complete!)
- ▶ Other reduction strategies
- ▶ Defining substitution

Concrete-Syntax Notes

We (and OCaml) resolve concrete-syntax ambiguities as follows:

1. $\lambda x. e_1 e_2$ is $(\lambda x. e_1 e_2)$, not $(\lambda x. e_1) e_2$
2. $e_1 e_2 e_3$ is $(e_1 e_2) e_3$, not $e_1 (e_2 e_3)$
 - ▶ Convince yourself application is not associative

More generally:

1. Function bodies extend to an unmatched right parenthesis
Example: $(\lambda x. y(\lambda z. z)w)q$
2. Application associates to the left
Example: $e_1 e_2 e_3 e_4$ is $((e_1 e_2) e_3) e_4$
 - ▶ Like in IMP, assume we really have ASTs (with non-leaves labeled λ or “application”)
 - ▶ Rules may seem strange at first, but it is the most convenient concrete syntax
 - ▶ Based on 70 years experience

Lambda Encodings

Fairly crazy: we left out constants, conditionals, primitives, and data structures

In fact, we are *Turing complete* and can *encode* whatever we need (just like assembly language can)

Motivation for encodings:

- ▶ Fun and mind-expanding
- ▶ Shows we are not oversimplifying the model (“numbers are syntactic sugar”)
- ▶ Can show languages are *too expressive* (e.g., unlimited C++ template instantiation)

Encodings are also just “(re)definition via translation”

Encoding booleans

The “Boolean ADT”

- ▶ There are two booleans and one conditional expression.
- ▶ The conditional takes 3 arguments (e.g., via currying). If the first is one boolean it evaluates to the second. If it is the other boolean it evaluates to the third.

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Here is one of an infinite number of encodings:

“true”	$\lambda x. \lambda y. x$
“false”	$\lambda x. \lambda y. y$
“if”	$\lambda b. \lambda t. \lambda f. b t f$

Example: “if” “true” $v_1 v_2 \rightarrow^* v_1$

Evaluation Order Matters

Careful: With CBV we need to “think”...

“if” “true” $(\lambda x. x) \underbrace{((\lambda x. x x)(\lambda x. x x))}_{\text{an infinite loop}}$

diverges, but

“if” “true” $(\lambda x. x) \underbrace{(\lambda z. ((\lambda x. x x)(\lambda x. x x)) z)}_{\text{a value that when called diverges}}$

does not

Encoding Pairs

The “pair ADT”:

- ▶ There is 1 constructor (taking 2 arguments) and 2 selectors
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“mkpair” $\lambda x. \lambda y. \lambda z. z x y$

“fst” $\lambda p. p(\lambda x. \lambda y. x)$

“snd” $\lambda p. p(\lambda x. \lambda y. y)$

Example:

“snd” (“fst” (“mkpair” (“mkpair” v_1 v_2) v_3)) \rightarrow^* v_2

Reusing Lambdas

Is it weird that the encodings of Booleans and pairs both used $\lambda x. \lambda y. x$ and $\lambda x. \lambda y. y$ for different purposes?

Is it weird that the same bit-pattern in binary code can represent an int, a float, an instruction, or a pointer?

Von Neumann: Bits can represent (all) code and data

Church (?): Lambdas can represent (all) code and data

Beware the “Turing tarpit”

Encoding Lists

Rather than start from scratch, notice that booleans and pairs are enough to encode lists:

- ▶ Empty list is “mkpair” “false” “false”
- ▶ Non-empty list is $\lambda h. \lambda t.$ “mkpair” “true” (“mkpair” h t)
- ▶ Is-empty is ...
- ▶ Head is ...
- ▶ Tail is ...

Note:

- ▶ Not too far from how lists are implemented
- ▶ Taking “tail” (“tail” “empty”) will produce some lambda
 - ▶ Just like, without page-protection hardware, `null->tail->tail` would produce some bit-pattern

Encoding Recursion

Some programs diverge, but can we write *useful* loops? Yes!

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- ▶ Write a function that takes an f and calls it in place of recursion
 - ▶ Example (in enriched language):

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 - ▶ The details, especially for CBV, are icky; the point is it is possible and you define “fix” only once
 - ▶ Not on exam:
“fix” $\lambda g. (\lambda x. g (\lambda y. x x y))(\lambda x. g (\lambda y. x x y))$

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How I would do it based on what we have so far:

- ▶ Lists of booleans for binary numbers
 - ▶ Zero can be the empty list
 - ▶ Use fix to implement adders, etc.
 - ▶ Like in hardware except fixed-width avoids recursion

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But instead everybody always teaches Church numerals. Why?

- ▶ Tradition? Some sense of professional obligation?
- ▶ Better reason: You do not need fix: Basic arithmetic is often encodable in languages where all programs terminate
- ▶ In any case, we will show some basics “just for fun”

Church Numerals

“0” $\lambda s. \lambda z. z$

“1” $\lambda s. \lambda z. s z$

“2” $\lambda s. \lambda z. s (s z)$

“3” $\lambda s. \lambda z. s (s (s z))$

...

- ▶ Numbers encoded with two-argument functions
- ▶ The “number i ” composes the first argument i times, starting with the second argument
 - ▶ z stands for “zero” and s for “successor” (think unary)
- ▶ The trick is implementing arithmetic by cleverly passing the right arguments for s and z

Church Numerals

"0"	$\lambda s. \lambda z. z$
"1"	$\lambda s. \lambda z. s z$
"2"	$\lambda s. \lambda z. s (s z)$
"3"	$\lambda s. \lambda z. s (s (s z))$
"successor"	$\lambda n. \lambda s. \lambda z. s (n s z)$

successor: take "a number" and return "a number" that (when called) applies s one more time

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“successor” $\lambda n. \lambda s. \lambda z. s (n s z)$
“plus” $\lambda n. \lambda m. \lambda s. \lambda z. n s (m s z)$

plus: take two “numbers” and return a “number” that uses one number as the zero argument for the other

Church Numerals

"0"	$\lambda s. \lambda z. z$
"1"	$\lambda s. \lambda z. s z$
"2"	$\lambda s. \lambda z. s (s z)$
"3"	$\lambda s. \lambda z. s (s (s z))$
"successor"	$\lambda n. \lambda s. \lambda z. s (n s z)$
"plus"	$\lambda n. \lambda m. \lambda s. \lambda z. n s (m s z)$
"times"	$\lambda n. \lambda m. m$ ("plus" n) "zero"

times: take two "numbers" m and n and pass to m a function that adds n to its argument (so this will happen m times) and "zero" (where to start the m iterations of addition)

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"successor"	$\lambda n. \lambda s. \lambda z. s (n s z)$
"plus"	$\lambda n. \lambda m. \lambda s. \lambda z. n s (m s z)$
"times"	$\lambda n. \lambda m. m$ ("plus" n) "zero"
"isZero"	$\lambda n. n$ ($\lambda x.$ "false") "true"

isZero: an easy one, see how the two arguments will lead to the correct answer

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"times"	$\lambda n. \lambda m. m$ ("plus" n) "zero"
"isZero"	$\lambda n. n$ ($\lambda x.$ "false") "true"
"predecessor"	(with 0 sticky) the hard one; see Wikipedia
"minus"	similar to times with pred instead of plus
"isEqual"	subtract and test for zero

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Then start type systems

- ▶ Later take a break from types to consider first-class continuations and related topics