A Sophomoric Introduction to Shared-Memory Parallelism and Concurrency

Lecture 3
Parallel Prefix, Pack, and Sorting

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For more information, see http://www.cs.washington.edu/homes/djg/teachingMaterials/

Outline

Done:

- Simple ways to use parallelism for counting, summing, finding
- Analysis of running time and implications of Amdahl's Law

Now: Clever ways to parallelize more than is intuitively possible

- Parallel prefix:
  - This "key trick" typically underlies surprising parallelization
  - Enables other things like packs
- Parallel sorting: quicksort (not in place) and mergesort
  - Easy to get a little parallelism
  - With cleverness can get a lot

The prefix-sum problem

Given **`int[] input`**, produce **`int[] output`** where **`output[i]`** is the sum of **`input[0]+input[1]+`**…**`+input[i]`**

Sequential can be a CS1 exam problem:

```
int[] prefix_sum(int[] input){
  int[] output = new int[input.length];
  output[0] = input[0];
  for(int i=1; i < input.length; i++)
    output[i] = output[i-1]+input[i];
  return output;
}
```

Does not seem parallelizable

- Work: $O(n)$, Span: $O(n)$
- This *algorithm* is sequential, but a *different algorithm* has Work: $O(n)$, Span: $O(\log n)$

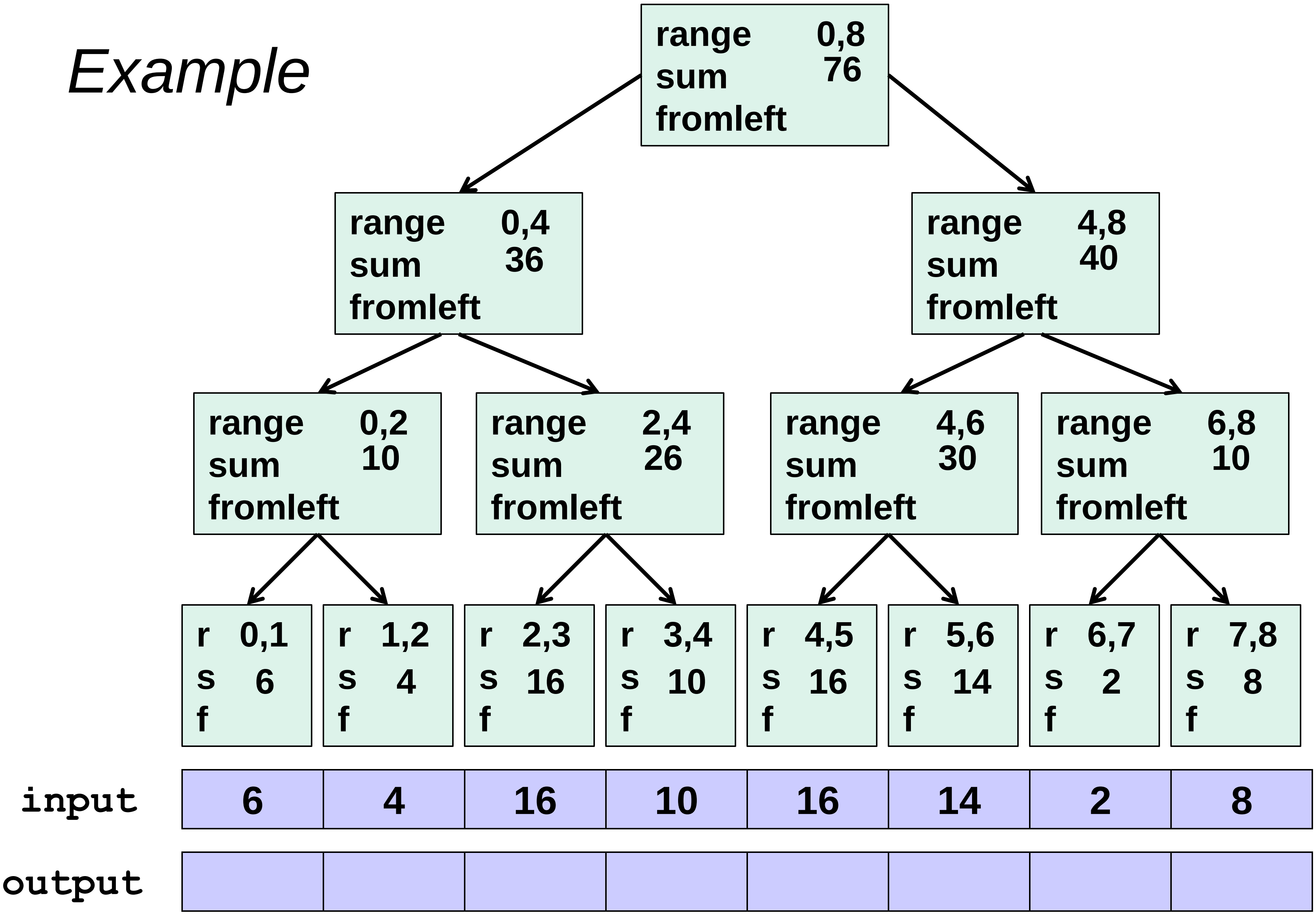
Parallel prefix-sum

- The parallel-prefix algorithm does two passes
  - Each pass has $O(n)$ work and $O(\texttt{log}\ n)$ span
  - So in total there is $O(n)$ work and $O(\texttt{log}\ n)$ span
  - So like with array summing, parallelism is $n/\texttt{log}\ n$
    - An exponential speedup
- First pass builds a tree bottom-up: the "up" pass
- Second pass traverses the tree top-down: the "down" pass

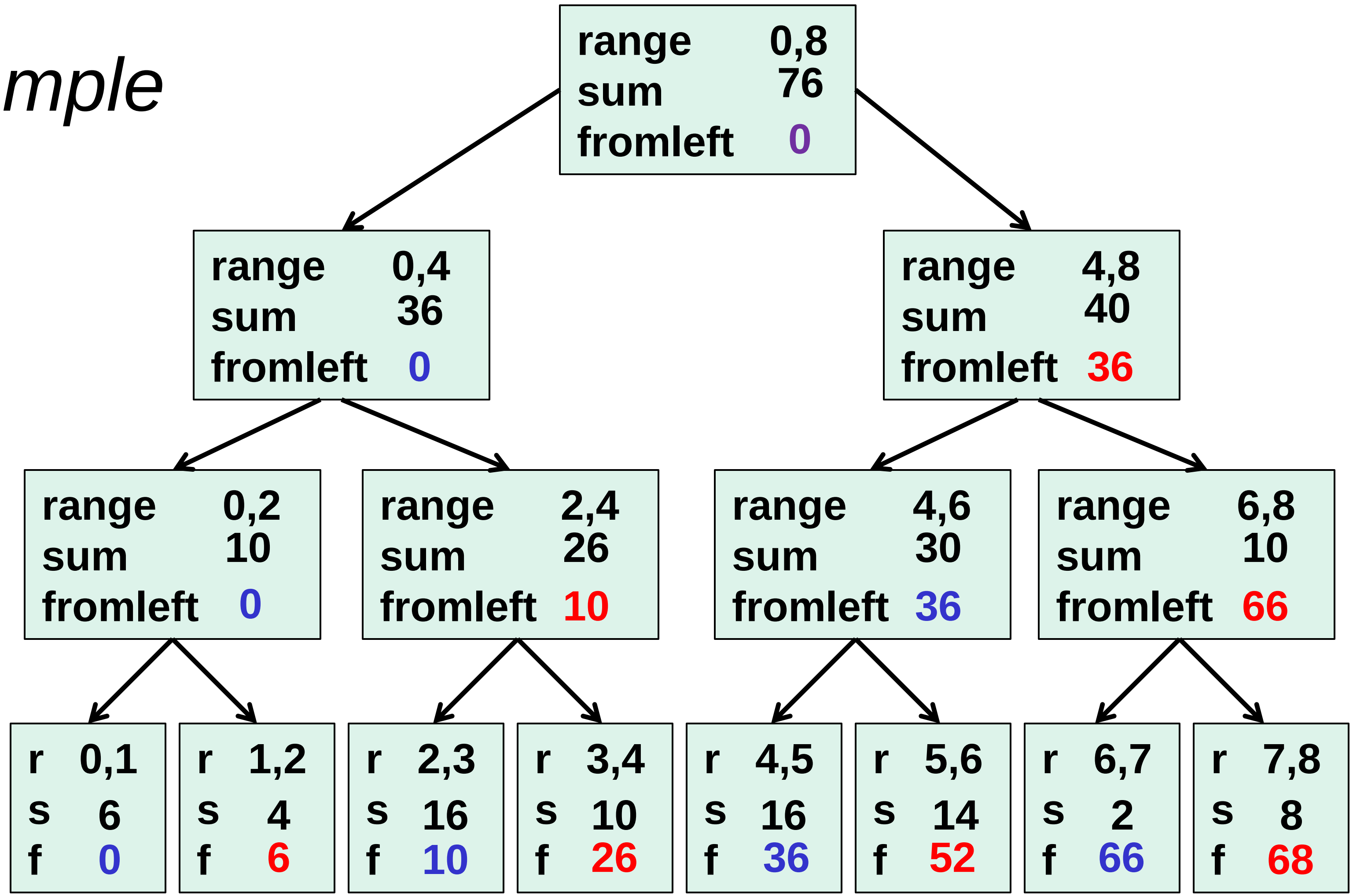
Historical note:

- Original algorithm due to R. Ladner and M. Fischer at the University of Washington in 1977

Example


range 0,8
sum 76
fromleft
range 0,4
sum 36
fromleft
range 4,8
sum 40
fromleft
range 0,2
sum 10
fromleft
range 2,4
sum 26
fromleft
range 4,6
sum 30
fromleft
range 6,8
sum 10
fromleft
r 0,1
s 6
f
r 1,2
s 4
f
r 2,3
s 16
f
r 3,4
s 10
f
r 4,5
s 16
f
r 5,6
s 14
f
r 6,7
s 2
f
r 7,8
s 8
f
input
6
4
16
10
16
14
2
8
output

Example



`input`	6	4	16	10	16	14	2	8
`output`	6	10	26	36	52	66	68	76

The algorithm, part 1

1. Up: Build a binary tree where
   - Root has sum of the range [**`x,y`**)
   - If a node has sum of [**`lo,hi`**) and **`hi>lo`**,
     - Left child has sum of [**`lo,middle`**)
     - Right child has sum of [**`middle,hi`**)
     - A leaf has sum of [**`i,i+1`**), i.e., **`input[i]`**

This is an easy fork-join computation: combine results by actually building a binary tree with all the range-sums

- Tree built bottom-up in parallel
- Could be more clever with an array like with heaps

Analysis: $O(n)$ work, $O($**`log`** $n)$ span

The algorithm, part 2

2. Down: Pass down a value **`fromLeft`**
   - Root given a **`fromLeft`** of **`0`**
   - Node takes its **`fromLeft`** value and
     - Passes its left child the same **`fromLeft`**
     - Passes its right child its **`fromLeft`** plus its left child's **`sum`** (as stored in part 1)
   - At the leaf for array position **`i`**, **`output[i]=fromLeft+input[i]`**

This is an easy fork-join computation: traverse the tree built in step 1 and produce no result

- Leaves assign to **`output`**
- Invariant: **`fromLeft`** is sum of elements left of the node's range

Analysis: $O(n)$ work, $O(\log n)$ span

Sequential cut-off

Adding a sequential cut-off is easy as always:

- Up:

  just a sum, have leaf node hold the sum of a range

- Down:

```
output[lo] = fromLeft + input[lo];
for(i=lo+1; i < hi; i++)
  output[i] = output[i-1] + input[i]
```

Parallel prefix, generalized

Just as sum-array was the simplest example of a common pattern, prefix-sum illustrates a pattern that arises in many, many problems

- Minimum, maximum of all elements to the left of **`i`**
- Is there an element to the left of **`i`** satisfying some property?
- Count of elements to the left of **`i`** satisfying some property
  - This last one is perfect for an efficient parallel pack…
  - Perfect for building on top of the "parallel prefix trick"
- We did an *inclusive* sum, but *exclusive* is just as easy

Pack

[Non-standard terminology]

Given an array **`input`**, produce an array **`output`** containing only elements such that **`f(elt)`** is **`true`**

Example:
```
input [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
f: is elt > 10
output [17, 11, 13, 19, 24]
```

Parallelizable?

- Finding elements for the output is easy
- But getting them in the right place seems hard

Parallel prefix to the rescue

1. Parallel map to compute a bit-vector for true elements

```
input  [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
bits   [1,  0, 0, 0,  1, 0,  1,  1, 0,  1]
```

2. Parallel-prefix sum on the bit-vector

```
bitsum [1,  1, 1, 1,  2, 2,  3,  4, 4,  5]
```

3. Parallel map to produce the output

```
output [17, 11, 13, 19, 24]
```

```
output = new array of size bitsum[n-1]
FORALL(i=0; i < input.length; i++){
  if(bits[i]==1)
    output[bitsum[i]-1] = input[i];
}
```

Pack comments

- First two steps can be combined into one pass
  - Just using a different base case for the prefix sum
  - No effect on asymptotic complexity

- Can also combine third step into the down pass of the prefix sum
  - Again no effect on asymptotic complexity

- Analysis: $O(n)$ work, $O(\texttt{log}\ n)$ span
  - 2 or 3 passes, but 3 is a constant

- Parallelized packs will help us parallelize quicksort…

Quicksort review

Recall quicksort was sequential, in-place, expected time $O(n \log n)$

	Best / expected case *work*
1. Pick a pivot element	**O(1)**
2. Partition all the data into:	**O(n)**
A. The elements less than the pivot	
B. The pivot	
C. The elements greater than the pivot	
3. Recursively sort A and C	**2T(n/2)**

How should we parallelize this?

Quicksort

	Best / expected case *work*
1. Pick a pivot element	**O(1)**
2. Partition all the data into:	**O(n)**
A. The elements less than the pivot	
B. The pivot	
C. The elements greater than the pivot	
3. Recursively sort A and C	**2T(n/2)**

Easy: Do the two recursive calls in parallel

- Work: unchanged of course $O(n \log n)$
- Span: now $T(n) = O(n) + 1T(n/2) = O(n)$
- So parallelism (i.e., work / span) is $O(\log n)$

Doing better

- *O*(`log` *n*) speed-up with an infinite number of processors is okay, but a bit underwhelming
  - Sort 10^9 elements 30 times faster

- Google searches strongly suggest quicksort cannot do better because the partition cannot be parallelized
  - The Internet has been known to be wrong ☺
  - But we need auxiliary storage (no longer in place)
  - In practice, constant factors may make it not worth it, but remember Amdahl's Law

- Already have everything we need to parallelize the partition…

Parallel partition (not in place)

Partition all the data into:

A. **The elements less than the pivot**
B. **The pivot**
C. **The elements greater than the pivot**

- This is just two packs!
  - We know a pack is $O(n)$ work, $O(\log n)$ span
  - Pack elements less than pivot into left side of **`aux`** array
  - Pack elements greater than pivot into right size of **`aux`** array
  - Put pivot between them and recursively sort
  - With a little more cleverness, can do both packs at once but no effect on asymptotic complexity
- With $O(\log n)$ span for partition, the total best-case and expected-case span for quicksort is

$$T(n) = O(\log n) + 1T(n/2) = O(\log^2 n)$$

Example

- Step 1: pick pivot as median of three

8	1	4	9	0	3	5	2	7	6

- Steps 2a and 2c (combinable): pack less than, then pack greater than into a second array
  - Fancy parallel prefix to pull this off not shown

1	4	0	3	5	2				

1	4	0	3	5	2	6	8	9	7

- Step 3: Two recursive sorts in parallel
  - Can sort back into original array (like in mergesort)

Now mergesort

Recall mergesort: sequential, not-in-place, worst-case $O(n \log n)$

1. **Sort left half and right half** **2T(n/2)**
2. **Merge results** **O(n)**

Just like quicksort, doing the two recursive sorts in parallel changes the recurrence for the span to $T(n) = O(n) + 1T(n/2) = O(n)$

- Again, parallelism is $O(\log n)$
- To do better, need to parallelize the merge
  - The trick won't use parallel prefix this time

Parallelizing the merge

Need to merge two *sorted* subarrays (may not have the same size)

0	1	4	8	9

2	3	5	6	7

Idea: Suppose the larger subarray has *m* elements. In parallel:

- Merge the first *m*/2 elements of the larger half with the "appropriate" elements of the smaller half
- Merge the second *m*/2 elements of the larger half with the rest of the smaller half

Parallelizing the merge

0	**4**	**6**	**8**	**9**

1	**2**	**3**	**5**	**7**

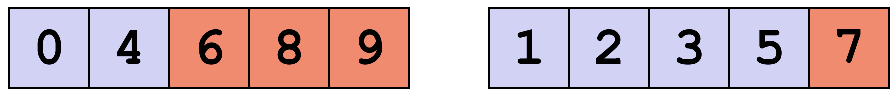
Parallelizing the merge

0	4	6	8	9

1	2	3	5	7

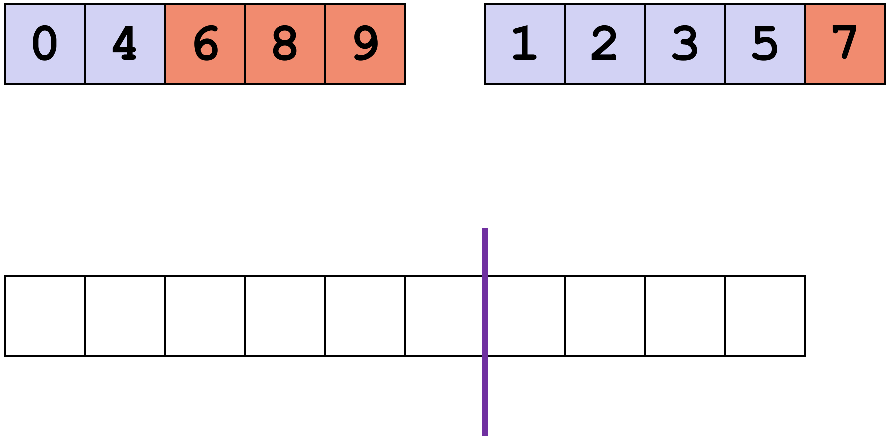
1. Get median of bigger half: *O*(1) to compute middle index

Parallelizing the merge



1. Get median of bigger half: *O*(1) to compute middle index
2. Find how to split the smaller half at the same value as the left-half split: *O*(**`log`** *n*) to do binary search on the sorted small half

Parallelizing the merge



1. Get median of bigger half: *O*(1) to compute middle index
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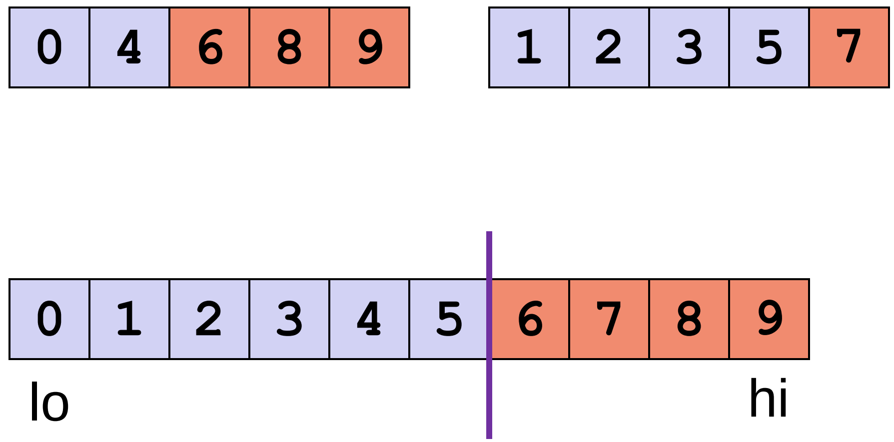
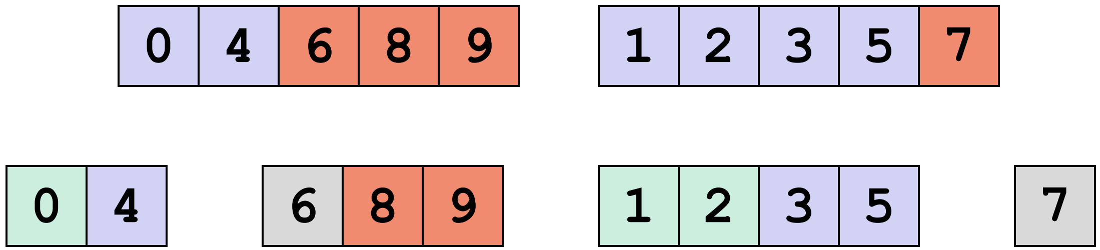
3. Size of two sub-merges conceptually splits output array: *O*(1)

Parallelizing the merge



1. Get median of bigger half: *O*(1) to compute middle index
2. Find how to split the smaller half at the same value as the left-half split: *O*(`log` *n*) to do binary search on the sorted small half
3. Size of two sub-merges conceptually splits output array: *O*(1)
4. Do two submerges in parallel

The Recursion



When we do each merge in parallel, we split the bigger one in half and use binary search to split the smaller one

Analysis

- Sequential recurrence for mergesort:

  $T(n) = 2T(n/2) + O(n)$ which is $O(n \log n)$

- Doing the two recursive calls in parallel but a sequential merge:

  Work: same as sequential   Span: $T(n)=1T(n/2)+O(n)$ which is $O(n)$

- Parallel merge makes work and span harder to compute
  - Each merge step does an extra $O(\log n)$ binary search to find how to split the smaller subarray
  - To merge *n* elements total, do two smaller merges of possibly different sizes
  - But worst-case split is (1/4)*n* and (3/4)*n*
    - When subarrays same size and “smaller” splits “all” / “none”

Analysis continued

For just a parallel merge of n elements:

- Work is $T(n) = T(3n/4) + T(n/4) + O(\mathtt{log}\ n)$ which is $O(n)$
- Span is $T(n) = T(3n/4) + O(\mathtt{log}\ n)$, which is $O(\mathtt{log}^2\ n)$
- (neither bound is immediately obvious, but "trust me")

So for mergesort with parallel merge overall:

- Work is $T(n) = 2T(n/2) + O(n)$, which is $O(n\ \mathtt{log}\ n)$
- Span is $T(n) = 1T(n/2) + O(\mathtt{log}^2\ n)$, which is $O(\mathtt{log}^3\ n)$

So parallelism (work / span) is $O(n\ /\ \mathtt{log}^2\ n)$

- Not quite as good as quicksort's $O(n\ /\ \mathtt{log}\ n)$
  - But worst-case guarantee
- And as always this is just the asymptotic result