

# Statistics on Wreath Products

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## 1 Introduction

The *colored permutation groups* are fundamental objects in much of today's mathematics. A better understanding of these groups may help to advance research in many fields. One method of studying these groups is by using numerical statistics and finding their generating functions. This method was successfully applied in the case of "one-colored" permutations groups, the *symmetric groups*. MacMahon [12] considered four different statistics for a permutation  $\pi$  in the *symmetric group*: the number of *descents* ( $des(\pi)$ ), the number of *excedance* ( $exc(\pi)$ ), the *length* statistic ( $\ell(\pi)$ ), and the *major index* ( $maj(\pi)$ ). MacMahon showed, algebraically, that *excedance* number is equidistributed with *descent* number, and that *length* is equidistributed with *major index* over the *symmetric groups*.

When we talk about permutation statistics we generally discuss about two main types of statistics: *Eulerian* statistics, which are equidistributed with the *descent* number, and *Mahonian* statistics which are equidistributed with *length*. Through the years many generalizations to MacMahon's results were found [6, 7, 8, 9] using combinations of statistics which are equidistributed with each other over the *symmetric groups*.

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Recently, Adin and Roichman [3] generalized MacMahon’s result on the *major index* to “two-colored” permutation groups, the *signed permutations groups*, by introducing a new Mahonian statistic, the *flag major index*. Bagno [4] introduced a new Mahonian statistic, *lmaj*, which is equidistributed with *length* over general *colored permutation groups*. These results open the gate for trying to generalize results that were obtained on *symmetric groups* to general *colored permutations groups*. In this paper we attempt to go in the above mentioned direction, and generalize known theorems from the *symmetric groups* to the *signed permutations groups*, and to general *colored permutation groups*.

The paper is organized as follows: we start by giving necessary background in Section 2. In Section 3 we present our main results. In Section 4 we study permutation statistics with respect to different linear orders, we prove that the *flag major index* is equidistributed with *length* over the *signed permutation groups* for every linear order, and also find a large collection of linear orders, which *sign* and *flag major index* are equidistributed with each other. In Section 5 we present a new method for calculating generating functions on the *signed permutations groups* in the *natural order*:

$$N : -n < -(n - 1) < \dots < -1 < 1 < \dots < n,$$

and also calculate generating functions on a well-known subgroup of the *signed permutation group*:  $D_n$  (to be defined below). In Section 6 we move to calculating generating functions for statistics, but this time on general *colored permutation groups* in the *friends order*:

$$F : 1^{[r-1]} < \dots < 1 < 2^{[r-1]} < \dots < 2 < \dots < n^{[r-1]} < \dots < n.$$

We present a new method of generalizing equidistributed statistics over the *symmetric groups* to general *colored permutations groups*. Using this method we find new Mahonian and Eulerian statistics, and generalize known theorems due to Foata; Zeilberger and Schützenberger. We conclude the paper in Section 7, where we introduce the *flag inversion number* and study its properties.

## 2 Background

### 2.1 Statistics on the Symmetric Group

In this subsection we present the main definitions, notation, and theorems on the *symmetric groups* (i.e., the Weyl groups of type A), denoted  $S_n$ .

**Definition 2.1** *Let be  $\mathbf{N}$  the set of all the natural numbers, a permutation of order  $n \in \mathbf{N}$  is a bijection  $\pi : \{1, 2, 3, \dots, n\} \rightarrow \{1, 2, 3, \dots, n\}$ .*

**Remark 2.2** *Permutations are traditionally written in a two-line notation of:*

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \pi(1) & \pi(2) & \pi(3) & \dots & \pi(n) \end{pmatrix},$$

*however for convenience we will use the shorter notation:*

$$\pi = [\pi(1), \pi(2), \pi(3), \dots, \pi(n)].$$

*For example:  $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix}$  will be written as  $\pi = [2, 4, 3, 1, 5]$ .*

**Definition 2.3** *The symmetric group of order  $n \in \mathbf{N}$  (denoted  $S_n$ ) is the group consisting of all the permutations of order  $n$ , with composition as the group operation.*

**Definition 2.4** *The Coxeter generators of  $S_n$  are  $s_1, s_2, \dots, s_{n-1}$  where  $s_i := [1, 2, \dots, i+1, i, \dots, n]$ .*

It is a well-known fact that the *symmetric group* is a *Coxeter group* with respect to the above generating set  $\{s_i \mid 1 \leq i \leq n-1\}$ . This fact gives rise to the following natural statistic of permutations in the symmetric group:

**Definition 2.5** *The length of a permutation  $\pi \in S_n$  is defined to be:*

$$\ell(\pi) := \min\{r \geq 0 \mid \pi = s_{i_1} \dots s_{i_r} \text{ for some } i_1, \dots, i_r \in [1, n]\}.$$

Here are other useful statistics on  $S_n$  that we are going to work with:

**Definition 2.6** *Let  $\pi \in S_n$ . Define the following:*

1. *The inversion number of  $\pi$ :*

$$\text{inv}(\pi) := |\{(i, j) \mid 1 \leq i < j \leq n, \pi(i) > \pi(j)\}|.$$

*Note that  $\text{inv}(\pi) = \ell(\pi)$ .*

2. *The descent set of  $\pi$ :  $\text{Des}(\pi) := \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$ .*

3. *The decent number of  $\pi$ :  $\text{des}(\pi) = |\text{Des}(\pi)|$ .*

4. *The major-index of  $\pi$ :  $\text{maj}(\pi) := \sum_{i \in \text{Des}(\pi)} i$ .*

5. *The sign of  $\pi$ :  $\text{sign}(\pi) := (-1)^{\ell(\pi)}$ .*

6. *The excedance number of  $\pi$ :  $\text{exc}(\pi) := |\{1 \leq i \leq n \mid \pi(i) > i\}|$ .*

**Example 2.7** Let  $\pi = [2, 3, 1, 5, 4] \in S_5$ . We can compute the above statistics on  $\pi$ , namely:

$$\begin{aligned} \text{inv}(\pi) = \ell(\pi) = 3, \quad \text{Des}(\pi) = \{2, 4\}, \quad \text{des}(\pi) = 2, \quad \text{maj}(\pi) = 6, \\ \text{sign}(\pi) = (-1)^3 = -1, \quad \text{and} \quad \text{exc}(\pi) = 3. \end{aligned}$$

**Remark 2.8** Throughout the paper we use the following notations for a nonnegative integer  $n$ :

$$\begin{aligned} [n]_q &:= \frac{1 - q^n}{1 - q}, \quad [n]_q! = [1]_q [2]_q \cdots [n]_q, \\ [n]_{\pm q}! &:= [1]_q [2]_{-q} [3]_q [4]_{-q} \cdots [n]_{(-1)^{n-1}q}, \quad \text{and also} \\ (a; q)_n &:= \begin{cases} 1, & \text{if } n = 0; \\ (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), & \text{otherwise.} \end{cases} \end{aligned}$$

MacMahon [12] was the first to find a connection between these statistics. He discovered that the *excedance* number is equidistributed with the *descent* number, and that the *inversion* number is equidistributed with the *major index*:

**Theorem 2.9** [12]

$$\sum_{\pi \in S_n} q^{\text{inv}(\pi)} = \sum_{\pi \in S_n} q^{\text{maj}(\pi)} = [1]_q [2]_q [3]_q \cdots [n]_q = [n]_q!.$$

**Theorem 2.10** [12]

$$\sum_{\pi \in S_n} q^{\text{exc}(\pi)} = \sum_{\pi \in S_n} q^{\text{des}(\pi)}.$$

Gessel and Simion gave a similar factorial type product formula for the *signed Mahonian*:

**Theorem 2.11** [14, Cor. 2]

$$\sum_{\pi \in S_n} \text{sign}(\pi) q^{\text{maj}(\pi)} = [n]_{\pm q}!.$$

A bivariate generalization of MacMahon's Theorem 2.9 was achieved during the 1970's by Foata and Schützenberger :

**Theorem 2.12** [8]

$$\sum_{\pi \in S_n} q^{\text{maj}(\pi) \ell^{\text{des}(\pi^{-1})}} = \sum_{\pi \in S_n} q^{\text{inv}(\pi) \ell^{\text{des}(\pi^{-1})}}.$$

In the same article Foata and Schützenberger also proved another bivariate connection between the different statistics:

**Theorem 2.13** [8]

$$\sum_{\pi \in S_n} q^{\text{maj}(\pi^{-1})} t^{\text{maj}(\pi)} = \sum_{\pi \in S_n} q^{\ell(\pi)} t^{\text{maj}(\pi)}.$$

In 1990, during her research of the genus zeta function Denert, found a new statistic which was also Mahonian:

**Definition 2.14** [6] *Let be  $\pi \in S_n$ , define the Denert's statistic to be:*

$$\begin{aligned} \text{den}(\pi) &:= |\{1 \leq l < k \leq n \mid \pi(k) < \pi(l) < k\}| \\ &+ |\{1 \leq l < k \leq n \mid \pi(l) < k < \pi(k)\}| \\ &+ |\{1 \leq l < k \leq n \mid k < \pi(k) < \pi(l)\}|. \end{aligned}$$

Later in the same year Foata and Zeilberger proved that the pair of statistics  $(\text{exc}, \text{den})$  is equidistributed with the pair  $(\text{des}, \text{maj})$ :

**Theorem 2.15** [9]

$$\sum_{\pi \in S_n} q^{\text{exc}(\pi)} t^{\text{den}(\pi)} = \sum_{\pi \in S_n} q^{\text{des}(\pi)} t^{\text{maj}(\pi)}.$$

## 2.2 Signed Permutations Groups

In this subsection we present the main definitions, notation and theorems for the classical Weyl groups of type B, also known as the *hyperoctahedral groups* or the *signed permutations groups*, and denoted  $B_n$ .

**Definition 2.16** *The hyperoctahedral group of order  $n \in \mathbf{N}$  (denoted  $B_n$ ) is the group consisting of all the bijections  $\sigma$  of the set  $[-n, n] \setminus \{0\}$  onto itself such that  $\sigma(-a) = -\sigma(a)$  for all  $a \in [-n, n] \setminus \{0\}$ , with composition as the group operation.*

**Remark 2.17** *There are different notations for a permutation  $\sigma \in B_n$ . We will use the notation  $\sigma = [\sigma(1), \dots, \sigma(n)]$ .*

We identify  $S_n$  as a subgroup of  $B_n$ , and  $B_n$  as a subgroup of  $S_{2n}$ . As in  $S_n$  we also have many different statistics; we will describe the main ones:

**Theorem 2.18** *Let  $\sigma \in B_n$ , define the following statistics on  $\sigma$ :*

1. *The inversion number of  $\sigma$ :  $\text{inv}(\sigma) := |\{(i, j) \mid 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}|$ .*

2. The descent set of  $\sigma$ :

$$Des(\sigma) := \{1 \leq i \leq n-1 \mid \sigma(i) > \sigma(i+1)\}.$$

3. The type A descent number of  $\sigma$ :  $des_A(\sigma) := |Des(\sigma)|$ .

4. The type B descent number of  $\sigma$ :

$$des_B(\sigma) := |\{0 \leq i \leq n-1 \mid \sigma(i) > \sigma(i+1)\}|, \text{ where here } \sigma(0) := 0.$$

5. The major index of  $\sigma$ :  $maj(\sigma) := \sum_{i \in Des(\sigma)} i$ .

6. The negative set of  $\sigma$ :  $Neg(\sigma) := \{i \in [1, \dots, n] \mid \sigma(i) < 0\}$ .

7. The negative number of  $\sigma$ :  $neg(\sigma) := |Neg(\sigma)|$ .

8. The negative number sum of  $\sigma$ :  $nsum(\sigma) := - \sum_{i \in Neg(\sigma)} \sigma(i)$ .

It is well known (see, e.g. [5, Proposition 8.1.3]) that  $B_n$  is a Coxeter group with respect to the generating set  $\{s_0, s_1, \dots, s_{n-1}\}$ , where  $s_i$ ,  $1 \leq i \leq n-1$ , are defined as in  $S_n$  (see 2.4), and  $s_0$  is defined as:

$$s_0 := [-1, 2, 3, \dots, n].$$

This gives rise to another natural statistic on  $B_n$ , the *length* statistic:

**Definition 2.19** For all  $\sigma \in B_n$  the length of  $\sigma$  is:

$$\ell(\sigma) := \min\{r \geq 0 \mid \sigma = s_{i_1} s_{i_2} \dots s_{i_r} \text{ for some } i_1, \dots, i_r \in [0, n-1]\}.$$

There is a well-known direct combinatorial way to compute this statistic:

**Theorem 2.20** ([5, Propositions 8.1.1 and 8.1.2]) For all  $\sigma \in B_n$  the length of  $\sigma$  can be computed as:

$$\ell(\sigma) = inv(\sigma) - \sum_{i \in Neg(\sigma)} \sigma(i).$$

Using the last definition we can define another natural statistic on  $B_n$ , the *sign* statistic:

**Definition 2.21** For all  $\sigma \in B_n$  the sign of  $\sigma$  is:

$$sign(\sigma) := (-1)^{\ell(\sigma)}.$$

The generating function of *length* is also called the Poincaré polynomial and can be presented in the following manner:

**Theorem 2.22** [11, §3.15]

$$\sum_{\sigma \in B_n} q^{\ell(\sigma)} = [2]_q [4]_q \dots [2n]_q = \prod_{i=1}^n [2i]_q.$$

Recently, Adin and Roichman generalized MacMahon's result Theorem 2.9 to  $B_n$ , by introducing a new Mahonian statistic, the *flag major index*:

**Definition 2.23** [3] *The flag major index of  $\sigma \in B_n$  is defined as:*

$$\text{flag-major}(\sigma) := 2\text{maj}(\sigma) + \text{neg}(\sigma),$$

where  $\text{maj}(\sigma)$  is calculated with respect to the linear order

$$-1 < -2 < \dots < -n < 1 < 2 < \dots < n.$$

**Theorem 2.24** [3, §2]

$$\sum_{\sigma \in B_n} q^{\ell(\sigma)} = \sum_{\sigma \in B_n} q^{\text{flag-major}(\sigma)} = [2]_q [4]_q \dots [2n]_q.$$

**Remark 2.25** *The previous result still holds if  $\text{maj}(\sigma)$  is calculated with respect to the natural order  $-n < -(n-1) < \dots < -2 < -1 < 1 < 2 < \dots < n-1 < n$ , see also [3].*

Adin, Brenti and Roichman introduced another statistic which was also Mahonian, the *nmaj* statistic:

**Definition 2.26** [1, §3.2] *Let  $\sigma \in B_n$  then the negative major index is defined as:*

$$\text{nmaj}(\sigma) := \text{maj}(\sigma) - \sum_{i \in \text{Neg}(\sigma)} \sigma(i) = \text{maj}(\sigma) + \text{nsun}(\sigma).$$

**Theorem 2.27** [1]

$$\sum_{\sigma \in B_n} q^{\ell(\sigma)} = \sum_{\sigma \in B_n} q^{\text{nmaj}(\sigma)}.$$

In the same article [1] they also defined a new descent multiset and new descent statistics, and found a new Euler-Mahonian bivariate distribution for these statistics:

**Definition 2.28** [1, §3.1 and §4.2] *Let  $\sigma \in B_n$  define:*

1. The negative descent multiset of  $\sigma$ :

$$NDes(\sigma) := Des(\sigma) \cup \{-\sigma(i) \mid i \in Neg(\sigma)\},$$

where  $\cup$  stands for multiset union.

2. The negative descent statistic of  $\sigma$ :  $ndes(\sigma) := |NDes(\sigma)|$ .

3. The flag-descent number of  $\sigma$ :  $fdes(\sigma) := des_A(\sigma) + des_B(\sigma) = 2des_A(\sigma) + \varepsilon(\sigma)$ ,  
where

$$\varepsilon(\sigma) := \begin{cases} 1, & \text{if } \sigma(1) < 0; \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 2.29** [1, §4.3]

$$\sum_{\sigma \in B_n} t^{ndes(\sigma)} q^{nmaj(\sigma)} = \sum_{\sigma \in B_n} t^{fdes(\sigma)} q^{flag-major(\sigma)}.$$

In their article from 2005 Adin, Gessel, and Roichman gave a type B analogue to the Gessel-Simion Theorem(e.g. [14, Cor. 2]):

**Theorem 2.30** [2, §5.1]

$$\sum_{\sigma \in B_n} sign(\sigma) q^{flag-major(\sigma)} = [2]_{-q} [4]_q \dots [2n]_{(-1)^n q}.$$

Where flag major index computed with respect to the linear order:

$$-1 < -2 < \dots < -n < 1 < 2 < \dots < n.$$

The group  $B_n$  has some well known subgroups [2, §7], in this paper we will only work with the subgroup of *elements with even neg*:

**Definition 2.31** The subgroup of elements with even neg in  $B_n$  (denoted  $D_n$ ) is defined as:

$$D_n := \{\sigma \in B_n \mid neg(\sigma) \equiv 0 \pmod{2}\}.$$



## 2.3 Colored Permutation Groups

In this subsection we will present the definitions, notations, and basic theorems for *the colored permutation groups* that we are going to work with throughout this paper. We start with the definition of the *colored permutation groups*.

**Definition 2.32** *Let  $r, n$  be positive integers. The group of colored permutations of  $n$  digits and  $r$  colors, denoted  $G_{r,n}$ , is the wreath product  $G_{r,n} = C_r \wr S_n$ , consisting of all the pairs  $(z, \pi)$  where  $z$  is a  $n$ -tuple of integers between 0 and  $r - 1$ , and  $\pi \in S_n$ . The group operation defined as follows: for  $z = (z_1, \dots, z_n)$ ,  $z' = (z'_1, \dots, z'_n)$ ,  $\pi, \pi' \in S_n$ ,*

$$(z, \pi) \cdot (z', \pi') = ((z_1 + z'_{\pi^{-1}(1)}, \dots, z_n + z'_{\pi^{-1}(n)}), \pi \circ \pi')$$

(where  $+$  is taken mod  $r$ ).

**Remark 2.33** *Let  $r$  and  $n$  be nonnegative integers. We will use the notation  $g, \bar{g} \in G_{r,n}$ , where  $g = (z, \pi)$  and  $\bar{g} = (z, \pi^{-1})$ .*

**Note 2.34** *Notice that for  $r = 1$ ,  $G_{1,n}$  isomorphic to the symmetric group  $S_n$ , and for  $r = 2$ ,  $G_{2,n}$  isomorphic to the signed permutation group  $B_n$ .*

In the following definitions we assume that the alphabet:

$$\{1, \dots, n, 1^{[1]}, \dots, n^{[1]}, \dots, 1^{[r-1]}, \dots, n^{[r-1]}\}$$

has some predefined linear order. As in the cases of the *symmetric group*, and the *signed permutation group* we can also define statistics on the *colored permutation groups*:

**Definition 2.35** *Let be  $g = (z, \pi) \in G_{r,n}$ . Define:*

1. *The inversion number of  $g$ :*

$$inv(g) := |\{(i, j) \mid 1 \leq i < j \leq n, \pi(i)^{[z_i]} > \pi(j)^{[z_j]}\}|.$$

2. *The Descent set of  $g$ :  $Des(g) := \{i \mid 1 \leq i \leq n - 1, \pi(i)^{[z_i]} > \pi(i + 1)^{[z_{i+1}]}\}$ .*

3. *The descent number of  $g$ :  $des(g) := |Des(g)|$ .*

4. *The major index of  $g$ :  $maj(g) := \sum_{i \in Des(g)} i$ .*

5. *The Negative set of  $g$ :  $Neg(g) := \{i \mid z_i \neq 0\}$ .*

6. *The negative number of  $g$ :  $neg(g) := |Neg(g)|$ .*

7. The color sum of  $g$ :  $csum(g) := \sum_{z_i \neq 0} z_i$ .

8. The negative color sum of  $g$ :  $ncsum(g) := \sum_{z_i \neq 0} (\pi(i) - 1)$ .

The *colored permutation group*  $G_{r,n}$  has a natural *length* statistic, defined with respect to generating set  $\{s_0, s_1, \dots, s_{n-1}\}$ , where  $s_i$ ,  $i \geq 1$  is defined as  $s_i := ((0, \dots, 0), [1, \dots, i+1, i, \dots, n])$  and  $s_0 := ((1, 0, \dots, 0), id)$ , see also [13] and [4]. There is a direct algebraic formula to calculate the *length* of  $g \in G_{r,n}$ , namely:

**Theorem 2.36** [4]

$$\ell(g) := inv(g) + \sum_{z_i \neq 0} (\pi(i) - 1) + \sum_{i=1}^n z_i,$$

where  $inv(g)$  is calculate according to the alphabet linear order:

$$n^{[r-1]} < \dots < n^{[1]} < \dots < 1^{[r-1]} < \dots < 1^{[1]} < 1 < \dots < n.$$

Adin and Roichman found a *flag major index* statistic on *colored permutation group*:

**Definition 2.37** [3, §3.1] Let be  $g \in G_{r,n}$ , define:

$$flag - major(g) := r \cdot maj(g) + \sum_{z_i \neq 0} z_i.$$

In the case of *colored permutation groups* we also have new Eulerian and Mahonian statistics which have been found in the passing year by Bagno:

**Definition 2.38** Let be  $g \in G_{r,n}$ , define:

$$1. ldes(g) := des(g) + \sum_{i=1}^n z_i.$$

$$2. lmaj(g) := maj(g) + \sum_{z_i \neq 0} (\pi(i) - 1) + \sum_{i=1}^n z_i.$$

**Theorem 2.39** [4, §5]

$$\sum_{g \in G_{r,n}} q^{lmaj(g)} = \sum_{g \in G_{r,n}} q^{\ell(g)} = [n]_q! \prod_{i=1}^n (1 + q^i [r-1]_q).$$

Bagno also found an interesting statistic over  $G_{r,n}$ , the  $nmaj$  statistic (see definition [4, §6]), which generalize the type B  $nmaj$  statistic (see 2.26), and proved the following equality:

**Theorem 2.40** [4, §7]

$$\sum_{g \in G_{r,n}} q^{nmaj(g)} = \sum_{g \in G_{r,n}} q^{flag-major(g)} = \prod_{i=1}^n [ri]_q.$$

### 3 Main Results

In this section we present the main results obtained in the paper according to their order of appearance:

**Remark 3.1** *Throughout this section we use the notation  $stat_{K_n}(g)$  when  $stat$  is a statistic on  $G_{r,n}$  and  $K_n$  is a predefined linear order on the alphabet of  $G_{r,n}$ . The notation  $stat_{K_n}(g)$  means that we calculate the statistic  $stat$  according to the linear order  $K_n$ .*

We begin Section 4 with Proposition 4.12, which states that the *flag major index* is equidistributed with the *length* over the *signed permutation group* for any linear order:

**Proposition 3.2** *(Proposition 4.12) For any linear order  $K_n$ :*

$$\sum_{\sigma \in B_n} q^{\text{flag-major}(\sigma)} = \sum_{\sigma \in B_n} q^{\text{flag-major}_{K_n}(\sigma)}.$$

The main results of Section 5 calculate generating functions according to *natural order*:

$$N : -n < -(n-1) < \dots < -1 < 1 < \dots < n-1 < n,$$

over the *signed permutations groups*. We begin with Theorem 5.7, which gives the generating function of the *signed Mahonian*:

**Theorem 3.3** *(Theorem 5.7)*

$$\sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_N(\sigma)} = (q; -1)_n [n]_{\pm q^2}!$$

The next result of this type:

**Theorem 3.4** *(Theorem 5.8)*

$$\sum_{\sigma \in B_n} \text{sign}(\sigma) q^{nmaj_N(\sigma)} = (q; -q)_n [n]_{\pm q}!$$

In Theorem 5.10, we calculate an interesting bivariate generating function of two Mahonian statistics:

**Theorem 3.5** *(Theorem 5.10)*

$$\sum_{\sigma \in B_n} q^{\text{flag-major}_N(\sigma)} t^{nmaj_N(\sigma)} = \prod_{i=1}^n (1 + qt^i) [n]_{q^2 t}!$$

In Theorem 5.12 we calculate another interesting bivariate generating function of *length* and *flag major index*:

**Theorem 3.6** (*Theorem 5.12*)

$$\sum_{\sigma \in B_n} q^{\text{flag-major}_N(\sigma)} t^{\ell(\sigma)} = \prod_{i=1}^n (1 + qt^i) A_n(q^2, t),$$

$$\text{where } A_n(q, t) = \sum_{\pi \in S_n} q^{\text{maj}(\pi)} t^{\ell(\pi)} = \sum_{\pi \in S_n} q^{\text{maj}(\pi)} t^{\text{inv}(\pi)}.$$

We finish reviewing Section 5 with Theorem 5.15, which presents a calculation of the generating function of *sign* and *flag major index* over  $D_n$ :

**Theorem 3.7** (*Theorem 5.15*)

$$\sum_{\sigma \in D_n} \text{sign}(\sigma) q^{\text{flag-major}_N(\sigma)} = (1 - q^2)^{\lfloor \frac{n}{2} \rfloor} [n]_{\pm q^2}!$$

According to the previous theorems we can conclude very interesting result, which is presented in Corollary 5.16:

**Corollary 3.8** (*Corollary 5.16*)

$$\sum_{\sigma \in D_{2n}} \text{sign}(\sigma) q^{\text{flag-major}_N(\sigma)} = \sum_{\sigma \in B_{2n}} \text{sign}(\sigma) q^{\text{flag-major}_N(\sigma)}.$$

We move on to Section 6, in which we calculate generating functions over the *colored permutation groups* according to the *friends order*:

$$F : 1^{[r-1]} < \dots < 1 < 2^{[r-1]} < \dots < 2 < \dots < n^{[r-1]} < \dots < n.$$

The first main result in this section is Lemma 6.4, which is used in almost all the proofs in this section:

**Lemma 3.9** (*Lemma 6.4*) *Let  $i \in [1, k]$  and let  $\text{ostat}_i : S_n \rightarrow \mathbf{Z}$ ,  $\text{cstat}_i : \mathbf{Z}_r^n \rightarrow \mathbf{Z}$  statistics on  $S_n$  and  $\mathbf{Z}_r^n$  respectively; define  $\text{stat}_i : G_{r,n} \rightarrow \mathbf{Z}$  by  $\text{stat}_i(z, \pi) := \text{ostat}_i(\pi) + \text{cstat}_i(z)$ . Then:*

$$\sum_{(z, \pi) \in G_{r,n}} q_1^{\text{stat}_1(z, \pi)} \dots q_k^{\text{stat}_k(z, \pi)} = \sum_{(z, id) \in \text{Color}_r^n} q_1^{\text{cstat}_1(z)} \dots q_k^{\text{cstat}_k(z)} \sum_{\pi \in S_n} q_1^{\text{ostat}_1(\pi)} \dots q_k^{\text{ostat}_k(\pi)},$$

where the set  $\text{Color}_r^n$  is defined in 6.1.

We continue on, define a new *friends color-sign* statistic (see Definition 6.8), and calculate the bivariate generating function of *friends color-sign* and *flag major index* over the *colored permutation groups*. The result of this calculation is presented in Theorem 6.10:

**Theorem 3.10** (Theorem 6.10)

$$\sum_{g \in G_{r,n}} csign_F(g) q^{flag-major_F(g)} = [r]_{-q}^n [n]_{\pm rq}!$$

We continue this section with the definitions of two new statistics over: the *colored permutations groups*: the *r-color excedance number*, denoted  $exc_r$  (see definition 6.12), and the *r-color Denert's statistic*, denoted  $den_r$  (see Definition 6.14). By using these new statistics we give in Theorem 6.16 a generalization of the Foata-Zeilberger Theorem 2.15:

**Theorem 3.11** (Theorem 6.16) *Let  $r, n$  be positive integers. Then:*

$$\sum_{g \in G_{r,n}} q^{den_r(g)} t^{exc_r(g)} = \sum_{g \in G_{r,n}} q^{flag-major_F(g)} t^{ldes_F(g)}.$$

We finish the paper in Section 7, where we define new statistic, the *flag-inversion* statistic (see Definition 7.2) which is equidistributed with *flag major index*, and by using the *flag-inversion* statistic we give in Theorem 7.7, and in Theorem 7.8 another generalizations to the Foata-Schützenberger Theorem 2.12, and Theorem 2.13:

**Theorem 3.12** (Theorem 7.7)

$$\sum_{g \in G_{r,n}} q^{flag-major_F(g)} t^{ldes_F(\bar{g})} = \sum_{g \in G_{r,n}} q^{inv_F(g)} t^{ldes_F(\bar{g})}.$$

**Theorem 3.13** (Theorem 7.8)

$$\sum_{\sigma \in B_n} q^{flag-major_F(\sigma^{-1})} t^{flag-major_F(\sigma)} = \sum_{\sigma \in B_n} q^{inv_F(\sigma)} t^{flag-major_F(\sigma)}.$$

## 4 Flag Major Index With Different Orders

In this section study statistics on the *signed permutation group* with respect to different linear orders. First, we give necessary definitions and introduce certain bijections (see Definition 4.5). Using these definitions we prove that the *flag major index* is equidistributed with the *length* statistic for any linear order; however, this result does not hold in the case of *signed Mahonian* distribution (it is easy to check this fact in the case of  $n = 2$ ), we finish this section indicating a large collection of linear orders which are *sign* and *flag major index* equidistributed with each other.

**Definition 4.1** *A linear order of length  $n$ , denoted  $K_n$ , is a bijection*

$$K_n : [1, 2n] \rightarrow [-n, n] \setminus \{0\}.$$

*For convenience we write a linear order in the following form:*

$$K_n : K_n(1) < K_n(2) < \dots < K_n(2n-1) < K_n(2n).$$

We can calculate permutation statistics according to a linear order  $K_n$ , we use the following notation:  $maj_{K_n}(\sigma)$ ,  $des_{K_n}(\sigma)$ ,  $flag - major_{K_n}(\sigma)$ ,  $nmaj_{K_n}(\sigma)$  etc, to indicate that the corresponding statistic is calculated with respect to the linear order  $K_n$ . We also use the notation:  $m >_{K_n} l$ , to indicate, that according to the linear order  $K_n$  'm' is larger than 'l', i.e. that  $m = K_n(s)$ ,  $l = K_n(r)$ , and  $s > r$ .

**Example 4.2** Let  $K_n$  be a linear order and let  $\sigma \in B_n$ . Then:

$$maj_{K_n}(\sigma) := \sum_{\sigma(i) >_{K_n} \sigma(i+1)} i.$$

**Note 4.3** Notice that for any linear order  $K_n$ , and for any  $\sigma \in B_n$ ,  $neg(\sigma) = neg_{K_n}(\sigma)$ . This also applies to the length statistic, because it is defined with respect to the Coxeter generators, which do not depend on the choice of linear order.

**Definition 4.4** Let  $K_n : K_n(1) < K_n(2) < \dots < K_n(2n-1) < K_n(2n)$ , be a linear order and let  $1 \leq j \leq 2n-1$ . Define  $K_{n,j}$  to be the following linear order:

$$K_{n,j} : K_n(1) < K_n(2) < \dots < K_n(j+1) < K_n(j) < \dots < K_n(2n-1) < K_n(2n).$$

**Definition 4.5** Let  $K_n$  be a linear order,  $j \in [1, 2n-1]$ , and  $s \in [1, n]$ . Define the function  $\psi^j(\sigma) : B_n \rightarrow B_n$  as:

$$\psi^j(\sigma)(s) := \begin{cases} K_n(j), & \sigma(s) = K_n(j+1), \\ & \exists p \in [1, n], \sigma(p) = K_n(j), \\ K_n(j+1), & \sigma(s) = K_n(j), \\ & \exists p \in [1, n], \sigma(p) = K_n(j+1), \\ \sigma(s), & \text{Otherwise.} \end{cases}$$

**Note 4.6** By the definition of  $\psi^j$  we can conclude that  $\psi^j(\psi^j(\sigma)) = \sigma$ .

**Lemma 4.7** Let  $K_n$  be a linear order and  $j \in [1, 2n-1]$ , then  $\psi^j : B_n \rightarrow B_n$  is a bijection, and

$$Des_{K_n}(\sigma) = Des_{K_{n,j}}(\psi^j(\sigma)), \forall \sigma \in B_n.$$

**Proof.** Let  $\sigma_1, \sigma_2 \in B_n$ , we need to prove that:  $\sigma = \psi^j(\sigma_1) = \psi^j(\sigma_2) \Rightarrow \sigma_1 = \sigma_2$ . We divide our proof into two cases:

1. If  $\exists p, m \in [1, n]$ , such that  $\sigma(p) = K_n(j)$ ,  $\sigma(m) = K_n(j+1)$ , then according to the definition of  $\psi^j$  we conclude that  $\sigma = \psi^j(\sigma_1) = \psi^j(\sigma_2) \Rightarrow \sigma_1(s) = \sigma_2(s), \forall s \neq p, m$ , and  $\sigma(p) = \sigma_1(m) = \sigma_2(m)$ ,  $\sigma(m) = \sigma_1(p) = \sigma_2(p)$ , therefore we conclude that:  $\sigma_1 = \sigma_2$ .

2. Otherwise, according to the definition of  $\psi^j$  we get  $\psi^j(\sigma_1) = \sigma_1 = \psi^j(\sigma_2) = \sigma_2 \Rightarrow \sigma_1 = \sigma_2$ .

We have proved that  $\psi^j$  is a bijection. Now we need to prove that:  $Des_{K_n}(\sigma) = Des_{K_{n,j}}(\psi^j(\sigma))$ . We will do so by checking all the possible cases of  $\psi^j(\sigma)(i)$  and  $\psi^j(\sigma)(i+1)$ , and proving that for each case  $\sigma(i) >_{K_n} \sigma(i+1) \Leftrightarrow \psi^j(\sigma)(i) >_{K_{n,j}} \psi^j(\sigma)(i+1)$ :

1.  $\sigma(i) = K_n(s)$ ,  $s \in [1, 2n]$ ,  $s \neq j, j+1$ , therefore by the definition of  $\psi^j$ :  
 $\psi^j(\sigma)(i) = \sigma(i) = K_n(s) = K_{n,j}(s)$ , and again we divide into cases:
  - (a)  $\sigma(i+1) = K_n(t)$ ,  $t \in [1, 2n]$ ,  $t \neq j, j+1$  therefore;  $\psi^j(\sigma)(i+1) = \sigma(i+1) = K_n(t) = K_{n,j}(t)$ , and now we can conclude that:  

$$\sigma(i) >_{K_n} \sigma(i+1) \Leftrightarrow K_n(s) >_{K_n} K_n(t) \Leftrightarrow s > t \Leftrightarrow \sigma(i) >_{K_{n,j}} \sigma(i+1) \Leftrightarrow \psi^j(\sigma)(i) >_{K_{n,j}} \psi^j(\sigma)(i+1).$$
  - (b)  $\sigma(i+1) = K_n(j)$ , again we divide into two cases according to the behavior of  $\psi^j$ :
    - i.  $\exists p \in [1, n]$ ,  $\sigma(p) = K_n(j+1)$ , in this case:  

$$\psi^j(\sigma)(i+1) = K_n(j+1) = K_{n,j}(j), \text{ and}$$

$$\psi^j(\sigma)(i) = K_n(s) = K_{n,j}(s), \text{ therefore;}$$

$$\sigma(i) >_{K_n} \sigma(i+1) \Leftrightarrow s > j \Leftrightarrow K_{n,j}(s) > K_{n,j}(j) \Leftrightarrow \psi^j(\sigma)(i) >_{K_{n,j}} \psi^j(\sigma)(i+1).$$
    - ii.  $\forall p \in [1, n]$ ,  $\sigma(p) \neq K_n(j+1)$ , in this case  $\psi^j(\sigma) = \sigma$   

$$\psi^j(\sigma)(i+1) = K_n(j) = K_{n,j}(j+1), \text{ therefore;}$$

$$\sigma(i) >_{K_n} \sigma(i+1) \Leftrightarrow s > j \Leftrightarrow_{s \neq j, j+1} s > j+1 \Leftrightarrow \psi^j(\sigma)(i) = K_{n,j}(s) >_{K_{n,j}} K_{n,j}(j+1) = \psi^j(\sigma)(i+1).$$
  - (c)  $\sigma(i+1) = K_n(j+1)$ , this case is almost identical to the previous case (b) with minor changes.

2.  $\sigma(i) = K_n(j)$ , as in the previous case, we will check all the possible options:
  - (a)  $\sigma(i+1) = K_n(t)$ ,  $t \in [1, 2n]$ ,  $t \neq j+1$ , and we divide into cases according to the behavior of  $\psi^j$ :
    - i.  $\exists p \in [1, n]$   $\sigma(p) = K_n(j+1)$ , then:  

$$\psi^j(\sigma)(i) = K_n(j+1) = K_{n,j}(j),$$

$$\psi^j(\sigma)(i+1) = K_n(t) = K_{n,j}(t), \text{ therefore;}$$

$$\sigma(i) >_{K_n} \sigma(i+1) \Leftrightarrow j > t \Leftrightarrow \psi^j(\sigma)(i) = K_{n,j}(j) >_{K_{n,j}} K_{n,j}(t) = \psi^j(\sigma)(i+1).$$

ii.  $\forall p \in [1, n] \sigma(p) \neq K_n(j+1)$ , then:

$$\begin{aligned} \psi^j(\sigma)(i) &= K_n(j) = K_{n,j}(j+1), \\ \psi^j(\sigma)(i+1) &= K_n(t) = K_{n,j}(t), \text{ therefore;} \\ \sigma(i) >_{K_n} \sigma(i+1) &\Leftrightarrow j > t \Leftrightarrow \\ j+1 > t &\Leftrightarrow \psi^j(\sigma)(i) = K_{n,j}(j+1) >_{K_{n,j}} K_{n,j}(t) = \\ &\psi^j(\sigma)(i+1). \end{aligned}$$

(b)  $\sigma(i+1) = K_n(j+1) = K_{n,j}(j)$ , then:

$$\begin{aligned} \psi^j(\sigma)(i) &= K_n(j+1) = K_{n,j}(j), \\ \psi^j(\sigma)(i+1) &= K_n(j) = K_{n,j}(j+1), \text{ therefore;} \\ \sigma(i+1) = K_n(j+1) &>_{K_n} \sigma(i) = K_n(j) \Leftrightarrow \\ \psi^j(\sigma)(i+1) &= K_{n,j}(j+1) >_{K_{n,j}} \psi^j(\sigma)(i) = K_{n,j}(j). \end{aligned}$$

3.  $\sigma(i) = K_n(j+1)$ . This case is similar to case 2 above.

We got that:

$$\forall \sigma \in B_n, \forall i \in [1, n-1], \sigma(i) >_{K_n} \sigma(i+1) \Leftrightarrow \psi^j(\sigma)(i) >_{K_{n,j}} \psi^j(\sigma)(i+1).$$

Now we can conclude that  $\forall \sigma, \in B_n \text{ Des}_{K_n}(\sigma) = \text{Des}_{K_{n,j}}(\psi^j(\sigma))$  •

**Corollary 4.8** *Let  $K_n$  be a linear order and  $1 \leq j \leq 2n-1$  then:*

$$\text{maj}_{K_{n,j}}(\psi^j(\sigma)) = \text{maj}_{K_n}(\sigma).$$

**Corollary 4.9** *Let  $K_n$  be linear order and  $1 \leq j \leq 2n-1$  then:*

$$\sum_{\sigma \in B_n} q^{\text{maj}_{K_n}(\sigma)} = \sum_{\sigma \in B_n} q^{\text{maj}_{K_{n,j}}(\sigma)}.$$

**Corollary 4.10** *Let  $K_n$  be linear order then:*

$$\sum_{\sigma \in B_n} q^{\text{maj}(\sigma)} = \sum_{\sigma \in B_n} q^{\text{maj}_{K_n}(\sigma)}.$$

**Proof.** We can transfer the given linear order  $K_n$  to the natural order:  $N : -n < -(n-1) < \dots < -1 < 1 < \dots < n-1 < n$ , by a finite number of adjacent transpositions.

**Remark 4.11** *According to the above proof for each linear order  $K_n$  there exists a bijection  $\phi : B_n \rightarrow B_n$  that,  $\text{maj}(\phi(\sigma)) = \text{maj}_{K_n}(\sigma)$ ,  $\forall \sigma$ , and  $\phi = \psi^{r_m} \circ \psi^{r_{m-1}} \circ \dots \circ \psi^{r_1}$  where  $\{r_i\}_{i=1}^m$  are positive integers.*



Now we can prove that for any linear order  $K_n$  the *flag major index* equidistributed with *length* statistic over the *signed permutations groups*:

**Proposition 4.12** *Let  $K_n$  be a linear order then:*

$$\sum_{\sigma \in B_n} q^{\text{flag-major}(\sigma)} = \sum_{\sigma \in B_n} q^{\text{flag-major}_{K_n}(\sigma)}.$$

**Proof.** Use the bijection  $\phi : B_n \rightarrow B_n$ , and note that

$$\text{neg}(\phi(\sigma)) = \text{neg}(\sigma), \quad \forall \sigma \in B_n.$$

$$\begin{aligned} \sum_{\sigma \in B_n} q^{\text{flag-major}_{K_n}(\sigma)} &= \sum_{\sigma \in B_n} q^{2\text{maj}_{K_n}(\sigma) + \text{neg}_{K_n}(\sigma)} \\ &= \sum_{\sigma \in B_n} q^{2\text{maj}_{K_n}(\phi(\sigma)) + \text{neg}(\phi(\sigma))} \\ &= \sum_{\sigma \in B_n} q^{2\text{maj}(\sigma) + \text{neg}(\sigma)} \\ &= \sum_{\sigma \in B_n} q^{\text{flag-major}(\sigma)} \quad \bullet \end{aligned}$$

**Lemma 4.13** *Let  $K_n$  be a linear order, such that, for some  $j \in [1, 2n - 1]$ ,  $K_n(j) + K_n(j + 1) = 0$ . Then:  $\psi^j(\sigma) = \sigma$ .*

**Proof.** Let be  $\sigma \in B_n$ , by the definition of  $B_n$ , we know that  $\exists i \in [1, n] \sigma(i) = \pm K_n(j)$ , therefore;  $\forall s \in [1, n] \sigma(s) \neq -\sigma(i)$ , and now according to the definition of  $\psi^j$  we conclude that:  $\psi^j(\sigma) = \sigma \quad \bullet$

**Corollary 4.14** *Let  $K_n$  be a linear order, such that, for some  $j \in [1, 2n - 1]$ ,  $K_n(j) + K_n(j + 1) = 0$ . Then:*

$$\sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_{K_n}(\sigma)} = \sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_{K_n, j}(\sigma)}.$$

**Proof.**

$$\begin{aligned} \sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_{K_n}(\sigma)} &= \sum_{\sigma \in B_n} \text{sign}(\psi^j(\sigma)) q^{\text{flag-major}_{K_n}(\psi^j(\sigma))} \\ &= \sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_{K_n, j}(\sigma)} \\ &= \sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_{K_n, j}(\sigma)} \quad \bullet \end{aligned}$$

**Remark 4.15** Let  $K_n$  be linear order, and let  $\{i_s\}_{s=1}^{s=m}$  be integers in  $[1, 2n - 1]$ , such that  $K_n(i_s) + K_n(i_{s+1}) = 0 \forall s$ , then it easy to prove by induction that for any linear order  $K'_n = K_{n, i_{j_1}, i_{j_2}, \dots, i_{j_w}}$ ,  $w \in \mathbf{N}$ ,  $i_{j_k} \in \{i_s\}_{s=1}^{s=m}$ ,  $k \in [1, w]$ , exist the following equality:

$$\sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_{K_n}(\sigma)} = \sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_{K'_n}(\sigma)}.$$

**Example 4.16** For example consider the linear order:

$$F : -1 < 1 < -2 < 2 < \dots < -n < n,$$

then we can create new linear order  $F^{2i-1} = F s_{2i-1}$  by swapping  $F(2i - 1)$  with  $F(2i)$  for all  $i \in [1, n]$ . And according to the last corollary we get that:

$$\sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_F(\sigma)} = \sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_{F^{2i-1}}(\sigma)}.$$

According to the last remark we can generate from the linear order  $F$ ,  $2^n$  linear orders which also satisfy the last equality.

## 5 Permutation Statistics in the Natural Order

In this section we will calculate generating functions of different statistics according to the *natural order*:

$$N : -n < -(n - 1) < \dots < -1 < 1 < \dots < n - 1 < n.$$

Most of the results of this section have been achieved by reducing the summation from  $B_n$  to  $U_n$  (see also [2, §7]), and calculating the generating function over  $U_n$  by recursion to  $U_{n-1}$ . Before we begin with our calculation and theorems, we need some helpful tools that will be presented in the following section.

### 5.1 Preliminaries

**Definition 5.1** Define the following set:

$$U_n := \{\tau \in B_n \mid \tau(1) < \tau(2) < \dots < \tau(n - 1) < \tau(n)\}.$$

There are several facts (see also [1],[2]) about the set  $U_n$  that can be directly concluded from the definition of  $U_n$ , namely: each  $\sigma \in B_n$  has a unique representation as:

$$\sigma = \tau \pi \quad (\tau \in U_n, \text{ and } \pi \in S_n).$$

The following properties are clear:

**Fact 5.2** 1.  $maj_N(\sigma) = maj_N(\pi)$ .

2.  $inv_N(\sigma) = inv_N(\pi)$ .

3.  $neg(\sigma) = neg(\tau)$ .

4.  $nsum(\sigma) = nsum(\tau)$ .

5.  $sign(\sigma) = sign(\tau)sign(\pi)$ .

We continue with our definitions.

**Definition 5.3** Define the following subsets of  $U_n$ :

1.  $U_{n1} := \{\tau \in U_n \mid \tau(1) = -n\}$ .

2.  $U_{n2} := \{\tau \in U_n \mid \tau(n) = n\}$ .

**Note 5.4**  $U_n = U_{n1} \uplus U_{n2}$ , where  $\uplus$  stands for disjoint union.

We also define two bijections from  $U_{n-1}$  one onto  $U_{n1}$ , and one onto  $U_{n2}$ :

**Definition 5.5** For  $i \in 1, 2$ , define  $\varphi_{ni} : U_{n-1} \rightarrow U_{ni}$  by:

1.  $\varphi_{n1}(\tau)(i) = \begin{cases} -n, & i=1; \\ \tau(i-1), & 2 \leq i \leq n. \end{cases}$

2.  $\varphi_{n2}(\tau)(i) = \begin{cases} \tau(i), & 1 \leq i \leq n-1; \\ n, & i = n. \end{cases}$

Clearly, for  $i \in 1, 2$ ,  $\varphi_{ni}$  is a bijection (for  $U_{n-1}$  onto  $U_{ni}$ ). Therefore, for all  $\tau_1 \in U_{n1}$ ,  $\tau_2 \in U_{n2}$  there exist unique  $\tau'_1 \in U_{n-1}$ ,  $\tau'_2 \in U_{n-1}$ , such that  $\varphi_{n1}(\tau'_1) = \tau_1$ ,  $\varphi_{n2}(\tau'_2) = \tau_2$ . The relations between the permutation statistics of  $\tau_1$ ,  $\tau_2$ , and  $\tau'_1$ ,  $\tau'_2$  are presented in the following equations:

**Fact 5.6** 1.  $neg(\tau_1) = neg(\varphi_{n1}(\tau'_1)) = neg(\tau'_1) + 1$ ;  $neg(\tau_2) = neg(\varphi_{n2}(\tau'_2)) = neg(\tau'_2)$ .

2.  $sign(\varphi_{n1}(\tau'_1)) = (-1)^n sign(\tau'_1)$ ;  $sign(\varphi_{n2}(\tau'_2)) = sign(\tau'_2)$ .

Now after we finished our preparations we can move on to the theorems:

## 5.2 Signed Mahonian

### Theorem 5.7

$$\sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_N(\sigma)} = (q; -1)_n [n]_{\pm q^2}!$$

**Proof.** We will use the facts mentioned above in *natural linear order* to reduce the summation from  $B_n$  into  $U_n$ :

$$\begin{aligned} \sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_N(\sigma)} &= \sum_{\tau\pi \in B_n} \text{sign}(\tau\pi) q^{\text{flag-major}_N(\tau\pi)} \\ &= \sum_{\tau \in U_n, \pi \in S_n} \text{sign}(\pi) \text{sign}(\tau) q^{2\text{maj}(\pi) + \text{neg}(\tau)} \\ &= \sum_{\tau \in U_n} \text{sign}(\tau) q^{\text{neg}(\tau)} \sum_{\pi \in S_n} \text{sign}(\pi) q^{2\text{maj}(\pi)}. \end{aligned}$$

Now according to Theorem 2.11 we know that:

$$\sum_{\pi \in S_n} \text{sign}(\pi) q^{2\text{maj}(\pi)} = [1]_{q^2} [2]_{-q^2} [3]_{q^2} [4]_{-q^2} \dots [n]_{(-1)^{n-1} q^2} = [n]_{\pm q^2}!$$

In order to finish the calculation we need to calculate:

$$a_n = \sum_{\tau \in U_n} \text{sign}(\tau) q^{\text{neg}(\tau)},$$

we will do so by recursion:

$$\begin{aligned} a_n &= \sum_{\tau \in U_n} \text{sign}(\tau) q^{\text{neg}(\tau)} \\ &= \sum_{\tau \in U_{n1}} \text{sign}(\tau) q^{\text{neg}(\tau)} + \sum_{\tau \in U_{n2}} \text{sign}(\tau) q^{\text{neg}(\tau)} \\ &= \sum_{\tau' \in U_{n-1}} \text{sign}(\varphi_{n1}(\tau')) q^{\text{neg}(\varphi_{n1}(\tau'))} + \sum_{\tau' \in U_{n-1}} \text{sign}(\varphi_{n2}(\tau')) q^{\text{neg}(\varphi_{n2}(\tau'))} \\ &= \sum_{\tau' \in U_{n-1}} (-1)^n \text{sign}(\tau') q^{\text{neg}(\tau')+1} + \sum_{\tau' \in U_{n-1}} \text{sign}(\tau') q^{\text{neg}(\tau')} \\ &= (-1)^n q a_{n-1} + a_{n-1}. \end{aligned}$$

In the end of the calculation we got the following recurrence equation:

$$a_n = (1 + (-1)^n q) a_{n-1}, \quad a_1 = 1 - q.$$

This is easy to solve recurrence equation, which its solution is:

$$a_n = \begin{cases} a_{2m} = (1 - q)^m (1 + q)^m = (1 - q^2)^m, & n=2m; \\ a_{2m+1} = (1 - q)^{m+1} (1 + q)^m = (1 - q^2)^m (1 - q), & n=2m+1. \end{cases}$$

And with the previous calculation we conclude that:

$$\begin{aligned} \sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_N(\sigma)} &= \begin{cases} (1 - q^2)^m [n]_{\pm q^2}!, & n=2m; \\ (1 - q)(1 - q^2)^m [n]_{\pm q^2}!, & n=2m+1. \end{cases} \\ &= (q; -1)_n [n]_{\pm q^2} \bullet \end{aligned}$$

We move on and calculate the generating function of another *signed Mahonian* statistic:

**Theorem 5.8**

$$\sum_{\sigma \in B_n} \text{sign}(\sigma) q^{nmaj_N(\sigma)} = (q; -q)_n [n]_{\pm q}!$$

**Proof.** We will use the same methods as in the previous theorem, reducing the problem to  $U_n$ :

$$\begin{aligned} \sum_{\sigma \in B_n} \text{sign}(\sigma) q^{nmaj_N(\sigma)} &= \sum_{\pi \in S_n, \tau \in U_n} \text{sign}(\tau\pi) q^{\text{maj}(\tau\pi) + n\text{sum}(\tau)} \\ &= \sum_{\pi \in S_n, \tau \in U_n} \text{sign}(\pi) \text{sign}(\tau) q^{\text{maj}(\pi) + n\text{sum}(\tau)} \\ &= \sum_{\pi \in S_n} \text{sign}(\pi) q^{\text{maj}(\pi)} \sum_{\tau \in U_n} \text{sign}(\tau) q^{n\text{sum}(\tau)}. \end{aligned}$$

By Theorem 2.11 we know that  $\sum_{\pi \in S_n} \text{sign}(\pi) q^{\text{maj}(\pi)} = [n]_{\pm q}!$ , and:

$$\begin{aligned} a_n &= \sum_{\tau \in U_n} \text{sign}(\tau) q^{n\text{sum}(\tau)} \\ &= \sum_{\tau \in U_{n1}} \text{sign}(\tau) q^{n\text{sum}(\tau)} + \sum_{\tau \in U_{n2}} \text{sign}(\tau) q^{n\text{sum}(\tau)} \\ &= \sum_{\tau' \in U_{n-1}} \text{sign}(\varphi_{n1}(\tau')) q^{n\text{sum}(\varphi_{n1}(\tau'))} + \sum_{\tau' \in U_{n-1}} \text{sign}(\varphi_{n2}(\tau')) q^{n\text{sum}(\varphi_{n2}(\tau'))} \\ &= \sum_{\tau' \in U_{n-1}} \text{sign}(\tau') (-1)^n q^{n\text{sum}(\tau') + n} + \sum_{\tau' \in U_{n-1}} \text{sign}(\tau') q^{n\text{sum}(\tau')} \\ &= (-1)^n q^n a_{n-1} + a_{n-1} = (1 + (-q)^n) a_{n-1}. \end{aligned}$$

We got a recurrence equation  $a_n = (1 + (-1)^n q^n) a_{n-1}$ ,  $a_1 = 1 - q$ , which its solution is:

$$a_n = \prod_{i=1}^n (1 + (-q)^i) = (q; -q)_n.$$

And after we multiply our results we get:

$$\sum_{\sigma \in B_n} \text{sign}(\sigma) q^{nmaj_N(\sigma)} = \prod_{i=1}^n (1 + (-q)^i) [n]_{\pm q}! = (q; -q)_n [n]_{\pm q}! \bullet$$

### 5.3 Bivariate Generating Functions

We can use the method of reducing the summation from  $B_n$  into  $U_n$ , for the purpose of generalizing and proving known theorems, we will start with generalizing Theorem 2.27:

#### Theorem 5.9

$$\sum_{\sigma \in B_n} q^{nmaj_N(\sigma)} t^{neg(\sigma)} = \sum_{\sigma \in B_n} q^{\ell(\sigma)} t^{neg(\sigma)}.$$

**Proof.**

$$\begin{aligned} \sum_{\sigma \in B_n} q^{nmaj_N(\sigma)} t^{neg(\sigma)} &= \sum_{\pi \in S_n, \tau \in U_n} q^{maj(\tau\pi) + nsum(\tau\pi)} t^{neg(\tau\pi)} \\ &= \sum_{\pi \in S_n, \tau \in U_n} q^{maj(\pi) + nsum(\tau)} t^{neg(\tau)} \\ &= \sum_{\tau \in U_n} q^{nsum(\tau)} t^{neg(\tau)} \sum_{\pi \in S_n} q^{maj(\pi)} \\ &= \sum_{\tau \in U_n} q^{nsum(\tau)} t^{neg(\tau)} \sum_{\pi \in S_n} q^{inv(\pi)} \\ &= \sum_{\pi \in S_n, \tau \in U_n} q^{inv(\tau\pi) + nsum(\tau\pi)} t^{neg(\tau\pi)} \\ &= \sum_{\sigma \in B_n} q^{\ell(\sigma)} t^{neg(\sigma)} \quad \bullet \end{aligned}$$

We can also calculate some interesting *Mahonian-Mahonian* generating function:

#### Theorem 5.10

$$\sum_{\sigma \in B_n} q^{flag-major_N(\sigma)} t^{nmaj_N(\sigma)} = \prod_{i=1}^n (1 + qt^i) [n]_{q^2 t}!$$

**Proof.** We will prove this theorem by reducing the problem to  $U_n$ :

$$\begin{aligned} \sum_{\sigma \in B_n} q^{flag-major_N(\sigma)} t^{nmaj_N(\sigma)} &= \sum_{\pi \in S_n, \tau \in U_n} q^{2maj(\pi) + neg(\tau)} t^{maj(\pi) + nsum(\tau)} \\ &= \sum_{\tau \in U_n} q^{neg(\tau)} t^{nsum(\tau)} \sum_{\pi \in S_n} q^{2maj(\pi)} t^{maj(\pi)} \\ &= \sum_{\tau \in U_n} q^{neg(\tau)} t^{nsum(\tau)} \sum_{\pi \in S_n} (q^2 t)^{maj(\pi)}. \end{aligned}$$

We know according to Theorem 2.9 that:  $\sum_{\pi \in S_n} (q^2 t)^{maj(\pi)} = [n]_{q^2 t}!$ , and by calculation we get:

$$\begin{aligned}
a_n &= \sum_{\tau \in U_n} q^{neg(\tau)} t^{nsum(\tau)} = \sum_{\tau \in U_{n1}} q^{neg(\tau)} t^{nsum(\tau)} + \sum_{\tau \in U_{n2}} q^{neg(\tau)} t^{nsum(\tau)} \\
&= \sum_{\tau' \in U_{n-1}} q^{neg(\varphi_{n1}(\tau'))} t^{nsum(\varphi_{n1}(\tau'))} \\
&\quad + \sum_{\tau' \in U_{n-1}} q^{neg(\varphi_{n2}(\tau'))} t^{nsum(\varphi_{n2}(\tau'))} \\
&= \sum_{\tau' \in U_{n-1}} q^{neg(\tau')+1} t^{nsum(\tau')+n} + \sum_{\tau' \in U_{n-1}} q^{neg(\tau')} t^{nsum(\tau')} \\
&= qt^n a_{n-1} + a_{n-1} = (1 + qt^n) a_{n-1}.
\end{aligned}$$

We got the recurrence equation:  $a_n = (1 + qt^n) a_{n-1}$ ,  $a_1 = 1 + qt$ , and the solution to this equation is:  $a_n = \prod_{i=1}^n (1 + qt^i)$ , and therefore; the general solution is:

$$\sum_{\sigma \in B_n} q^{flag-major_N(\sigma)} t^{nmaj_N(\sigma)} = [n]_{q^2 t}! \prod_{i=1}^n (1 + qt^i) \quad \bullet$$

**Note 5.11** Notice that if we put  $t = 1$  in the previous Theorem 5.10, we get Theorem 2.24 and the equation:  $[n]_{q^2} (1 + q)^n = \prod_{i=1}^n [2i]_q!$ .

We can also calculate the generating function of *length* and *flag major index* by using a similar method:

**Theorem 5.12**

$$\sum_{\sigma \in B_n} q^{flag-major_N(\sigma)} t^{\ell(\sigma)} = A_n(q^2, t) \prod_{i=1}^n (1 + qt^i),$$

where  $A_n(q, t) = \sum_{\pi \in S_n} q^{maj(\pi)} t^{\ell(\pi)} = \sum_{\pi \in S_n} q^{maj(\pi)} t^{inv(\pi)}$ .

**Proof.** We will start with reducing  $B_n$  to  $U_n$ :

$$\begin{aligned}
\sum_{\sigma \in B_n} q^{flag-major(\sigma)} t^{\ell(\sigma)} &= \sum_{\tau \in U_n, \pi \in S_n} q^{2maj(\pi) + neg(\tau)} t^{inv(\pi) + nsum(\tau)} \\
&= \sum_{\tau \in U_n} q^{neg(\tau)} t^{nsum(\tau)} \sum_{\pi \in S_n} q^{2maj(\pi)} t^{inv(\pi)} \\
&= A_n(q^2, t) \sum_{\tau \in U_n} q^{neg(\tau)} t^{nsum(\tau)}.
\end{aligned}$$

Now we need to calculate  $\sum_{\tau \in U_n} q^{neg(\tau)} t^{nsum(\tau)}$ , we calculate this exact equation in the proof of the previous Theorem 5.10, and got the recurrence equation:  $a_n = (1 +$

$qt^n)a_{n-1}$ ,  $a_1 = 1 + qt$ , which its solution is:  $a_n = \prod_{i=1}^n (1 + qt^i)$ , therefore; the final solution is:

$$\sum_{\sigma \in B_n} q^{\text{flag-major}_N(\sigma)} t^{\ell(\sigma)} = \prod_{i=1}^n (1 + qt^i) A_n(q^2, t) \quad \bullet$$

## 5.4 The Signed Mahonian Over $D_n$

In this subsection we are going to calculate *signed Mahonian* on  $D_n$  the subgroup of  $B_n$  (see definition 2.31). We open this subsection with few definitions that we are going to use in our proofs:

**Definition 5.13** *Define the following subset of  $D_n$ :*

$$UD_n := U_n \cap D_n = \{\tau \in D_n \mid \tau(1) < \tau(2) < \dots < \tau(n)\}.$$

**Remark 5.14** *Let be  $\tau \in UD_n$ , if  $\tau(n) = n$ , then exists unique  $\tau' \in UD_{n-1}$ , where  $\varphi_{n1}(\tau') = \tau$ , and if  $\tau(1) = -n$  then exists unique  $\tau' \in \overline{UD}_{n-1} := U_{n-1} - UD_{n-1}$ , where  $\varphi_{n2}(\tau') = \tau$ .*

Notice that for every  $\sigma \in D_n$  there exists a unique decomposition:  $\sigma = \tau\pi$ ,  $\pi \in S_n$ ,  $\tau \in UD_n$ . Now we use the last remark, and the constructions in the beginning of this subsection to prove the following theorem:

**Theorem 5.15**

$$\sum_{\sigma \in D_n} \text{sign}(\sigma) q^{\text{flag-major}_N(\sigma)} = (1 - q^2)^{\lfloor \frac{n}{2} \rfloor} [n]_{\pm q^2}!$$

**Proof.**

$$\begin{aligned} \sum_{\sigma \in D_n} \text{sign}(\sigma) q^{\text{flag-major}_N(\sigma)} &= \sum_{\pi \in S_n, \tau \in UD_n} \text{sign}(\tau) \text{sign}(\pi) q^{2\text{maj}(\pi) + \text{neg}(\tau)} \\ &= \sum_{\pi \in S_n} \text{sign}(\pi) q^{2\text{maj}(\pi)} \sum_{\tau \in UD_n} \text{sign}(\tau) q^{\text{neg}(\tau)} \\ &= [n]_{\pm q^2}! \sum_{\tau \in UD_n} \text{sign}(\tau) q^{\text{neg}(\tau)}. \end{aligned}$$

Now we need to calculate  $\sum_{\tau \in UD_n} \text{sign}(\tau) q^{\text{neg}(\tau)}$ , we will do so by recursion to  $UD_{n-1}$ , before doing so, we notice the fact that:

$$\sum_{\tau \in U_n} \text{sign}(\tau) q^{\text{neg}(\tau)} = \sum_{\tau \in UD_n} \text{sign}(\tau) q^{\text{neg}(\tau)} + \sum_{\tau \in \overline{UD}_n} \text{sign}(\tau) q^{\text{neg}(\tau)},$$



therefore;

$$\begin{aligned} \sum_{\tau \in \overline{UD_n}} \text{sign}(\tau)q^{\text{neg}(\tau)} &= \sum_{\tau \in U_n} \text{sign}(\tau)q^{\text{neg}(\tau)} - \sum_{\tau \in UD_n} \text{sign}(\tau)q^{\text{neg}(\tau)} \\ &= \begin{cases} (1 - q^2)^m - \sum_{\tau \in \overline{UD_{2m}}} \text{sign}(\tau)q^{\text{neg}(\tau)}, & n=2m \\ (1 - q)(1 - q^2)^m - \sum_{\tau \in \overline{UD_{2m+1}}} \text{sign}(\tau)q^{\text{neg}(\tau)}, & n=2m+1. \end{cases} \end{aligned}$$

Now we are ready to do the calculation:

$$\begin{aligned} a_n &= \sum_{\tau \in UD_n} \text{sign}(\tau)q^{\text{neg}(\tau)} \\ &= \sum_{\tau \in UD_n, \tau(n)=n} \text{sign}(\tau)q^{\text{neg}(\tau)} + \sum_{\tau \in UD_n, \tau(1)=-n} \text{sign}(\tau)q^{\text{neg}(\tau)} \\ &= \sum_{\tau' \in \overline{UD_{n-1}}} \text{sign}(\varphi_{n2}(\tau'))q^{\text{neg}(\varphi_{n2}(\tau'))} + \sum_{\tau' \in \overline{UD_{n-1}}} \text{sign}(\varphi_{n1}(\tau'))q^{\text{neg}(\varphi_{n1}(\tau'))} \\ &= \sum_{\tau' \in \overline{UD_{n-1}}} \text{sign}(\tau')q^{\text{neg}(\tau')} + \sum_{\tau' \in \overline{UD_{n-1}}} (-1)^n \text{sign}(\tau')q^{\text{neg}(\tau')+1} \\ &= a_{n-1} + (-1)^n q[(1 - q)^{\lceil \frac{n-1}{2} \rceil} (1 + q)^{\lfloor \frac{n-1}{2} \rfloor} - a_{n-1}]. \end{aligned}$$

We got the recurrence equations:

$$a_n = \begin{cases} (1 - q)a_{2m-1} + q(1 - q^2)^{m-1}(1 - q), & n=2m; \\ (1 + q)a_{2m} - q(1 - q^2)^m, & n=2m+1. \end{cases}, a_1 = 1.$$

The solutions to these equations are:

$$\begin{aligned} a_{2m+1} &= (1 + q)a_{2m} - q(1 - q^2)^m \\ &= (1 + q)[(1 - q)a_{2m-1} + q(1 - q^2)^{m-1}(1 - q)] - q(1 - q^2)^m \\ &= (1 - q^2)a_{2m-1} + q(1 - q^2)^m - q(1 - q^2)^m = (1 - q^2)a_{2m-1} \\ &= (1 - q^2)^m. \end{aligned}$$

$$\begin{aligned} a_{2m} &= (1 - q)(1 - q^2)^{m-1} + q(1 - q)(1 - q^2)^{m-1} \\ &= (1 - q)(1 - q^2)^{m-1}(1 + q) \\ &= (1 - q^2)^m. \end{aligned}$$

In the end of the equation we got the solution:

$$a_n = (1 - q^2)^{\lfloor \frac{n}{2} \rfloor} = \begin{cases} (1 - q^2)^m, & n=2m; \\ (1 - q^2)^m, & n=2m+1. \end{cases}$$

Now we can write the final solution:

$$\sum_{\sigma \in D_n} \text{sign}(\sigma) q^{\text{flag-major}_N(\sigma)} = (1 - q^2)^{\lfloor \frac{n}{2} \rfloor} [n]_{\pm q^2}! \quad \bullet$$

**Corollary 5.16** *From the last Theorem 5.15, and from Theorem 5.7 we can conclude that for even  $n$ :*

$$\sum_{\sigma \in D_n} \text{sign}(\sigma) q^{\text{flag-major}_N(\sigma)} = \sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{flag-major}_N(\sigma)}.$$

## 6 Colored Permutation Groups

In this section we present a method, which takes known equalities from  $S_n$  and generalize them to  $G_{r,n}$ . The main idea of this method is to rely on the fact that our alphabet:

$$\{1^{[r-1]}, \dots, 1, 2^{[r-1]}, \dots, 2, \dots, n^{[r-1]}, \dots, n\}$$

is organized according to the *friends order*:

$$F : 1^{[r-1]} < \dots < 1 < 2^{[r-1]} < \dots < 2 < \dots < n^{[r-1]} < \dots < n.$$

In the *friends order* we can notice, that the colors have no effect on many statistics.

### 6.1 Preliminaries

We begin with the definition of the  $Color_r^n$  groups:

**Definition 6.1** *Let  $r, n$  be nonnegative integers. Define the  $Color_r^n$  group to be:*

$$Color_r^n := \{(z, id) \in G_{r,n}\} \cong \mathbf{Z}_r^n.$$

We define the following subsets of  $Color_r^n$ :

**Definition 6.2** *Let  $i \in [1, r]$ . Define:*

$$Color_{r,i}^n := \{(z, id) \in Color_r^n \mid z_n = i\}.$$

**Remark 6.3** *The group  $Color_r^n$  is a disjoint union:*

$$Color_r^n = \bigsqcup_{i=0}^{r-1} Color_{r,i}^n,$$

where  $\bigsqcup$  donates for disjoint union.

Every  $(z, \pi) \in G_{r,n}$  has two a unique compositions as products  $(z, \pi) = (z, id)(0, \pi) = (0, \pi')(z', \pi'')$ , where here  $0 := (0, \dots, 0)$ .

**Lemma 6.4** *Let  $i \in [1, k]$  and let  $ostat_i : S_n \rightarrow \mathbf{Z}$ ,  $cstat_i : \mathbf{Z}_r^n \rightarrow \mathbf{Z}$  statistics on  $S_n$  and  $\mathbf{Z}_r^n$  respectively; define  $stat_i : G_{r,n} \rightarrow \mathbf{Z}$  by  $stat_i(z, \pi) := ostate_i(\pi) + cstat_i(z)$ . Then:*

$$\sum_{(z, \pi) \in G_{r,n}} q_1^{stat_1(z, \pi)} \dots q_k^{stat_k(z, \pi)} = \sum_{(z, id) \in Color_r^n} q_1^{cstat_1(z)} \dots q_k^{cstat_k(z)} \sum_{\pi \in S_n} q_1^{ostat_1(\pi)} \dots q_k^{ostat_k(\pi)}.$$

**Proof.** By using the fact that every  $(z, \pi) = (z, id)(0, \pi)$ ,

$$\begin{aligned} \sum_{(z, \pi) \in G_{r,n}} q_1^{stat_1(z, \pi)} \dots q_k^{stat_k(z, \pi)} &= \sum_{(z, \pi) \in G_{r,n}} q_1^{ostat_1(z, \pi) + cstat_1(z, \pi)} \dots q_k^{ostat_k(z, \pi) + cstat_k(z, \pi)} \\ &= \sum_{(z, id) \in Color_r^n} \sum_{(0, \pi) \in G_{r,n}} q_1^{ostat_1(\pi) + cstat_1(z)} \dots q_k^{ostat_k(\pi) + cstat_k(z)} \\ &= \sum_{(z, id) \in Color_r^n} q_1^{cstat_1(z)} \dots q_k^{cstat_k(z)} \sum_{(0, \pi) \in G_{r,n}} q_1^{ostat_1(\pi)} \dots q_k^{ostat_k(\pi)} \\ &= \sum_{(z, id) \in Color_r^n} q_1^{cstat_1(z)} \dots q_k^{cstat_k(z)} \sum_{\pi \in S_n} q_1^{ostat_1(\pi)} \dots q_k^{ostat_k(\pi)}. \bullet \end{aligned}$$

We now define a set of bijections  $Color_r^{n-1} \rightarrow Color_{r,j}^n$ ,  $j \in [0, r-1]$ :

**Definition 6.5** *Let  $j \in [0, r-1]$ . Define the bijections  $\xi^j : Color_r^{n-1} \rightarrow Color_{r,j}^n$  by:*

$$\xi^j((z, id)) := (z', id), \text{ where } \begin{cases} z'_i = z_i, & 1 \leq i \leq n-1; \\ z'_i = j, & i = n. \end{cases} \quad (\forall (z, id) \in Color_r^{n-1}).$$

The relations between  $(z, id)$ , and  $\xi^j((z', id))$  are presented in the following equation that can be easily concluded from the definition of  $\xi^j$ :

$$1. \quad ncsun(\xi^j((z, id))) = \begin{cases} ncsun((z, id)) + n - 1, & 1 \leq j \leq r - 1; \\ ncsun((z, id)), & j=0. \end{cases}$$

$$\text{Recall that } ncsun((z, \pi)) := \sum_{z_i \neq 0} (\pi(i) - 1).$$

$$2. \quad csum(\xi^j((z, id))) = csum((z, id)) + j.$$

## 6.2 Friends Color-Signed Mahonian

In this subsection we calculate the bivariate generating function of the *flag major index* and *friends color sign* statistics. We open this subsection with the definition of the *color sign* statistic:

**Definition 6.6** Let  $r, n$  be nonnegative integers. Define the color sign of  $g \in G_{r,n}$  to be:

$$csign(g) := (-1)^{\ell(g)}.$$

**Note 6.7** Notice that in the general case of  $r$ -colors the color-sign statistic is not necessarily a character. For example in the case of  $n = 3$ ,  $r = 3$ , we take the following permutations:

$$g_1 = ((1, 0, 0), [1, 2, 3]), \quad g_2 = ((2, 0, 0), [1, 2, 3]) \text{ therefore; } g_1 g_2 = ((0, 0, 0), [1, 2, 3]), \text{ and} \\ csign(g_1) = -1, \quad csign(g_2) = 1, \text{ but } csign(g_1)csign(g_2) = -1 \neq csign(g_1 g_2) = 1.$$

**Definition 6.8** Let  $g \in G_{r,n}$ . Define the friends color-sign of  $g \in G_{r,n}$  to be:

$$csign_F(g) := (-1)^{inv_F(g) + csum(g)}.$$

**Proposition 6.9** Let  $g = (z, \pi) \in G_{r,n}$  then:

$$csign_F(g) = csign_F((0, \pi))csign_F((z, id)).$$

**Proof.** According to Definition 6.8 and Lemma 6.4 we conclude:

$$\begin{aligned} csign_F((z, \pi)) &= (-1)^{inv_F((z, \pi)) + csum((z, \pi))} \\ &= (-1)^{inv_F((0, \pi)) + csum((z, id))} \\ &= (-1)^{inv_F((0, \pi)) + csum((0, \pi))} (-1)^{inv_F((z, id)) + csum((z, id))} \\ &= csign_F((0, \pi))csign_F((z, id)) \quad \bullet \end{aligned}$$

Now by using the above facts, we can calculate the generating function of *friends color sign* and *flag major index*:

**Theorem 6.10**

$$\sum_{g \in G_{r,n}} csign_F(g) q^{flag-major_F(g)} = [r]_{-q}^n [n]_{\pm r q}!$$

**Proof.** According to Lemma 6.4 and Theorem 2.11 we get:

$$\begin{aligned} \sum_{g \in G_{r,n}} csign_F(g) q^{flag-major_F(g)} &= \sum_{(z, id) \in Color_r^n} csign_F((z, id)) q^{csum((z, id))} \sum_{\pi \in S_n} sign(\pi) q^{r \cdot maj(\pi)} \\ &= \sum_{(z, id) \in Color_r^n} csign_F((z, id)) q^{csum((z, id))} [n]_{\pm r q}!. \end{aligned}$$

Now we need to calculate:  $a_n = \sum_{(z, id) \in Color_r^n} csign_F((z, id))q^{csum((z, id))}$ . We will do so by using the  $\xi^j$  bijections to find recurrence equation to  $a_n$ :

$$\begin{aligned}
a_n &= \sum_{(z, id) \in Color_r^n} csign_F((z, id))q^{csum((z, id))} \\
&= \sum_{i=0}^{r-1} \sum_{(z, id) \in Color_{r,i}^n} csign_F((z, id))q^{csum((z, id))} \\
&= \sum_{i=0}^{r-1} \sum_{(z, id) \in Color_r^{n-1}} csign_F(\xi^i((z, id)))q^{csum(\xi^i((z, id)))} \\
&= \sum_{i=0}^{r-1} \sum_{(z, id) \in Color_r^{n-1}} (-1)^i csign_F((z, id))q^{csum((z, id))+i} \\
&= \sum_{i=0}^{r-1} (-q)^i a_{n-1} \\
&= [r]_{-q} a_{n-1}.
\end{aligned}$$

In the end we got the recurrence equation:

$$a_n = [r]_{-q} a_{n-1}, \quad a_1 = [r]_{-q},$$

which its solution is:

$$a_n = [r]_{-q}^n.$$

Our final solution is:

$$\sum_{g \in G_{r,n}} csign_F(g)q^{flag-major_F(g)} = [r]_{-q}^n [n]_{\pm rq}! \quad \bullet$$

**Note 6.11** Notice that if we put  $r = 1$  in Theorem 6.10, we get Theorem 2.11.

### 6.3 Colored Excedance and Denert's Statistics

In this subsection we give a generalized version of the *excedance number* and *Denert's statistic* for the *colored permutation groups*, by using the *r-color excedance number* (denoted  $exc_r$ ) and the *r-color Denert's statistic* (denoted  $den_r$ ), similar type results appear also in [13].

**Definition 6.12** Let  $r, n$  be a nonnegative integers and  $g = (z, \pi) \in G_{r,n}$ . Define the *r-color excedance number* of  $g$  to be:

$$exc_r(g) := exc(\pi) + csum(g).$$

We prove that the  $r$ -color *excedance number* is *Eulerian*:

**Theorem 6.13** *Let  $r, n$  be nonnegative integers. Then:*

$$\sum_{g \in G_{r,n}} q^{\text{exc}_r(g)} = \sum_{g \in G_{r,n}} q^{\text{l}_{\text{des}_F}(g)}.$$

**Proof.** According to Lemma 6.4 and Theorem 2.10 we get:

$$\begin{aligned} \sum_{g \in G_{r,n}} q^{\text{exc}_r(g)} &= \sum_{g \in G_{r,n}} q^{\text{exc}(\pi) + \text{csum}(g)} \\ &= \sum_{(z, id) \in \text{Color}_r^n} q^{\text{csum}((z, id))} \sum_{\pi \in S_n} q^{\text{exc}(\pi)} \\ &= \sum_{(z, id) \in \text{Color}_r^n} q^{\text{csum}((z, id))} \sum_{\pi \in S_n} q^{\text{des}(\pi)} \\ &= \sum_{(z, id) \in \text{Color}_r^n} \sum_{\pi \in S_n} q^{\text{des}(\pi) + \text{csum}((z, id))} \\ &= \sum_{g \in G_{r,n}} q^{\text{l}_{\text{des}_F}(g)} \quad \bullet \end{aligned}$$

**Definition 6.14** *Let  $r, n$  be nonnegative integers. Define for  $g \in G_{r,n}$  the  $r$ -color Denert's statistic to be:*

$$\text{den}_r(g) = r \cdot \text{den}(\pi) + \text{csum}(g).$$

We prove that the  $r$ -color *Denert's statistic* is equidistributed with the *flag major index* over  $G_{r,n}$ :

**Theorem 6.15** *Let  $r, n$  be a nonnegative integers. Then:*

$$\sum_{g \in G_{r,n}} q^{\text{den}_r(g)} = \sum_{g \in G_{r,n}} q^{\text{flag-major}_F(g)}.$$

**Proof.** Using Lemma 6.4, and Theorem 2.15 we get:

$$\begin{aligned} \sum_{g \in G_{r,n}} q^{\text{den}_r(g)} &= \sum_{g \in G_{r,n}} q^{r \cdot \text{den}(\pi) + \text{csum}(g)} \\ &= \sum_{(z, id) \in \text{Color}_r^n} q^{\text{csum}((z, id))} \sum_{\pi \in S_n} q^{r \cdot \text{den}(\pi)} \\ &= \sum_{(z, id) \in \text{Color}_r^n} q^{\text{csum}((z, id))} \sum_{\pi \in S_n} q^{r \cdot \text{maj}(\pi)} \\ &= \sum_{(z, id) \in \text{Color}_r^n} \sum_{\pi \in S_n} q^{r \cdot \text{maj}(\pi) + \text{csum}((z, id))} \\ &= \sum_{g \in G_{r,n}} q^{\text{flag-major}_F(g)} \quad \bullet \end{aligned}$$

Now we prove that the pair of statistics  $(den_r, exc_r)$  is equidistributed with  $(flag - major_F, ldes_F)$  and therefore, the pair of statistics  $(den_r, exc_r)$  generalizes the Foata-Zeilberger Theorem 2.15.

**Theorem 6.16** *Let  $r, n$  be a nonnegative integers. Then:*

$$\sum_{g \in G_{r,n}} q^{den_r(g)} t^{exc_r(g)} = \sum_{g \in G_{r,n}} q^{flag - major_F(g)} t^{ldes_F(g)}.$$

**Proof.** Using Lemma 6.4 and Theorem 2.15 we can conclude:

$$\begin{aligned} \sum_{g \in G_{r,n}} q^{den_r(g)} t^{exc_r(g)} &= \sum_{g \in G_{r,n}} q^{r \cdot den(\pi) + csum(g)} t^{exc(g) + csum(g)} \\ &= \sum_{(z, id) \in Color_r^n} q^{csum((z, id))} t^{csum((z, id))} \sum_{\pi \in S_n} q^{r \cdot den(\pi)} t^{exc(\pi)} \\ &= \sum_{(z, id) \in Color_r^n} q^{csum((z, id))} t^{csum((z, id))} \sum_{\pi \in S_n} q^{r \cdot maj(\pi)} t^{des(\pi)} \\ &= \sum_{g \in G_{r,n}} q^{r \cdot maj(\pi) + csum(g)} t^{des(g) + csum(g)} \\ &= \sum_{g \in G_{r,n}} q^{flag - major_F(g)} t^{ldes_F(g)} \quad \bullet \end{aligned}$$

## 6.4 Flag-Excedance and Flag-Denert's Statistic of Type B

In this subsection we present the *flag-Denert's* statistic (denoted  $fden$ ) and the *flag-excedance* (denoted  $fexc$ ) statistic. We prove that the pair of statistics  $(fden, fexc)$  are equidistributed with  $(flag - major, fdes)$  over  $B_n$  and, therefore, the *flag-Denert* and *flag-excedance* statistics gives a type B generalization to the Foata-Zeilberger Theorem 2.15.

**Definition 6.17** *Define the type b excedance number of  $\sigma \in B_n$  to be:*

$$exc_B(\sigma) := |\{ 1 \leq i \leq n \mid i < |\sigma(i)| \}|.$$

**Definition 6.18** *Define the flag-excedance of  $\sigma \in B_n$  to be:*

$$fexc(\sigma) := 2exc_B(\sigma) + \varepsilon(\sigma).$$

**Definition 6.19** *Let  $n$  be a nonnegative integer. Define the following subset of  $B_n$ :*

$$Color_2^n := \{\sigma \in B_n \mid \sigma(i) = \pm i, \forall i \in [1, n]\}.$$

*This is another form of definition 6.1 for  $r = 2$ .*

We prove that the *flag-excedance* statistics is Eulerian.

**Remark 6.20** *In the following theorems we use the fact that if  $\sigma = \pi\tau$  and where  $\pi = [|\sigma(1)|, \dots, |\sigma(n)|]$  and  $\tau \in \text{Color}_2^n$ , then  $\varepsilon(\sigma) = \varepsilon(\tau)$ .*

**Theorem 6.21**

$$\sum_{\sigma \in B_n} q^{\text{fexc}(\sigma)} = \sum_{\sigma \in B_n} q^{\text{fdes}_F(\sigma)}.$$

**Proof.** By using Lemma 6.4 and Remark 6.20 we can conclude that:

$$\begin{aligned} \sum_{\sigma \in B_n} q^{\text{fexc}(\sigma)} &= \sum_{\tau \in \text{Color}_2^n} \sum_{\pi \in S_n} q^{2\text{exc}_B(\pi\tau) + \varepsilon(\pi\tau)} \\ &= \sum_{\tau \in \text{Color}_2^n} q^{\varepsilon(\tau)} \sum_{\pi \in S_n} q^{2\text{exc}(\pi)} \\ &= \sum_{\tau \in \text{Color}_2^n} q^{\varepsilon(\tau)} \sum_{\pi \in S_n} q^{2\text{des}(\pi)} \\ &= \sum_{\sigma \in B_n} q^{\text{fdes}_F(\sigma)} \quad \bullet \end{aligned}$$

We define the type B Denert's statistic (denoted  $\text{den}_B$ ):

**Definition 6.22** *Let  $\sigma \in B_n$ . Define the type B Denert's statistic to be:*

$$\begin{aligned} \text{den}_B(\sigma) &= |\{1 \leq l < k \leq n \mid |\sigma(k)| < |\sigma(l)| < k\}| \\ &+ |\{1 \leq l < k \leq n \mid |\sigma(l)| < k < |\sigma(k)|\}| \\ &+ |\{1 \leq l < k \leq n \mid k < |\sigma(k)| < |\sigma(l)|\}|. \end{aligned}$$

**Remark 6.23** *According to the definition of  $\text{den}_B$  we can see that:*

$$\text{den}_B(\sigma) = \text{den}_B(\tau\pi) = \text{den}_B(\pi), \quad \forall \sigma \in B_n, \tau \in \text{Color}_2^n, \pi \in S_n.$$

We define the *flag-Denert's* statistic (denoted  $\text{fden}_B$ ), and prove that it is equidistributed with the *flag major index* over the *signed permutations groups*:

**Definition 6.24** *Let  $\sigma \in B_n$ . Define the flag-Denert's statistic to be:*

$$\text{fden}(\sigma) := 2\text{den}_B(\sigma) + \text{neg}(\sigma).$$

**Theorem 6.25**

$$\sum_{\sigma \in B_n} q^{\text{fden}(\sigma)} = \sum_{\sigma \in B_n} q^{\text{flag-major}_F(\sigma)}.$$



**Proof.** We use the Definition 6.24, Lemma 6.4, and Remark 6.23 and conclude the following equality:

$$\begin{aligned}
\sum_{\sigma \in B_n} q^{fden(\sigma)} &= \sum_{\sigma \in B_n} q^{2den_B(\sigma) + neg(\sigma)} \\
&= \sum_{\tau \in Color_2^n} q^{neg(\tau)} \sum_{\pi \in S_n} q^{2den(\pi)} \\
&= \sum_{\tau \in Color_2^n} q^{neg(\tau)} \sum_{\pi \in S_n} q^{2maj(\pi)} \\
&= \sum_{\sigma \in B_n} q^{flag-major_F(\sigma)} \quad \bullet
\end{aligned}$$

We prove that the pair of statistics  $(fden, fexc)$  is equidistributed with  $(flag-major, fdes)$ .

### Theorem 6.26

$$\sum_{\sigma \in B_n} q^{fden(\sigma)} t^{fexc(\sigma)} = \sum_{\sigma \in B_n} q^{flag-major_F(\sigma)} t^{fdes_F(\sigma)}.$$

**Proof.** We use the Definitions 6.18, 6.24, Lemma 6.4, and Theorem 2.15 and conclude the following equality:

$$\begin{aligned}
\sum_{\sigma \in B_n} q^{fden(\sigma)} t^{fexc(\sigma)} &= \sum_{\sigma \in B_n} q^{2den_B(\sigma) + neg(\sigma)} t^{2exc_B(\sigma) + \varepsilon(\sigma)} \\
&= \sum_{\tau \in Color_2^n} q^{neg(\tau)} t^{\varepsilon(\tau)} \sum_{\pi \in S_n} q^{2den(\pi)} t^{2exc(\pi)} \\
&= \sum_{\tau \in Color_2^n} q^{neg(\tau)} t^{\varepsilon(\tau)} \sum_{\pi \in S_n} q^{2maj(\pi)} t^{2des(\pi)} \\
&= \sum_{\sigma \in B_n} q^{flag-major_F(\sigma)} t^{fdes_F(\sigma)} \quad \bullet
\end{aligned}$$

## 7 The Flag-Inversion Statistic

In this section we define a new statistic, *flag-inversion* (denoted  $finv$ ). We show that this new statistic is also equidistributed with the *flag major index*, and therefore it is equidistributed with the *length* statistic for  $r = 1, 2$ . Using  $finv$  we calculate interesting generating functions and present a generalization of Foata and Schützenberger’s Theorem 2.12 to the groups  $G_{r,n}$ . We finish this section by giving an algebraic interpretation of  $finv$ .

**Remark 7.1** *Throughout this section we assume that our alphabet is:*

$$\{1, \dots, 1^{[r-1]}, 2, \dots, 2^{[r-1]}, \dots, \dots, n, \dots, n^{[r-1]}\},$$

and it is ordered by the friends order:

$$F : 1 < \dots < 1^{[r-1]} < 2 < \dots < 2^{[r-1]} < \dots < n < \dots < n^{[r-1]}.$$

For convenience we define  $i^{[0]} := i$ ,  $\forall i \in [1, n]$ .

## 7.1 A New Mahonian Statistic

**Definition 7.2** Let  $g \in G_{r,n}$ . Define the flag-inversion statistic of  $g$  to be

$$finv(g) := r \cdot inv(g) + csum(g).$$

**Remark 7.3** For  $\sigma \in B_n$  this can be written as:

$$finv(\sigma) := 2 \cdot inv(\sigma) + neg(\sigma).$$

We prove that the *flag inversion* statistic is equidistributed with the *flag-major index* over the *colored permutation group*  $G_{r,n}$ :

**Theorem 7.4**

$$\sum_{g \in G_{r,n}} q^{flag-major_F(g)} = \sum_{g \in G_{r,n}} q^{finv_F(g)}.$$

**Proof.** By using the proof of Lemma 6.4, we can conclude:

$$\begin{aligned} \sum_{g \in G_{r,n}} q^{finv_F(g)} &= \sum_{(z, id) \in Color_r^n} q^{csum((z, \pi))} \sum_{\pi \in S_n} q^{r \cdot inv(\pi)} \\ &= \sum_{(z, id) \in Color_r^n} q^{csum((z, id))} \sum_{\pi \in S_n} q^{r \cdot maj(\pi)} \\ &= \sum_{g \in G_{r,n}} q^{flag-major_F(\sigma)} \quad \bullet \end{aligned}$$

**Remark 7.5** There exists another direct proof of Theorem 7.4. One can show by induction on  $n$  that  $a_n := \sum_{g \in G_{r,n}} q^{finv_F(g)}$  satisfies:

$$a_n = a_{n-1} [r]_q [n]_{q^r} = a_n [rn]_q,$$

and therefore

$$\sum_{g \in G_{r,n}} q^{finv_F(g)} = \prod_{i=1}^n [ri]_q.$$

According to Theorem 2.40 we can conclude:

$$\sum_{g \in G_{r,n}} q^{flag-major(g)} = \sum_{g \in G_{r,n}} q^{finv_F(g)}.$$

**Corollary 7.6** *The flag-inversion statistic is equidistributed with the length statistic over the signed permutation groups  $B_n$ .*

**Proof.** Combine to the Theorem 7.4 (in the case of  $r = 2$ ), with Proposition 4.12 and Theorem 2.24.

By using the *flag-inversion* statistic we can generalize the Foata-Schützenberger Theorem 2.12 to the *colored permutation groups*:

**Theorem 7.7**

$$\sum_{g \in G_{r,n}} q^{\text{flag-major}_F(g)} t^{\text{ldes}_F(\bar{g})} = \sum_{g \in G_{r,n}} q^{\text{inv}_F(g)} t^{\text{ldes}_F(\bar{g})},$$

where  $\bar{g} := (z, \pi^{-1})$ .

**Proof.** By using Lemma 6.4 and Theorem 2.13 we get:

$$\begin{aligned} \sum_{g \in G_{r,n}} q^{\text{flag-major}_F(g)} t^{\text{ldes}_F(\bar{g})} &= \sum_{(z,\pi) \in G_{r,n}} q^{\text{flag-major}_F((z,\pi))} t^{\text{ldes}_F((z,\pi^{-1}))} \\ &= \sum_{(z,id) \in \text{Color}_r^n} q^{\text{csum}((z,id))} t^{\text{csum}((z,id))} \sum_{\pi \in S_n} q^{r \cdot \text{maj}(\pi)} t^{\text{des}(\pi^{-1})} \\ &= \sum_{(z,id) \in \text{Color}_r^n} q^{\text{csum}((z,id))} t^{\text{csum}((z,id))} \sum_{\pi \in S_n} q^{r \cdot \text{inv}(\pi)} t^{\text{des}(\pi^{-1})} \\ &= \sum_{(z,\pi) \in G_{r,n}} q^{\text{inv}_F(g)} t^{\text{ldes}_F(\bar{g})} \quad \bullet \end{aligned}$$

In the case of  $r = 2$  by using *flag-inversion* statistic, we can get the following equality, which is a type B generalization of the Foata-Schützenberger Theorem 2.13 :

**Theorem 7.8**

$$\sum_{\sigma \in B_n} q^{\text{flag-major}_F(\sigma^{-1})} t^{\text{flag-major}_F(\sigma)} = \sum_{\sigma \in B_n} q^{\text{inv}_F(\sigma)} t^{\text{flag-major}_F(\sigma)}.$$

**Proof.** Using Lemma 6.4, Theorem 2.13, and the fact that in the case of  $r = 2$ ,  $\tau = \tau^{-1} \in \text{Color}_2^n$  we get:

$$\begin{aligned} \sum_{\sigma \in B_n} q^{\text{flag-major}_F(\sigma^{-1})} t^{\text{flag-major}_F(\sigma)} &= \sum_{\tau \in \text{Color}_2^n} q^{\text{neg}(\tau)} t^{\text{neg}(\tau)} \sum_{\pi \in S_n} q^{2\text{maj}(\pi^{-1})} t^{2\text{maj}(\pi)} \\ &= \sum_{\tau \in \text{Color}_2^n} q^{\text{neg}(\tau)} t^{\text{neg}(\tau)} \sum_{\pi \in S_n} q^{2\text{inv}(\pi)} t^{2\text{maj}(\pi)} \\ &= \sum_{\sigma \in B_n} q^{\text{inv}_F(\sigma)} t^{\text{flag-major}_F(\sigma)} \quad \bullet \end{aligned}$$

## 7.2 Algebraic Interpretation

In this subsection we give an algebraic interpretation to the *flag-inversion* statistic, as length in the group  $S_{r,n}$  with respect to a new set of generators. We open this subsection with the definition of the new generators:

**Definition 7.9** *Let  $r \geq 1$ ,  $n \geq 1$  be integers. Define the set of two dimensional Coxeter generators of the Symmetric group  $S_{r,n}$  by :*

$$C_2 := \{d_{i,j,R} \mid 1 \leq i \leq n, 0 \leq j \leq r-2\} \cup \{d_{i,j,C} \mid 1 \leq i \leq n-1, 0 \leq j \leq r-1\},$$

where:

$$d_{i,j,R} := [1, \dots, 1^{[r-1]}, \dots, i^{[j-1]}, i^{[j+1]}, i^{[j]}, i^{[j+2]}, \dots, n, \dots, n^{[r-1]}],$$

and:

$$d_{i,j,C} := [\dots, i^{[j-1]}, (i+1)^{[j]}, i^{[j+1]}, \dots, (i+1)^{[j-1]}, i^{[j]}, (i+1)^{[j+1]}, \dots].$$

There is an easy geometrical way to understand these generators; represent a permutation  $\pi \in S_{r,n}$  by a matrix  $A = (a_{ij})_{i=1, j=0}^{i=n, j=r-1} \in M_{n \times r}$ , where  $\forall i \in [1, n], k \in [1, r]$ ,  $a_{i,k} = \pi(ir + k - 1)$ . For example, the identity permutation can be represented by the following matrix:

$$\begin{pmatrix} 1 & 1^{[1]} & \dots & 1^{[r-2]} & 1^{[r-1]} \\ 2 & 2^{[1]} & \dots & 2^{[r-2]} & 2^{[r-1]} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & (n-1)^{[1]} & \dots & (n-1)^{[r-2]} & (n-1)^{[r-1]} \\ n & n^{[1]} & \dots & n^{[r-2]} & n^{[r-1]} \end{pmatrix}.$$

The generator  $d_{i,j,R}$  swaps two consecutive numbers in the same row, while  $d_{i,j,C}$  swaps two consecutive numbers in the same column.

$$\begin{pmatrix} 1 \xleftrightarrow{d_{1,0,R}} & 1^{[1]} \xleftrightarrow{d_{1,1,R}} & \dots & d_{1,r-3,R} & 1^{[1]} \xleftrightarrow{d_{1,r-2,R}} & 1^{[r-1]} \\ \updownarrow d_{1,0,C} & \updownarrow d_{1,1,C} & & & \updownarrow d_{1,r-2,C} & \updownarrow d_{1,r-1,C} \\ 2 \xleftrightarrow{d_{2,0,R}} & 2^{[1]} \xleftrightarrow{d_{2,1,R}} & \dots & d_{2,r-3,R} & 2^{[r-2]} \xleftrightarrow{d_{2,r-2,R}} & 2^{[r-1]} \\ \updownarrow d_{2,0,C} & \updownarrow d_{2,1,C} & & & \updownarrow d_{2,r-2,C} & \updownarrow d_{2,r-1,C} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \updownarrow d_{n-1,0,C} & \updownarrow d_{n-1,1,C} & & & \updownarrow d_{n-1,r-2,C} & \updownarrow d_{n-1,r-1,C} \\ n \xleftrightarrow{d_{n,0,R}} & n^{[1]} \xleftrightarrow{d_{n,1,R}} & \dots & d_{n,r-3,R} & n^{[r-2]} \xleftrightarrow{d_{n,r-2,R}} & n^{[r-1]} \end{pmatrix}$$

figure 1. The Two dimensional Coxeter generators  $d_{i,j,k}$ ,  $k \in \{R, C\}$ .

**Remark 7.10** It is easy to check that for  $r = 1$  the two dimensional Coxeter generators  $d_{i,0,C}$  are equal to the usual Coxeter generators (Definition 2.4).

We now define a bijection  $\Gamma : G_{r,n} \rightarrow S_{r \cdot n}$ , and calculate the length of  $\Gamma(g)$  according to the *two dimensional Coxeter generators*:

**Definition 7.11** Define a bijection  $\Gamma(g) : G_{r,n} \rightarrow S_{r \cdot n}$  as follows: for  $g \in G_{r,n}$

$$\Gamma(g)(i^{[k]}) := \begin{cases} \pi(i)^{[z_i]}, & k = 0; \\ \pi(i)^{[k-1]}, & 1 \leq k \leq z_i; \\ \pi(i)^{[k]}, & z_i < k \leq r-1; \end{cases}, \quad \forall k \in [0, r-1], i \in [1, n].$$

Here we identify  $[1, \dots, r \cdot n]$  with  $[1, \dots, 1^{[r-1]}, \dots, n, \dots, n^{[r-1]}]$ .

**Example 7.12** Let  $n = 3$ ,  $r = 4$ ,  $(z, \pi) = ((0, 3, 2), [2, 1, 3])$ . Then  $\Gamma : G_{4,3} \rightarrow S_{12}$ :

$$\Gamma(g) = [2, 2^{[1]}, 2^{[2]}, 2^{[3]}, 1^{[3]}, 1, 1^{[1]}, 1^{[2]}, 3^{[2]}, 3, 3^{[1]}, 3^{[3]}] = \begin{pmatrix} 2 & 2^{[1]} & 2^{[2]} & 2^{[3]} \\ 1^{[3]} & 1 & 1^{[1]} & 1^{[2]} \\ 3^{[2]} & 3 & 3^{[1]} & 3^{[3]} \end{pmatrix}.$$

**Definition 7.13** Let  $g \in G_{r,n}$ . Define the length of  $g$  according to the two dimensional Coxeter generators to be:

$$\ell_D(g) = \min\{t \in \mathbf{N} \mid \Gamma(g) = d_{i_1, j_1, k_1} \dots d_{i_t, j_t, k_t}, \text{ for some } i_s, j_s, k_s\}.$$

Our main result here is the next theorem.

**Theorem 7.14** Let  $g \in G_{r,n}$ . Then

$$\ell_D(g) = r \cdot \text{inv}(g) + \text{csum}(g) = \text{finv}(g).$$

**Proof.** We give an algorithm which proves the theorem. Consider  $\Gamma(g)$  as a matrix  $A \in M_{n \times r}$ . We do the following steps to create  $A$  from the identity matrix using the generators  $d_{i,j,k}$ :

1. View every column of the matrix as a  $\pi \in S_n$  use  $d_{i,j,R}$  to generate it from the identity matrix.

$$\begin{pmatrix} 1 & \dots & 1^{[r-1]} \\ \vdots & \ddots & \vdots \\ n & \dots & n^{[r-1]} \end{pmatrix} \longrightarrow \begin{pmatrix} \pi(1) & \dots & \pi(1)^{[r-1]} \\ \vdots & \ddots & \vdots \\ \pi(n) & \dots & \pi(n)^{[r-1]} \end{pmatrix}$$

2. In every row  $j \in [1, n]$  move  $j^{z_j}$  from column  $z_j$  to column 1 using  $d_{i,j,C}$ .

$$\begin{pmatrix} \pi(1) & \dots & \pi(1)^{[r-1]} \\ \vdots & \ddots & \vdots \\ \pi(n) & \dots & \pi(n)^{[r-1]} \end{pmatrix} \longrightarrow \begin{pmatrix} \pi(1)^{[z_1]} & \dots & \pi(1)^{[r-1]} \\ \vdots & \ddots & \vdots \\ \pi(n)^{[z_n]} & \dots & \pi(n)^{[r-1]} \end{pmatrix}.$$

The first step of the algorithm uses  $r \cdot \text{inv}(\pi)$  generators (to organize every column we use  $\text{inv}(\pi)$  Coxeter generators of  $S_n$ ) and the second step uses  $\text{csum}(g)$  generators (in every column  $i$  we move  $\pi(i)^{[z_i]}$  from column  $z_i$  to the first column, this action uses  $z_i$  generators). We conclude that  $\text{finv}(g) \geq \ell_D(g)$ . Now we need to prove that  $\ell_D(g) \geq \text{finv}(g)$ . We look at  $\Gamma(g)$  as a matrix  $(a_{ij})_{i=1, j=0}^{i=n, j=r-1}$ , where  $a_{ij} = \pi(i)^{[x_{ij}]}$ ,  $x_{ij} \in [0, r-1]$ . It easy to see that:

$$\begin{aligned} \ell_D(g) &\geq r \cdot \ell(\pi) + \frac{1}{2} \sum_{i=1}^n \sum_{j=0}^{r-1} |x_{ij} - j| = r \cdot \ell(\pi) + \frac{1}{2} \sum_{i=1}^n (z_i + \sum_{j=0}^{z_i} 1) \\ &= r \cdot \ell(\pi) + \sum_{i=1}^n z_i = \text{finv}(g). \end{aligned}$$

therefore;

$$\ell_D(g) = \text{finv}(g) \bullet$$

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