Spectral clustering

$G=(V,E)$

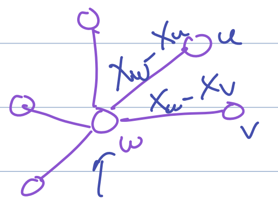
$V=\{1,2,\dots,n\}$

$(L_G\vec{x})_w = \sum_{v:\{w,v\}\in E}(x_w-x_v)$

Laplacian matrix: $\boxed{L_G = D - A}$ adjacency

$D_{ii}=\deg(i)$

$n\times n$ matrix



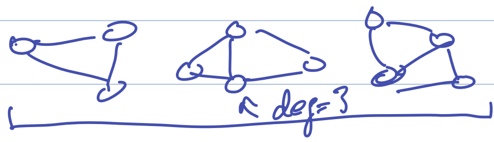
$\vec{x}=(1,1,\dots,1)$

$L_G\vec{x}=(0,0,\dots,0)$

$0=\lambda_1\le\lambda_2\le\dots\le\lambda_n$

Thm: # conn. comp. in G = multiplicity of 0 as an eigenvalue

$(\underbrace{0=\lambda_1=\lambda_2=\dots=\lambda_k}_{\text{multiplicity } k}<\lambda_{k+1})$



$0=\lambda_1=\lambda_2=\lambda_3,\ \lambda_4>0$

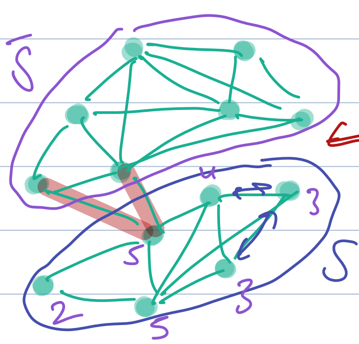
$\boxed{\text{Fact: } \lambda_2=0 \iff G \text{ is not a connected graph}}$

Robust: λ_2 small $\iff$ G has a "nice" cluster

NP-hard to optimize

Def: For $S\subseteq V$, $vol(S) := \sum_{u\in S}\deg(u)$

Conductance of S: $\boxed{\Phi_G(S) := \frac{|E(S,\bar{S})|}{\min(vol(S), vol(\bar{S}))}}$



$E(S,\bar{S})$

:= fraction of edges incident to S that leave S

$\boxed{\Phi_G(S) \text{ small} \iff S \text{ a "nice" cluster}}$

$|E(S,\bar{S})|=2$

$vol(S)=22$

$\Phi_G(S)=\frac{2}{22}=\frac{1}{11}$

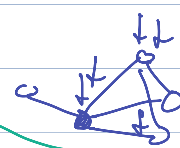
$$\rho_2(G) = \underbrace{\min_{\emptyset \neq S \subseteq V} \Phi_G(S)}_{S^*} = \min_{\substack{S_1, S_2 \subseteq V \\ S_1 \cap S_2 = \emptyset}} \max(\underbrace{\Phi_G(S_1)}_{\le \varepsilon}, \underbrace{\Phi_G(S_2)}_{\le \varepsilon})$$

Graph expansion,
Sparsest Cut problem (...)

$$\boxed{\begin{aligned} S_1 &= S^* \\ S_2 &= \overline{S^*} \end{aligned}} \qquad \Phi_G(S_1) = \Phi_G(S_2) = \Phi_G(S^*)$$

Assume G is d-regular (Every vertex has degree d)

$$\boxed{\mathcal{L}_G := \frac{1}{d} L_G} = \frac{1}{d}(D - A) = I - \frac{1}{d} A \qquad \text{norm. Laplacian}$$



$$(\mathcal{L}_G := I - D^{-1/2} A D^{-1/2})$$

$$\rho_2 \le \varepsilon \Rightarrow \lambda_2 \le 2\varepsilon \Rightarrow \sqrt{2\lambda_2} \le 2\sqrt{\varepsilon}$$

$$\rho_2(G) = \min_{\emptyset \neq S \subseteq V} \Phi_G(S)$$

Discrete Cheeger Ineq:

$$\frac{1}{2}\lambda_2(G) \le \overbrace{\rho_2(G)}^{\varepsilon} \le \boxed{\sqrt{2\lambda_2(G)}}$$

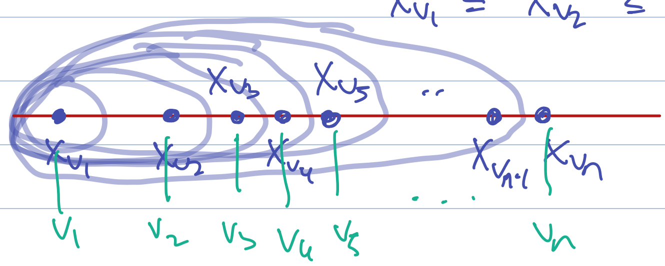
where $\lambda_2(G)$ is the 2nd smallest e.v of $\mathcal{L}_G$.

$$\lambda_2(G) \le \rho_2(G) \, \cancel{\le} \, O(\lambda_2(G))$$

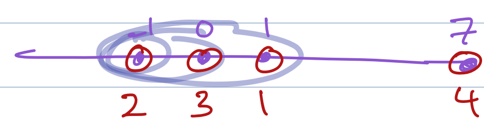
SWEEP($\vec{x}$):

Sweep Algorithm: Given a vector $\vec{x} \in \mathbb{R}^n$, sort the vertices $v_1, v_2, \dots, v_n$ s.t.

$$x_{v_1} \le x_{v_2} \le \cdots \le x_{v_n}$$



$$\vec{x} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ (1, & -1, & 0, & 7) \end{array}$$



$\{2\}, \{2,3\}, \{2,3,1\}$

Output the set of min. conductance

among $\{v_1\}, \{v_1, v_2\}, \ldots, \{v_1, v_2, \ldots, v_{n-1}\}$

Clustering Alg: Run SWEEP($\vec{x}_2$) where $\vec{x}_2$ is the e.v. of $\mathcal{L}_G$ corresponding to λ_2

Analysis: (1) For any $\vec{x} \in \mathbb{R}^n$, SWEEP($\vec{x}$) returns a cut $S \subseteq V$ s.t.

$$\Phi_G(S) \leq \sqrt{2 R_G(\vec{x})}$$

Cauchy-Schwarz

$$R_G(\vec{x}) := \frac{\frac{1}{d} \sum_{\{u,v\} \in E} (x_u - x_v)^2}{\left(\sum_{u \in V} (x_u - \bar{x})^2 \right.}, \quad \bar{x} = \frac{1}{n} \sum_{u \in V} x_u$$

(2): $\boxed{R_G(\vec{x}_2) = \lambda_2(G)}$ ← 2^{nd} smallest e.v. of $\mathcal{L}_G$

$$\vec{x}_2 = \operatorname{argmin} \{ R_G(\vec{x}) : \vec{x} \neq 0 \}$$

Recall: $\langle \vec{x}, L_G \vec{x} \rangle = \sum_{\{u,v\} \in E} (x_u - x_v)^2$

$$\langle \vec{x}, \mathcal{L}_G \vec{x} \rangle = \frac{1}{d} \sum_{\{u,v\} \in E} (x_u - x_v)^2$$

$$R_G(\vec{x}) = \frac{\langle \vec{x}, \mathcal{L}_G \vec{x} \rangle}{\| \vec{x} - \langle \vec{x}_1, \vec{x} \rangle \vec{x}_1 \|^2}$$

$\langle \vec{x}_1, \vec{x} \rangle = \bar{x}$

$\boxed{\vec{x}_1 = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right)}$

$$\langle \vec{x}_1, x \rangle = \bar{x}\sqrt{n}$$

$$\underline{\langle \vec{x}_1, x \rangle \vec{x}_1} = \left(\frac{\bar{x}\sqrt{n}}{\sqrt{n}}, \frac{\bar{x}\sqrt{n}}{\sqrt{n}}, \ldots \right)$$

$$= (\bar{x}, \bar{x}, \ldots, \bar{x})$$

$$R_G(\vec{x}) = \frac{\langle \vec{x}, L_G \vec{x} \rangle}{\|\vec{x} - \langle \vec{x}_1, \vec{x} \rangle \vec{x}_1\|^2}$$

$$\min_{\vec{x} \neq 0} R_G(\vec{x}) = \min_{\vec{x} \neq 0} \frac{\langle \vec{x}, L_G \vec{x} \rangle}{\underbrace{\|\vec{x} - \langle \vec{x}_1, \vec{x} \rangle \vec{x}_1\|^2}}$$

$$\stackrel{?}{=} \min_{\vec{x} \perp \vec{x}_1} \frac{\langle \vec{x}, L_G \vec{x} \rangle}{\|\vec{x}\|^2}$$

$$\boxed{\vec{x}_1 = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right)}$$

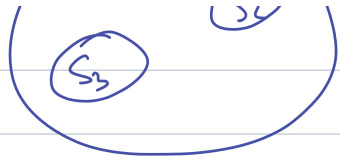
variational char of eigenvalues $\rightarrow$

$$= \boxed{\min_{\substack{\vec{x} \perp \vec{x}_1 \\ \|\vec{x}\| = 1}} \langle \vec{x}, L_G \vec{x} \rangle} = \lambda_2 \text{ of } L_G$$

$$\rho_k(G) = \min_{\substack{S_1, S_2, \ldots, S_k \subseteq V \\ \{S_i\} \text{ non-empty and} \\ \text{pairwise disjoint}}} \max\left(\Phi_G(S_1), \ldots, \Phi_G(S_k) \right)$$

G

Thm:



$$\frac{1}{2}\lambda_k(G) \le \rho_k(G) \le O(k^2)\sqrt{\lambda_k(G)}$$

$$\sqrt{\lambda_{1.1k}(G)\,\log(k)}$$

$F: V \to \mathbb{R}^k$

$F(v) = (\vec{x}_1(v), \dots, \vec{x}_k(v))$

k-means cluster on

$\{F(v) : v \in V\}$