CSE 5999: Lecture 6
Bell inequalities and the EPR paradox

Experimental validation \Rightarrow 2022 Nobel prize in Physics

Partial measurements: U unitary (U \otimes I) |\Psi> 

|\Psi>| = x_{00} |00> + x_{01} |01> + x_{10} |10> + x_{11} |11>

Alice measures her qubit in \{ |0>, |1> \} basis

P_0 = P[Alice measures |0>] = \frac{x_{00} |00> + x_{01} |01>}{\sqrt{|x_{00}|^2 + |x_{01}|^2}}

P_1 = P[Alice measures |1>] = \frac{x_{10} |10> + x_{11} |11>}{\sqrt{|x_{10}|^2 + |x_{11}|^2}}

Suppose we measure |2> in the \{ |00>, |01>, |10>, |11> \} basis.
Then P[measure "|00>"] = |x_{00}|^2 \rightarrow state collapses to |00>
P[measure "|01>"] = |x_{01}|^2 \rightarrow state collapses to |01>
P[measure "|10>" in the 1st bit] = P[ measure 00 ] + P[ measure 01 ]

Random Two coins A, B \in \{ 0, 1 \}

P[B = 0 | A = 0] = \frac{P[ A, B = 0, 0 ]}{P[A = 0]}

\sum P[B = 1 | A = 0] = P[A = 0]
\[ P[A=0] = \frac{\alpha_{10} |0\rangle + \alpha_{01} |1\rangle}{\sqrt{|\alpha_{10}|^2 + |\alpha_{01}|^2}} \]

\[ \frac{\alpha_{10} |0\rangle + \alpha_{01} |1\rangle}{\sqrt{|\alpha_{10}|^2 + |\alpha_{01}|^2}} \sim \frac{|\alpha_{10}| |0\rangle + |\alpha_{01}| |1\rangle}{\sqrt{|\alpha_{10}|^2 + |\alpha_{01}|^2}} \]

\[
\frac{|\alpha_{10}|^2 + |\alpha_{01}|^2}{|\alpha_{10}|^2 + |\alpha_{01}|^2} = \frac{|\alpha_{10}|^2 + |\alpha_{01}|^2}{(|\alpha_{10}|^2 + |\alpha_{01}|^2)^2}
\]

Born rule for measurements

\[ \langle \psi | = \frac{\alpha_{10} |0\rangle + \alpha_{01} |1\rangle + \alpha_{10} |0\rangle + \alpha_{11} |1\rangle}{\sum_i |\psi_i|} \]

\[ p_0 = P[\text{Alice measures } 0] = |\alpha_{10}|^2 + |\alpha_{01}|^2 \quad \rightarrow \quad \frac{\alpha_{10} |0\rangle + \alpha_{01} |1\rangle}{\sqrt{|\alpha_{10}|^2 + |\alpha_{01}|^2}} \]

\[ \{ \psi \} = \frac{\alpha_{10} |0\rangle + \alpha_{01} |1\rangle}{\sqrt{p_0}} \]

\[ p_1 = P[\text{Alice measures } 1] = |\alpha_{11}|^2 + |\alpha_{11}|^2 \]

\[ \langle \psi | \text{ collapses to } \frac{\alpha_{10} |0\rangle + \alpha_{11} |1\rangle}{\sqrt{p_1}} \]

\[ \left\{ \left| \psi \right\rangle = \frac{1}{\sqrt{3}} |00\rangle + \frac{1}{\sqrt{3}} |11\rangle + \frac{1}{\sqrt{3}} |01\rangle \right\} \]

Measure qubits 1 and 3

\[ A e C^{4 \times 4} \text{ unitary} \]

\[ I_2 \otimes A < 8 \times 8 \text{ matrix} \]

\[ P[\text{measure } 1^{st} \text{ qubit } 1 \text{ and } 3^{rd} \text{ qubit } 1] = \frac{2}{3} \]
\[ |u_k \rangle \rightarrow \frac{1}{\sqrt{3}} \left( |101\rangle + \frac{1}{\sqrt{2}} |111\rangle \right) = \frac{1}{\sqrt{2}} |101\rangle + \frac{1}{\sqrt{2}} |111\rangle \]

\[ e_1 \otimes e_2 \otimes e_3 = (e_1 \otimes (e_2 \otimes e_3)) = ((e_1 \otimes e_2) \otimes e_3) \]

\[ e_1^A \otimes e_1^B \otimes e_1^C \]

\[ B \]

\[ \text{mental} \]

\[ A \quad B \quad C \]

\[ A \quad B \quad C \]

\[ A \quad B \quad C \]

\[ A \quad B \quad C \]

\[ A \quad B \quad C \]

\[ A \quad B \quad C \]

\[ A \quad B \quad C \]

\[ A \quad B \quad C \]

\[ A \quad B \quad C \]

\[ A \quad B \quad C \]
(EPR paradox) CHSH game:

Alice

Referee

Bob

A(x) \in \{0, 1\}

B(y) \in \{0, 1\}

Referee checks if $A(x) \oplus B(y) = x \cdot y$.

If Alice and Bob are classical, then

$P[\text{win}] \leq 75\%$

For randomized strategies:

$P[\text{win}] = 75\%$

Next time: If Alice and Bob share a bell state:

$|\text{Bell}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + \frac{1}{\sqrt{2}} |11\rangle)$
Then $\exists$ strategy wins w/ $\text{prob} \geq \boxed{0.875} > \frac{3}{4}$

\[
|0\rangle \otimes (\alpha_0 |0\rangle + \alpha_1 |1\rangle) \otimes |1\rangle
\]

\[
= \alpha_0 (|0\rangle \otimes |0\rangle \otimes |1\rangle) + \alpha_1 (|0\rangle \otimes |1\rangle \otimes |1\rangle)
\]

\[
= \alpha_0 |001\rangle + \alpha_1 |011\rangle
\]

\[
\vec{p} = (V_1, V_2, \ldots, V_n)
\]

\[
\vec{v} = (V_2, V_1, V_3, \ldots, V_n)
\]