Optimal spectral sparsification

$A_1, A_2, \dots, A_m \succeq 0$ PSD

$A = A_1 + \cdots + A_m$ $\quad m \gg n$

$a_1 a_1^T + \cdots + a_m a_m^T$

$$\left\{ \begin{array}{l} A \preceq B \\ \iff B - A \text{ is PSD} \\ \iff \langle x, Ax \rangle \le \langle x, Bx \rangle \quad \forall x \in \mathbb{R}^n \end{array} \right.$$

$\rightarrow \quad (1-\varepsilon) A \preceq \sum_{i=1}^{m} c_i A_i \preceq (1+\varepsilon) A$

$c_1, c_2, \dots, c_m \ge 0$

$\#\{i : c_i \ne 0\} \le s$

$f_i(x) := \|A_i^{1/2} x\|_2^2$

$F(x) = f_1(x) + \cdots + f_m(x) = \|Ax\|_2^2$

$$\boxed{(1-\varepsilon) F(x) \le \tilde{F}(x) \le (1+\varepsilon) F(x) \quad \forall x}$$

[BSS]: $s \le O(n/\varepsilon^2)$

$A = A_1 + \cdots + A_m$

$e_1 e_1^T + e_2 e_2^T + \cdots + e_n e_n^T = I$

$I = A^{-1/2} A_1 A^{-1/2} + \cdots + A^{-1/2} A_m A^{-1/2}$

$p_1 = \cdots = p_n = \frac{1}{n}$ $\quad$ #samples $\gtrsim n \log n$

$$\boxed{I = A_1 + \cdots + A_m}$$

Goal: Find $x \in [-1,1]^m$ s.t.

(i) At least half of the x_i are ± 1

(ii) $\left\| \sum_i x_i A_i \right\|_{op} \le C \sqrt{\frac{n}{m}}$

$$(1 - C\sqrt{\tfrac{n}{m}}) I \preceq \underbrace{\sum_{i=1}^{m} (1+x_i) A_i}_{\frac{3m}{4}\text{-sparse}} \preceq (1 + C\sqrt{\tfrac{n}{m}}) I$$

$-C\sqrt{\frac{n}{m}} I \preceq \sum_i x_i A_i \preceq C\sqrt{\frac{n}{m}} I$

$\ge \frac{m}{4}$ of the x_i are -1

$A^{(0)} = I, \quad A^{(1)} = \underbrace{\sum_i (1+x_i) A_i}_{\frac{3m}{4}\text{-sparse}}, \quad A^{(2)} = \underbrace{\cdots}_{(\frac{3}{4})^2 m\text{-sparse}}, \quad \dots, \quad A^{(k)}$

$\uparrow m \asymp \frac{n}{\varepsilon^2}$

$m_j := \left(\frac{3}{4}\right)^j m$

$$\|A^{(k)} - A^{(0)}\|_{op} \le \sum_j \underbrace{\|A^{(j+1)} - A^{(j)}\|_{op}}_{C\sqrt{\frac{n}{m_j}}} \le O(\varepsilon)$$

$$\|A^{(k)} - I\|_{op}$$

$$\sqrt{\frac{n}{n/\varepsilon^2}} = \varepsilon$$

$$\|A^{(k)} - I\|_{op} \le \varepsilon \iff (1-\varepsilon) I \preceq A^{(k)} \preceq (1+\varepsilon) I$$

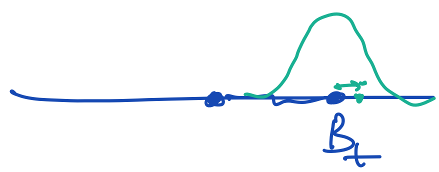
Goal: Find $\underline{x \in [-1,1]^m}$ s.t. (i) At least half of the x_i are ± 1

(ii) $\left\| \sum_i x_i A_i \right\|_{op} \le C\sqrt{\frac{n}{m}}$

$[-1,1]^m$

$\{B_t : t \ge 0\}$

$dB_t = B_{t+dt} - B_t \sim N(0, dt)$



$B_t = (B_t^1, B_t^2, \dots, B_t^m) \in \mathbb{R}^m$

$$\boxed{dX_t = I_{J_t}\, dB_t}$$

$X_t \in \mathbb{R}^m$

$$\begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} dB_t^1 \\ \vdots \\ dB_t^m \end{bmatrix}$$

$J_t \subseteq \{1, \dots, m\}$

T_i = first time when $|X_t^i| = 1$

$:= \inf\{t > 0 : |X_t^i| \ge 1\}$

$S_t := \{ i \in \{1, \dots, m\} : T_i \le t \}$

$T := \sup\{t > 0 : |S_t| \le m/2\}$

$X_T \in [-1,1]^m$

$$\sum_{i=1}^m x_i A_i \preceq C\sqrt{\frac{n}{m}}\, I$$

$$\tilde{A}_i = \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix}$$

$$\Rightarrow -C\sqrt{\frac{n}{m}}\, I \preceq \sum_i x_i A_i \preceq C\sqrt{\frac{n}{m}}\, I$$

Goal: Find $x \in [-1,1]^m$ s.t. (i) At least half of the x_i are ± 1

(ii) $\sum_i x_i A_i \preceq C\sqrt{\frac{n}{m}}\, I_n$

$$\boxed{|A_1| + \cdots + |A_m| \preceq I}$$

$$\sum_{i=1}^{m} x_t^i A_i$$

$$U(x) := \theta I_n + \lambda \|x\|_2^2 I_n - \sum_i x_i A_i \qquad \theta, \lambda > 0$$

$X_0 = 0$

$$\Phi(x) := \mathrm{tr}(U(x)^{-1}) \qquad \{X_t\} \quad U_t = U(X_t)$$

$$U_0 \succ 0, \quad \Phi(X_0) = \Phi(0) = \mathrm{tr}(U_0^{-1}) = n/\theta$$

If $\{\Phi(X_t) : t \in [0,T]\}$ remains bounded,

then $U_t \succ 0$ for all $t \in [0,T]$. $X_T \in [-1,1]^m$

$$U_T \succ 0 \iff (\theta + \lambda \|X_T\|_2^2) I_n - \sum_i X_T^i A_i \succ 0$$

$$\iff \sum_i X_T^i A_i \prec (\theta + \lambda \|X_T\|_2^2) I_n$$

$$\prec (\theta + \lambda m) I_n$$

$A_1, \dots, A_m$ real symmetric and $|A_1| + \dots + |A_m| \preceq I$

Goal: (i) $T < \infty$ w/ prob 1

(ii) $\sum_i X_T^i A_i \preceq C \sqrt{\frac{n}{m}}\, I$

$$\boxed{dX_t = I_{\mathcal{J}_t}\, dB_t}$$

$\mathcal{J}_t$ set of "live" coordinates

$U(x) := \theta I_n + \lambda \|x\|_2^2 I_n - \sum_i x_i A_i \qquad \theta, \lambda > 0$

$X_0 = 0$

$\Phi(x) := \mathrm{tr}\left(U(x)^{-1}\right) \qquad \{X_t\} \quad U_t = U(X_t)$

Track: $d\Phi(X_t) = \Phi(X_{t+dt}) - \Phi(X_t)$

$\cancel{d\Phi(X_t) = \langle \nabla \Phi(X_t), dX_t \rangle}$

Ito derivative

$d\Phi(x) = \Phi(x + dx) - \Phi(x)$

$= \langle \nabla \Phi(x), dx \rangle$