Fusible HSTs and the randomized $k$-server conjecture

James R. Lee
University of Washington

Abstract

We show that a potential-based algorithm for the fractional $k$-server problem on hierarchically separated trees (HSTs) with competitive ratio $f(k)$ can be used to obtain a randomized algorithm for any metric space with competitive ratio $f(k)^2 O((\log k)^2)$. Employing the $O((\log k)^2)$-competitive algorithm for HSTs from our joint work with Bubeck, Cohen, Lee, and Mądry (2017), this yields an $O((\log k)^6)$-competitive algorithm for the $k$-server problem on general metric spaces.

The best previous result independent of the geometry of the underlying metric space is the $2k - 1$ competitive ratio established for the deterministic work function algorithm by Koutsoupias and Papadimitriou (1995). Even for the special case when the underlying metric space is the real line, the best known competitive ratio was $k$. Since deterministic algorithms can do no better than $k$ on any metric space with at least $k + 1$ points, this establishes that for every metric space on which the problem is non-trivial, randomized algorithms give an exponential improvement over deterministic algorithms.

Contents

1 Introduction
   1.1 The limitations of oblivious HST embeddings ........................................ 2
   1.2 The limitations of dynamic HST embeddings ..................................... 4
   1.3 Augmenting the model with cluster fusion ........................................ 5
   1.4 Bird’s-eye view: Embeddings, isoperimetry, and scales ....................... 6
   1.5 Preliminaries ...................................................................................... 8

2 The online HST model
   2.1 Universal HSTs ................................................................................. 9
   2.2 Online HST embeddings ..................................................................... 10
   2.3 The potential axioms ......................................................................... 12

3 Construction of the embedding
   3.1 Embedding components ..................................................................... 14
   3.2 The embedding algorithm .................................................................. 15

4 Analysis
   4.1 Interesting scales ............................................................................... 18
   4.2 Auxiliary potential functions ............................................................. 20
   4.3 Potential calculus ............................................................................... 21
   4.4 Proof outline ...................................................................................... 23
   4.5 Stretch analysis .................................................................................. 25
   4.6 Tracking the algorithm ..................................................................... 27

5 Monotonicity under fusion
   5.1 Online rounding .................................................................................. 28
   5.2 Verification of the potential axioms for [BCL+17] .................................. 30

1
1 Introduction

An online algorithm is one that receives a sequence of inputs \(x_1, x_2, \ldots\) at discrete times \(t \in \{1, 2, \ldots\}\). At every time step \(t\), the algorithm takes some feasible action based only on the inputs \(x_1, x_2, \ldots, x_t\) it has seen so far. There is a cost associated with every feasible action, and the objective of an algorithm is to minimize the average cost per time step. This performance can be compared to the optimal offline algorithm which is allowed to decide on a sequence of feasible actions given the entire input sequence in advance.

Roughly speaking, an online algorithm is \(C\)-competitive if, on any valid input sequence, its average cost per time step is at most a factor \(C\) more than that of the optimal offline algorithm for the same sequence. The best achievable factor \(C\) is referred to as the competitive ratio of the underlying problem. It bounds the detrimental effects of uncertainty on optimization. Algorithms designed in the online model tend to tradeoff the benefits of acting locally to minimize cost while hedging against uncertainty in the future. We refer to the book [BE98].

The \(k\)-server problem. Perhaps the most well-studied problem in this area is the \(k\)-server problem proposed by Manasse, McGeoch, and Sleator [MMS90] as a significant generalization of various other online problems. The authors of [BBN10] refer to it as the “holy grail” of online algorithms.

Fix an integer \(k \geq 1\) and let \((X, d_X)\) denote an arbitrary metric space. We will assume that all metric spaces occurring in the paper have at least two points. The input is a sequence \(\langle \sigma_t \in X : t \geq 0 \rangle\) of requests. At every time \(t\), an online algorithm maintains a state \(\rho_t \in X^k\) which can be thought of as the location of \(k\) servers in the space \(X\). At time \(t\), the algorithm is required to have a server at the requested site \(\sigma_t \in X\). In other words, a feasible state \(\rho_t\) is one that services \(\sigma_t\):

\[
\sigma_t \in \{(\rho_1)_1, \ldots, (\rho_k)_k\).
\]

Formally, an online algorithm is a sequence of mappings \(\rho = \langle \rho_1, \rho_2, \ldots, \rangle\) where, for every \(t \geq 1\), \(\rho_t : X^t \rightarrow X^k\) maps a request sequence \(\langle \sigma_1, \ldots, \sigma_t \rangle\) to a \(k\)-server state that services \(\sigma_t\). In general, \(\rho_0 \in X^k\) will denote some initial state of the algorithm.

The cost of the algorithm \(\rho\) in servicing \(\sigma = \langle \sigma_t : t \geq 1 \rangle\) is defined as the sum of the movements of all the servers:

\[
\text{cost}_\rho(\sigma; k, \rho_0) := \sum_{t \geq 1} d_{X^k}((x_1, \ldots, x_t), (y_1, \ldots, y_t)) = \sum_{i=1}^k d_X(x_i, y_i) \quad \forall x_1, \ldots, x_t, y_1, \ldots, y_t \in X.
\]

For a given request sequence \(\sigma = \langle \sigma_t : t \geq 1 \rangle\) and initial configuration \(\rho_0\), denote the cost of the offline optimum by

\[
\text{cost}^*(\sigma; k, \rho_0) := \inf_{\rho_1, \rho_2, \ldots} \sum_{t \geq 1} d_{X^k}((x_1, \ldots, x_t), (y_1, \ldots, y_t)),
\]

where the infimum is over all sequences \(\langle \rho_1, \rho_2, \ldots \rangle\) such that \(\rho_t\) services \(\sigma_t\) for each \(t \geq 1\).

An online algorithm \(\rho\) is said to be \(\alpha\)-competitive if, for every initial configuration \(\rho_0 \in X^k\), there is a constant \(c > 0\) such that

\[
\text{cost}_\rho(\sigma; k, \rho_0) \leq \alpha \cdot \text{cost}^*(\sigma; k, \rho_0) + c
\]

for all request sequences \(\sigma\). A randomized online algorithm \(\rho\) is a random online algorithm that is feasible with probability one. Such an algorithm is said to be \(\alpha\)-competitive if for every \(\rho_0 \in X^k\), there is a constant \(c > 0\) such that for all \(\sigma\):

\[
\mathbb{E}[\text{cost}_\rho(\sigma; k, \rho_0)] \leq \alpha \cdot \text{cost}^*(\sigma; k, \rho_0) + c.
\]
The initial configuration $\rho_0$ will play a very minor role in our arguments, and we will usually leave it implicit, using instead the notations $\text{cost}_p(\sigma; k)$ and $\text{cost}^*(\sigma; k)$. Let $D_k(X, d_X)$ denote the infimum of competitive ratios achievable by deterministic online algorithms, and let $R_k(X, d_X)$ denote the infimum over randomized online algorithms. When the metric $d_X$ on $X$ is clear from context, we will often omit it from our notation.

The authors of [MMS90] showed that if $(X, d_X)$ is an arbitrary metric space and $|X| > k$, then $D_k(X) \geq k$. They conjectured that this is tight.

**Conjecture 1.1** (k-server conjecture, [MMS90]). For every metric space $X$ with $|X| > k \geq 1$, it holds that $D_k(X) = k$.

Fiat, Rabani, and Ravid [FRR94] were the first to show that $D_k(X) < \infty$ for every metric space; they gave the explicit bound $D_k(X) \leq k^{O(k)}$. While Conjecture 1.1 is still open, it is now known to be true within a factor of 2.

**Theorem 1.2** (Koutsoupias-Papadimitriou, [KP95]). For every metric space $X$ and $k \geq 1$, it holds that $D_k(X) \leq 2k - 1$.

**Paging and randomization.** Let $U_n$ denote the metric space on $\{1, 2, \ldots, n\}$ equipped with the uniform metric $d(i, j) = 1_{(i \neq j)}$. The special case of the $k$-server problem when $X = U_n$ is called $k$-paging. Note that an adversarial request sequence for a deterministic online algorithm can be constructed by basing future requests on the current state of the algorithm. Consider, for instance, the following lower bound for $U_{k+1} \subseteq U_n$ (for $n > k$). For any deterministic algorithm $A$, define the request sequence that at time $t \geq 1$ makes a request at the unique site in $U_{k+1}$ at which $A$ does not have a server.

Clearly $A$ incurs movement cost exactly $t$ up to time $t$. On the other hand, the algorithm that starts with its servers at $k$ uniformly random points in $\{1, 2, \ldots, k + 1\}$ and moves a uniformly random server to service the request (whenever there is not already a server there) has expected movement cost $t/k$. Thus there is some (deterministic) offline algorithm with cost $t/k$ up to time $t$. Moreover, manifestly there is also a randomized online algorithm that achieves cost $1/k$ per time step in expectation.

And indeed, in the setting of $k$-paging, it was show that allowing an online algorithm to make random choices helps dramatically in general.

**Theorem 1.3** ([FKL+91, MS91]). For every $n > k \geq 1$:

$$R_k(U_n) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}.$$

Work of Karloff, Rabani, and Ravid [KRR94] exploited a “metric Ramsey dichotomy” to give a lower bound on the randomized competitive ratio for any sufficiently large metric space. The works [BBM06, BLMN05] made substantial advances along this front, obtaining the following.

**Theorem 1.4.** For any metric space $X$ and $k \geq 2$ such that $|X| > k$, it holds that

$$R_k(X) \geq \Omega\left(\frac{\log k}{\log \log k}\right).$$

In light of a lack of further examples, a folklore conjecture arose.

**Conjecture 1.5** (Randomized k-server conjecture). For every metric space $X$ and $k \geq 2$:

$$R_k(X) \leq O(\log k).$$

3
The possibility that \( R_k(X) \leq (\log k)^{O(1)} \) is stated explicitly, for instance, in [BBK99], although this is certainly not its first appearance. See also Conjecture 2 in the survey [Kou09]. Our main theorem asserts that, indeed, randomization helps dramatically for every metric space.

**Theorem 1.6 (Main theorem).** For every metric space \( X \) and \( k \geq 2 \):

\[
R_k(X) \leq O((\log k)^6).
\]

Even when \( X = \mathbb{R} \), the best previous upper bound was inherited from the deterministic setting [CKPV91]: \( R_k(\mathbb{R}) \leq D_k(\mathbb{R}) = k \).

**Theorem 1.6** owes much to three recent works that each dramatically improve our understanding of the \( k \)-server problem. The first is the successful resolution of the randomized \( k \)-server conjecture for an important special case called *weighted paging*. Consider a set \( X \) and a non-negative weight \( w: X \to \mathbb{R}_+ \). Define the distance \( d_w(x, y) := \max\{w(x), w(y)\} \). We refer to this as a *weighted star metric*.

**Theorem 1.7 (Bansal-Buchbinder-Naor, [BBN12]).** If \( X \) is a weighted star metric and \( k \geq 2 \), then

\[
R_k(X) \leq O((\log k)^{3 \log \log |X|}).
\]

The second recent breakthrough shows that when \( X \) is finite, the competitive ratio can be bounded by polylogarithmic factors in \( |X| \).

**Theorem 1.8 (Bansal-Buchbinder-Mądry-Naor, [BBMN15]).** For every \( k \geq 2 \) and finite metric space \( X \), it holds that

\[
R_k(X) \leq O\left((\log k)^2(\log |X|)^3 \log \log(e|X|)\right).
\]

Finally, in joint work with Bubeck, Cohen, Lee, and Mądry [BCL+17], we obtain a cardinality-independent bound when \( X \) is an ultrametric. This result will be an essential component of our arguments.

**Theorem 1.9 ([BCL+17]).** For every \( k \geq 2 \) and every ultrametric space \( X \), it holds that

\[
R_k(X) \leq O\left((\log k)^2\right).
\]

### 1.1 The limitations of oblivious HST embeddings

The significance of ultrametrics in **Theorem 1.9** stems from their pivotal role in online algorithms for \( k \)-server. Consider a rooted tree \( T = (V, E) \) equipped with non-negative vertex weights \( \{w_u \geq 0 : u \in V\} \) such that the weights are non-increasing along every root-leaf path. Let \( L \subseteq V \) denote the set of leaves of \( T \), and define an ultrametric on \( L \) by

\[
d_w(\ell, \ell') := w_{\text{lca}(\ell, \ell')},
\]

where \( \text{lca}(u, v) \) denotes the least common ancestor of \( u, v \in V \) in \( T \).

If it holds for some \( \tau \geq 1 \) that \( w_v \leq w_u/\tau \) whenever \( v \) is a child of \( u \), then \((T, w)\) is called a \( \tau \)-hierarchically separated tree (\( \tau \)-HST) and \((L, d_w)\) is referred to as a \( \tau \)-HST metric space. (For finite metric spaces, the notion of an ultrametric and a 1-HST are equivalent.)

This notion was introduced in a seminal work of Bartal [Bar96, Bar98] along with the powerful tool of probabilistic embeddings into random HSTs. Moreover, he showed that every \( n \)-point metric space embeds into a distribution over random HSTs with \( O(\log n \log \log n) \) distortion. Using the optimal \( O(\log n) \) distortion bound from [FRT04] yields the following consequence.
Theorem 1.10. Suppose that \((X, d)\) is a finite metric space. Then for every \(k \geq 2\):

\[
R_k(X, d) \leq O(\log |X|) \cdot \sup_{(L, d')} R_k(L, d') ,
\]

where the supremum is over all ultrametrics \((L, d')\) with \(|L| = |X|\).

Clearly in conjunction with Theorem 1.9, this yields \(R_k(X) \leq O((\log k)^2 \log |X|)\) for any finite metric space \(X\). The reduction from general finite metric spaces to ultrametrics implicit in Theorem 1.10 is oblivious to the request sequence; one chooses a single random embedding, and then simulates an online algorithm for the corresponding ultrametric. This is both useful and problematic, as no such approach can yield a bound that does not depend on the cardinality of \(X\); there are many families of metric spaces for which the \(O(\log |X|)\) distortion bound is tight.

In [BCL+17], we showed how a dynamic embedding of a metric space into ultrametrics could overcome the distortion barrier.

Theorem 1.11 ([BCL+17]). For every \(k \geq 2\) and every finite metric space \((X, d)\):

\[
R_k(X, d) \leq O((\log k)^3 \log(1 + A_X)) ,
\]

where

\[
A_X := \max_{x, y \in X} d(x, y) / \min_{x, y \in X} d(x, y) .
\]

The dependence of the competitive ratio on \(A_X\) is still problematic, but one should note that the resulting bound could not be achieved with an oblivious embedding. Indeed, suppose that \(\{G_n\}\) is a family of expander graphs with uniformly bounded degrees and such that \(G_n\) has \(n\) vertices. Let \((V_n, d_n)\) denote the induced shortest-path metric on the vertices of \(G_n\). It is well-known that a probabilistic embedding into ultrametrics incurs distortion \(\Omega(\log n)\), while (1.1) yields \(R_k(V_n, d_n) \leq O((\log k)^3 \log \log n)\).

1.2 The limitations of dynamic HST embeddings

Consider a bounded metric space \((X, d_X)\) (i.e., one with finite diameter) and a fixed (possibly infinite) \(\tau\)-HST metric space \((L, d_{\text{hst}})\) with \(\tau \geq 2\). We may consider a fixed HST because one can choose a universal target space without loss of generality; see Section 2.1. We will assume that every leaf has a unique preimage, i.e., there is a surjection \(\beta : L \to X\), and that the map \(\beta\) is 1-Lipschitz:

\[
d_X(\beta(x), \beta(y)) \leq d_{\text{hst}}(x, y) \quad \forall x, y \in L .
\]

Given a request sequence \(\sigma = \langle \sigma_1, \sigma_2, \ldots \rangle\) in \(X\), one can consider a (random) sequence \(\alpha = \langle \alpha_1, \alpha_2, \ldots \rangle\) of points in \(L\) with the property that \(\beta(\alpha_t) = \sigma_t\) for each \(t \geq 1\). Say that \(\alpha\) is oblivious if there is a single random map \(F : X \to L\) chosen independently of \(\sigma\) and \(\alpha_t := F(\sigma_t)\). Say that \(\alpha\) is adapted to the request sequence if \(\alpha_t\) depends only on \(\langle \sigma_1, \sigma_2, \ldots, \sigma_t \rangle\).

Finally, say that \(\alpha\) has \(k\)-server distortion at most \(D\) if there is a constant \(c > 0\) such that for every request sequence \(\sigma\):

\[
\mathbb{E} \left[ \text{cost}_{\text{hst}}^\alpha(\sigma; k) \right] \leq D \cdot \text{cost}_X^\sigma(\sigma; k) + c .
\]

If \(\alpha\) is adapted to the request sequence and has \(k\)-server distortion at most \(D\), then a \(C\)-competitive \(k\)-server algorithm on \((L, d_{\text{hst}})\) yields a \(CD\)-competitive algorithm for the \(k\)-server problem on \((X, d_X)\) since (1.2) allows us to pull the server trajectories back to \(X\) at no additional cost.

In [BCL+17], it is shown that such adapted sequences \(\alpha\) with \(k\)-server distortion \(D \leq O(\log(k) \log(1 + A_X))\). Unfortunately, this model is too weak to obtain Theorem 1.6 even when \(X\) is the unit circle (or the real line), even for the case of \(k = 1\) server. This is for a simple reason: Even if we don’t require the sequence \(\alpha\) to be adapted (i.e., we are given the entire request sequence in advance), there are request sequences \(\sigma\) so that (1.3) holds, then \(D \geq \Omega(\log A_X)\).
Lemma 1.12. For every $A \geq 2$, there is a set of points $X$ on the unit circle with $A_X \leq A$ and so that for any $\alpha$ satisfying (1.3) for every $\sigma$ with $k = 1$, it holds that

$$D \geq \Omega(\log A) \geq \Omega(\log |X|).$$

We sketch the straightforward proof, as it will motivate our modification of the dynamic embedding model and its subsequent analysis. Fix some $n \geq 2$ and consider a request sequence $\sigma = \langle \sigma_1, \sigma_2, \ldots, \sigma_n \rangle$, where $\sigma_t = e^{-2\pi t/n} \in S^1$, and $S^1$ denotes the unit circle in the complex plane equipped with its radial metric $d_{S^1}$.

Clearly $\text{cost}_{S^1}(\sigma; 1) \leq O(1)$. We claim that for any sequence of leaves $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ satisfying

$$d_{\text{hst}}(\alpha_i, \alpha_j) \geq d_{S^1}(\sigma_i, \sigma_j) \quad \forall i, j,$$

it holds that

$$\sum_{i=1}^{n-1} d_{\text{hst}}(\alpha_i, \alpha_{i+1}) \geq \Omega(\log n).$$

Indeed, this is immediate: For every $1 \leq j \leq \lceil \log \tau n \rceil$, by (1.4), the sequence $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ of leaves must exit a subtree of label (at least) $\tau^{-j}$ at least $\Omega(\tau^j)$ times, implying that

$$\sum_{i=1}^{n-1} d_{\text{hst}}(\alpha_i, \alpha_{i+1}) \geq \sum_{j=1}^{\lceil \log \tau n \rceil} \tau^{-j} \geq \Omega(\log \tau n).$$

1.3 Augmenting the model with cluster fusion

Consider again the example of the preceding section, but now it will be helpful to think about a continuous path: Suppose that $\sigma : [0, \infty) \to S^1$ is a point that moves clockwise at unit speed. Recall that $(\mathcal{L}, d_{\text{hst}})$ is a $\tau$-HST metric.

A non-contractive embedding $\alpha : S^1 \to \mathcal{L}$ induces a sequence of partitions $\{P_j : j \geq 0\}$ of $S^1$, where $P_0 = S^1$, for every $j \geq 0$, $P_{j+1}$ is a refinement of $P_j$, and where every set $S \in P_j$ has diameter at most $2\pi \tau^{-j}$. When $\sigma(t)$ approaches the boundary of $P_j$, the image $\alpha(\sigma(t))$ stands to incur $d_{\text{hst}}$ movement $\approx \tau^{-j}$ as $\alpha(\sigma(t))$ switches sets of the partition $P_j$. In order to prevent this, we fuse together the two sets of $P_j$ whose boundary $\sigma(t)$ is about to cross. See Figure 1.

When $\sigma(t)$ is safely past the boundary, we need to unfuse these sets so that we are prepared to fuse across the next $P_j$ boundary. Failing to do this, we might start fusing a long chain of sets; having sets of unbounded diameter in $P_j$ would prevent us from maintaining a non-contractive
embedding into $\mathcal{L}$. We will soon describe a model where fusion and fission of sets in the target HST is possible.

**Potential-based algorithms for HSTs.** Once we allow ourselves such operations, it no longer seems possible to use a competitive HST algorithm as a black box. Indeed, such an algorithm maintains internal state, and there is no reason it should continue to operate meaningfully under a sudden unexpected change to this state.

Thus we will assume the existence of an HST algorithm that maintains a configuration $\chi$ and whose operation can be described as a function mapping a pair $(\chi, \sigma)$ to a new configuration $\chi'$, where $\sigma \in \mathcal{L}$ is the request to be serviced, and $\chi'$ induces a fractional $k$-server measure $\mu_{\chi'}$ that services $\sigma$. (See Section 2 for a discussion of fractional $k$-server measures; for the present discussion, one can think of $\mu_{\chi'}$ as simply a $k$-server state.) Moreover, we will assume that the HST algorithm’s competitiveness is witnessed by a potential function $\Phi(\theta^*; \chi)$ that tracks the “discrepancy” between the server state induced by $\chi$ and the server state $\theta^*$ of the optimal offline algorithm.

Crucially, we will assume that $\Phi$ decreases monotonically under fusion operations applied simultaneously to both $\theta^*$ and (the measure underlying) $\chi$. If $\Phi$ is thought of as a measure of discrepancy with respect to the underlying HST, then this makes sense: When two clusters are fused, the corresponding notion of discrepancy becomes more coarse (meaning that it is less able to distinguish $\theta^*$ from $\mu_{\chi'}$).

We state the required properties formally in Section 2.3. In Section 5.2, we confirm that the algorithm establishing Theorem 1.9 satisfies these properties.

**The cost and difficulty of fusion and fission.** Whereas cluster fusion will be “free,” the inverse process of unfusing (i.e., “fission”) of clusters will be quite expensive—proportional to the cost of moving all the servers in those clusters a distance proportional to the cluster diameters. Thus deciding when to fuse clusters and when to undo this fusion will be the primary difficulty we confront.

Of course, the case of linear movement of a single server is trivial. Suppose that $\sqrt{k}$ is an integer, and imagine that one takes a “hard” (recall Theorem 1.4) request sequence for the $\sqrt{k}$-server problem: $\langle \sigma_1, \sigma_2, \ldots \rangle$. Let $\{B_t\}$ denote a one-dimensional Brownian motion and consider the randomly rotated request sequence

$$\tilde{\sigma}_j := e^{2\pi i B_t} \sigma_j,$$

where

$$t_j := \frac{c_j^2}{k},$$

and $c_j$ is the movement cost incurred by the optimal offline algorithm up to request $j$. It is not difficult to show that the expected offline optimal cost increases by only a constant factor.

Now suppose we interleave $\sqrt{k}$ independent copies of this request sequence (intended to be serviced by $k$ total servers):

$$\tilde{\sigma}_1^{(1)}, \tilde{\sigma}_1^{(2)}, \ldots, \tilde{\sigma}_1^{(\sqrt{k})}, \tilde{\sigma}_2^{(1)}, \tilde{\sigma}_2^{(2)}, \ldots, \tilde{\sigma}_2^{(\sqrt{k})}, \ldots.$$

It is not difficult to construct hard request sequences that can be serviced cheaply using $\sqrt{k}$ servers, but such that no algorithm using only $\sqrt{k} - 1$ servers can be competitive with respect to movement cost. As these $\sqrt{k}$ independent request sequences overlap and intersect (at many different scales), one can start to see the difficulty in deciding which clusters should be fused in the partitions at every scale.
1.4 Bird’s-eye view: Embeddings, isoperimetry, and scales

Following [Bar96], one generally constructs a $\tau$-HST embedding as follows. Suppose that $(X, d_X)$ has diameter at most one. Let $P = \{P_j : j \geq 0\}$ denote a sequence of partitions of $X$ so that for each $j \geq 0$, if $S \in P_j$ then $\text{diam}_X(S) \leq \tau^{-j}$. For a partition $P$ of $X$ and $x \in X$, let $P(x)$ denote the unique set of $P$ containing $x$.

One can define a $\tau$-HST metric on $X$ by

$$d_{\text{hst}}^P(x, y) := \tau^{-\min\{j \geq 0 : P_j(x) \neq P_j(y)\}}.$$  

If $X$ is finite, then by choosing the partitions $P_j$ appropriately at random, one can additionally obtain the property that

$$\mathbb{P}[P_j(x) \neq P_j(y)] \leq \frac{d_X(x, y)}{\tau^{-j}} O(\log |X|) \quad \forall j \geq 0.$$  

In particular, this implies that for any $x, y \in X$,

$$\mathbb{E}[d_{\text{hst}}^P(x, y)] \leq O(\log |X|) \cdot O(\log \mathcal{A}_X),$$  

where we recall the aspect ratio of $X$ from (1.1).

Both distortion factors in (1.6) are troublesome, but there is now a well-understood theory of how they arise. See, for instance, the elegant argument of [FRT04] which indicates that they cannot arise simultaneously. The $O(\log |X|)$ factor inherited from (1.5) might be called the “isoperimetric” obstruction. For instance, it can be replaced by a universal constant if $X = \mathbb{R}$, but it is necessary if $(X, d_X)$ is the shortest-path metric on an expander graph or the $\ell_1$ metric on $\{0, 1\}^d$ for some $d \geq 1$. The $O(\log \mathcal{A}_X)$ factor could be called the “multiscale” obstruction, and it arises whenever the underlying metric space contains paths (i.e., coarse geodesics).

**Bypassing the obstructions.** If, instead of choosing a static embedding, we imagine maintaining an embedding for the purposes of solving the $k$-server problem, then it is not unreasonable to expect that the HST embedding only needs to “track” $O(k)$ regions at every scale. Intuitively, if there are many more than $k$ regions being used in an interesting way at a given scale $\tau^{-j}$, then the optimum offline algorithm must be incurring significant “$\tau^{-j}$ movement” to service the underlying request sequence (because it only has the use of $k$ servers).

Morally, this allows us to track only $O(k)$ regions, and pay some cost whenever we have to alter the embedding to incorporate new regions (and discard old ones). This suggests one might replace $O(\log |X|)$ by $O(\log k)$ and, indeed, that was the motivation for the dynamic embeddings constructed in [BCL+17]. Here, we employ a more sophisticated version of that scheme to avoid the “overcharging” of the optimum that occurs in [BCL+17], but this is not the major new obstacle we face.

It is the multiscale obstruction that motivates a model in which we can “fuse” together sibling clusters in the HST embedding. The underlying idea is simple: Suppose that $\nu_t$ is some measure on $X$ with the property that for $S \subseteq X$, the value $\nu_t(S)$ indicates approximately how many servers we should have in (or near) $S$ at time $t$. Consider a ball $B$ in $X$ and the ball $\lambda B$ (with the same center, and with a $\lambda$ times larger radius, $\lambda \gg 1$). If it holds that for some small $\delta > 0$,

$$\nu_t(B) \geq (1 - \delta)\nu_t(\lambda B),$$

it indicates that almost all the servers allocated to $\lambda B$ should, in fact, be in the much smaller ball $B$.

This suggests that the fidelity of $B$ in the HST embedding is very important, and we should prevent partitions (at the scale of $B$) from cutting $B$ into pieces. This is easily achieved by “fusing”
together all the sets in \( P_j \) that intersect \( B \) into one supercluster (one might look ahead to Figure 2). The condition (1.7) directly implies that disjoint heavy balls must be far apart, and thus we avoid the problem of having chains of fusions that produce sets of unbounded diameter.

This also immediately addresses the multiscale obstruction: At every scale \( \tau^{-j} \) where the ball \( B_x(x, \tau^{-j}) \) is heavy, we fuse the clusters near \( x \), and therefore do not pay the separation penalty in (1.5). At how many scales \( j \in \{0, 1, 2, \ldots\} \) can the ball \( B_x(x, \tau^{-j}) \) be light? It is easy to see that the answer is \( O(\log v_t(x)/v_t(x)) \) since, at every light scale, a \( \delta \)-fraction of the mass is lost when zooming into \( x \) from radius \( \lambda \tau^{-j} \) to radius \( \tau^{-j} \). Since \( v_t \) is a proxy for servers, we have \( v_t(x) \leq k \), and when \( x \) is an “interesting” point, it will hold that \( v_t(x) \geq 1/2 \), say. Thus the number of non-trivial scales at which \( x \) is not automatically fused with its neighbor clusters is only \( O(\log k) \).

**Paying for cluster fission.** As mentioned previously, the difficulty comes when a ball that was once heavy becomes light, and then we must “unfuse” the underlying clusters. This cost is charged against the transportation cost of the sequence of measures \( \langle v_t : t \geq 0 \rangle \). We only unfuse the clusters corresponding to a heavy ball \( B \) when eventually some ball \( B' \) becomes heavy with \( B' \cap \lambda B \neq \emptyset \) and \( d_{t_{\delta}}(B, B') \geq \sqrt{\lambda} \text{diam}_X(B) \). It is intuitively clear that this requires significant movement of the measure \( v_t \) on which heaviness is based. By making \( \delta \) sufficiently small and \( \lambda \) sufficiently large, we increase the number of non-trivial scales, but correspondingly increase substantially the amount of movement required to create a new heavy ball. This facilitates the charging argument which is done formally through the “fission potential” introduce in Section 4.2.

The indicative measure \( v_t \). Our overview is complete except for one crucial detail: What is \( v_t \)? It is natural to have \( v_t(S) \) represent the number of servers we currently have in \( S \) at time \( t \). Problematically, this number depends on the random choices made so far in constructing an embedding, and thus we cannot use it to make future decisions. This is not just a technical matter; there are feedback loops that can arise if an HST embedding/algorithm bases its choices on its own server configuration. (We make a bad guess about where servers should go, act on it, then use that as an indicator of where servers should go, . . .)

Instead, \( v_t(S) \) will represent the *average* location of servers over the random choices made so far in constructing an HST embedding. In particular, this will mean that certain fusion and fission operations look “unnatural” from the perspective of a given run of the randomized algorithm, but there is a global charging scheme that connects the many parallel runs together. This averaging creates enough stability to avoid feedback looks from the rare events in which some random partitions are chosen poorly.

### 1.5 Preliminaries

Let us write \( \mathbb{R}_+ := [0, \infty) \) and \( \mathbb{Z}_+ := \mathbb{Z} \cap \mathbb{R}_+ \). Consider a set \( X \). We use \( \mathcal{M}(X) \) to denote the space of measures on \( X \) whose support is at most countable. Denote by \( \mathcal{M}_k(X) \subseteq \mathcal{M}(X) \) the subset of countably-additive measures \( \mu \in \mathcal{M}(X) \) that satisfy \( \mu(X) = k \). Since our measures have at most countable support, when \( x \in X \), we will often write \( \mu(x) \) for \( \mu(\{x\}) \). Denote by \( \hat{\mathcal{M}}(X) \) the set of integral measures on \( X \), i.e., those \( \mu \in \mathcal{M}(X) \) which take values in \( \mathbb{Z}_+ \), and similarly \( \hat{\mathcal{M}}_k(X) := \mathcal{M}_k(X) \cap \hat{\mathcal{M}}(X) \).

If \( X, Y \) are two spaces, \( F : X \to Y \), and \( \mu \in \mathcal{M}(X) \), then we use \( F\#\mu \) to denote the pushforward measure:

\[
F\#\mu(S) := \mu(F^{-1}(S)) \quad \forall S \subseteq Y.
\]

Note that if \( \mu \) is integral, then so is \( F\#\mu \). If \( \mu = \langle \mu_1, \mu_2, \ldots \rangle \) is a sequence of measures, we define \( F\#\mu := (F\#\mu_1, F\#\mu_2, \ldots) \).
If $\mu$ is a sequence of measures in $M_{\ell_1}(X)$, we write

$$\text{cost}_X(\mu) := \sum_{t \geq 1} W_{X}^1(\mu_t, \mu_{t+1}),$$

where $W_{X}^1(\mu, \nu)$ is the $L^1$-transportation distance between $\mu$ and $\nu$ in $X$. This is sometimes referred to as the Wasserstein 1-distance or the Earthmover distance. One can consult the book [Vil03] for an introduction to the geometry of optimal transportation. Note that we only deal here with countably-supported measures, so our considerations are elementary. The following claim is straightforward. (The reader should also note that with slightly more notational overhead, one could assume that all encountered measures have finite support.)

If $(X, d_X)$ and $(Y, d_Y)$ are two metric spaces and $F : X \rightarrow Y$, one defines

$$\|F\|_{\text{lip}} := \sup_{x \neq y \in X} \frac{d_Y(F(x), F(y))}{d_X(x, y)}.$$

**Claim 1.13.** For any sequence $\mu$, it holds that

$$\text{cost}_Y(F \# \mu) \leq \|F\|_{\text{lip}} \cdot \text{cost}_X(\mu).$$

For $x \in X$ and $r \geq 0$, we denote the ball $B_X(x, r) := \{ y \in X : d_X(x, y) \leq r \}$ and for $S \subseteq X$, the neighborhood $B_S(S, r) := \bigcup_{x \in S} B_X(x, r)$.

## 2 The online HST model

Fix a metric space $(X, d_X)$ with diameter at most one. Consider a global filtration $F = (F_1, F_2, \ldots)$ where $F_1 \subseteq F_2 \subseteq \cdots$, and $F_t$ represents information about the request sequence up to time $t$. Denote the request sequence $\sigma = (\sigma_1, \sigma_2, \ldots)$ with $\sigma_t \in X$ for all $t \geq 1$. We use $\sigma_{[s,t]}$ to denote the subsequence $(\sigma_s, \sigma_{s+1}, \ldots, \sigma_t)$. Say that a sequence $\rho = (\rho_0, \rho_1, \rho_2, \ldots)$ is $F$-adapted if each object $\rho_t$ is possibly a function of $\sigma_{[1:t]}$ (but not the future $\sigma_{t+1}, \sigma_{t+2}, \ldots$).

A notable observation is that in many cases it suffices to maintain a fractional $k$-server state, as opposed to a (random) integral state; one then rounds, in an $F$-adapted manner, the fractional solution to a random integral solution without blowing up the expected cost. This idea appears in [BBK99] and is made explicit in [BBN12] for weighted star metrics. In [BBMN15], it is extended to HST metrics. See Theorem 2.6 below for a variant tailored to our setting.

An offline fractional $k$-server algorithm (for $\sigma$) is a sequence of measures $\mu = (\mu_0, \mu_1, \mu_2, \ldots)$ such that $\mu_t \in M_{\ell_1}(X)$ for all $t \geq 1$, and such that $\mu_t(\sigma_t) \geq 1$ holds for every $t \geq 1$. We say that $\mu$ is integral if each measure $\mu_t$ takes values in $\mathbb{Z}_+$. An online fractional $k$-server algorithm is such a sequence $\mu$ that is additionally $F$-adapted. We will use the term fractional $k$-server algorithm to mean an online algorithm and explicitly use “offline” for the former notion.

### 2.1 Universal HSTs

It will be convenient for us to have a fixed HST into which our embeddings map requests. To accommodate request sequences of arbitrary length, the HST will be infinite, but the measure maintained by our algorithm will always be supported on a finite set of leaves (which are themselves a subset of the request sequence seen so far).

Fix some number $\tau \geq 2$. A sequence of subsets $\xi = (\xi_0, \xi_1, \xi_2, \ldots)$ of $X$ is a $\tau$-chain if

$$X = \xi_0 \supseteq \xi_1 \supseteq \xi_2 \supseteq \cdots,$$
and \( \text{diam}_X(\xi_j) \leq \tau^{-j} \) for all \( j \geq 0 \). If \( \xi \) is a finite sequence, we refer to \( \xi \) as a \textit{finite chain} and let \( \text{len}(\xi) \) denote its length (otherwise set \( \text{len}(\xi) := +\infty \)). Define the \textit{bottom} of \( \xi \) by \( b(\xi) := \bigcap_{i \geq 1} \xi_i \). Observe that for a finite \( \tau \)-chain \( \xi \),

\[
\text{diam}_X(b(\xi)) \leq \tau^{-\text{len}(\xi)} . \tag{2.1}
\]

A \textit{decorated} \( \tau \)-chain is a sequence \( \hat{\xi} = ( (\xi_0; 0), \hat{\xi}_1, \hat{\xi}_2, \ldots ) \) where \( \hat{\xi}_i = (\xi_i; \eta_i) \) for \( i \geq 1, \langle \xi_0, \xi_1, \ldots \rangle \) is a \( \tau \)-chain, and \( \langle \eta_i \in \mathbb{Z}_+ : i \geq 1 \rangle \) are arbitrary labels. We use \( \text{len}(\hat{\xi}) \) and \( b(\hat{\xi}) \) to denote the corresponding quantities for the underlying undecorated chain. We denote \( \eta((\xi_i, \eta_i)) := \eta_i \).

**Remark 2.1** (The decorations). We note that the decorations \( \{ \eta_i \} \) will play a minor role in our arguments. One could take \( \eta_i \in \{0, 1\} \) for all \( i \geq 1 \). We emphasize, in Section 3.1.4 and Section 5, the two places where they are used. One could do without them entirely, but they make some arguments substantially shorter.

Let \( V_T \) denote the set of finite decorated \( \tau \)-chains in \( X \). Define a rooted tree structure on \( T \) as follows. The root of \( T \) is the length-one chain \( (X, 0) \) (with label 0). For two chains \( \xi, \xi' \in V_T \): \( \xi' \) is a child of \( \xi \) if \( \xi \) is a prefix of \( \xi' \) and \( \text{len}(\xi') = \text{len}(\xi) + 1 \). Let \( T \) denote the rooted tree structure with vertex set \( V_T \). Let \( V_T^j \subseteq V_T \) denote the set of \( \tau \)-chains of length \( j \). A decorated \( \tau \)-chain \( \xi \) is a \textit{leaf chain} if \( \text{len}(\xi) = \infty \) and \( |b(\xi)| = 1 \). Let \( L_T \) denote the set of leaf chains. We denote the \textit{extended vertex set} \( \hat{V}_T := L_T \cup V_T \).

For two distinct chains \( \xi, \xi' \in V_T \), define their \textit{least common ancestor} \( \text{lca}(\xi, \xi') \in V_T \) as the maximal finite chain \( (\xi_0, \xi_1, \ldots, \xi_L) \) that is a prefix of both \( \xi \) and \( \xi' \). This allows us to define a \( \tau \)-HST metric on \( V_T \) by

\[
\text{dist}_T(\xi, \xi') := \tau^{-\text{len}(\text{lca}(\xi, \xi'))} .
\]

We call the pair \((T, \text{dist}_T)\) the \textit{universal} \( \tau \)-HST on \((X, d_X)\). For succinctness, we will employ the notations \( \text{cost}_T := \text{cost}_{V_T, \text{dist}_T} \) and \( W^1_T := W^1_{(V_T, \text{dist}_T)} \). We use \( V_T^0 \subseteq \hat{V}_T \) to denote the subset of chains whose decorations are identically 0 and \( L_T^0 := L_T \cap V_T^0 \).

**Pushing measures to** \( X \). Define the map \( \beta : L_T \to X \) as follows: \( \beta(\xi) \) is the unique element in \( b(\xi) \).

**Claim 2.2.** \( \beta \) is 1-Lipschitz as a map from \((L_T, \text{dist}_T)\) to \((X, d_X)\).

**Proof.** Consider \( \xi, \xi' \in L_T \). Let \( \hat{\xi} := \text{lca}(\xi, \xi') \). Then by definition, \( \beta(\xi), \beta(\xi') \in b(\hat{\xi}) \), hence

\[
d_X(\beta(\xi), \beta(\xi')) \leq \text{diam}_X(b(\hat{\xi})) \leq \tau^{-\text{len}(\hat{\xi})} = \text{dist}_T(\xi, \xi') .
\]

If one considers a measure \( \mu \in \mathcal{M}(L_T) \), then the pushforward \( \beta \# \mu \) gives a canonical way of transporting that measure to \( X \). We will want to think more generally about measures \( \mu \in \mathcal{M}(V_T) \) that also allow mass to sit at internal nodes of \( T \). But for each finite chain \( \xi \in V_T \), we have only an associated subset \( b(\xi) \subseteq X \), and indeed, the intent is that mass at \( \xi \) is only “specified” up to membership in \( b(\xi) \).

Thus for \( \mu \in \mathcal{M}(V_T) \), we define the \textit{coarse projection} of \( \mu \) as the map \( \Pi_X \mu : 2^X \to \mathbb{R}_+ \) defined by

\[
\Pi_X \mu(S) := \sum_{\xi \in V_T : b(\xi) \subseteq S} \mu(\xi) .
\]

Such an infinite sum is well-defined precisely because the measure \( \mu \) has at most countable support. The quantity \( \Pi_X \mu(S) \) measures the amount of projected mass that falls “inside \( S \).” Note that \( \Pi_X \mu \) is not a measure, but it is a \textit{supermeasure} in the sense that \( \Pi_X \mu(S \cup S') \geq \Pi_X \mu(S) + \Pi_X \mu(S') \) for all disjoint \( S, S' \subseteq X \). Moreover, if \( \mu \in \mathcal{M}(L_T) \), then \( \Pi_X \mu = \beta \# \mu \). The measure \( \Pi_X \mu \) is “coarsely subadditive” at the correct scale, which the following lemma formalizes.
Lemma 2.3 (Coarse subadditivity). If \( \xi \in \mathcal{V}_T \), then
\[
\Pi_X \mu(b(\xi)) \geq \mu(\mathcal{V}_T(\xi)).
\]
Moreover, if one defines \( \mu^{\geq j} \in \mathbb{M}(\mathcal{V}_T) \) by
\[
\mu^{\geq j}(\xi) := \mu(\xi) \mathbb{1}_{\{\text{len}(\xi) \geq j\}} \quad \forall \xi \in \mathcal{V}_T,
\]
then for any \( S_1, \ldots, S_m \subseteq X \):
\[
\Pi_X \mu^{\geq j}(S_1 \cup \cdots \cup S_m) \leq \sum_{i=1}^m \Pi_X \mu^{\geq j}(B_X(S_i, \tau^{-j})).
\]

The proof follows immediately from the definition of \( \Pi_X \) and the fact that \( \text{len}(\xi) \geq j \) implies \( \text{diam}_X(b(\xi)) \leq \tau^{-j} \).

Fusion maps and canonical injections. For \( \xi \in \mathcal{V}_T \), define
\[
\mathcal{V}_T(\xi) := \{ \xi' \in \mathcal{V}_T : \xi \text{ is a prefix of } \xi' \}
\]
\[
\mathcal{L}_T(\xi) := \mathcal{V}_T(\xi) \cap \mathcal{L}_T.
\]
Consider \( j \geq 1 \) and siblings \( \xi, \xi' \in \mathcal{V}_T^j \) with \( b(\xi) \subseteq b(\xi') \). Then there is a canonical mapping \( \varphi_{\xi \leftarrow \xi'} : \mathcal{V}_T \to \mathcal{V}_T \) defined as follows: \( \varphi_{\xi \leftarrow \xi'}|_{\mathcal{V}_T(\xi)} \) is the identity, and
\[
\langle \xi_0, \xi_1, \ldots, \xi_{j-1}, (b(\xi); \eta(\xi)), \xi_{j+1}, \xi_{j+2}, \ldots \rangle \in \mathcal{V}_T(\xi)
\]
is mapped to
\[
\langle \xi_0, \xi_1, \ldots, \xi_{j-1}, (b(\xi'); \eta(\xi')), \xi_{j+1}, \xi_{j+2}, \ldots \rangle \in \mathcal{V}_T(\xi').
\]
We refer to \( \varphi_{\xi \leftarrow \xi'} \) as the canonical injection of \( \xi \) into \( \xi' \). A map \( \varphi : \mathcal{V}_T \to \mathcal{V}_T \) is called a fusion map if it is the composition of finitely many canonical injections (in particular, the identity map is a fusion map).

The importance of fusion maps is encapsulated in the following lemma. It asserts that transporting a leaf measure under a fusion map does not induce movement when the measure is pushed from \( \mathcal{L}_T \) to \( X \). Its truth is immediate from the fact that if \( \varphi \) is a fusion map, then for every \( \xi \in \mathcal{L}_T \), \( \beta(\varphi(\xi)) = \beta(\xi) \).

Lemma 2.4. If \( \mu \in \mathbb{M}(\mathcal{L}_T) \) and \( \varphi \) is a fusion map, then \( \beta \# \varphi \# \mu = \beta \# \mu \).

Remark 2.5 (Tree terminology). Despite the orientation of trees found in nature, we will sometimes informally refer to the root as at the “top” of the tree and the leaves at the “bottom.”

2.2 Online HST embeddings

Let \( \mathbb{T} \) denote the universal \( \tau \)-HST for \( (X, d_X) \) and some \( \tau \geq 6 \). A stochastic HST embedding from \( X \) into \( \mathbb{T} \) is a random \( \mathcal{F} \)-adapted sequence \( \alpha = \langle \alpha_t : X \to \mathcal{L}_{\mathbb{T}}^0 \mid t \geq 0 \rangle \) such that with probability one:
\[
b(\alpha_t(x)) = \{x\} \quad \forall x \in X, t \geq 0
\]
\[
\alpha_t(\sigma_t) \in \mathcal{L}_{\mathbb{T}}^0 \quad \forall t \geq 1.
\]
This yields a (random) request sequence \( \alpha(\sigma) = \langle \alpha_1(\sigma_1), \alpha_2(\sigma_2), \ldots \rangle \) in \( \mathcal{L}_{\mathbb{T}}^0 \). We remark that requests are restricted to lie in the 0-decorated leaves simply because we will use the decorations for “bookkeeping.”
The cost modulo fusion. We will consider fractional $k$-server algorithms $\mu$ with $\mu_t \in M_k(V_T)$. While a $k$-server algorithm for $\alpha(\sigma)$ never needs to place mass at the internal nodes $V_T \setminus L_T$, (since all the requests lie in $L^0_T \subseteq L_T$), it will be helpful to allow this.

For two measures $\mu, \mu' \in M(V_T)$, say that $\mu$ dominates $\mu'$ if the following are satisfied:

1. $\mu(V_T) = \mu'(V_T)$.
2. $\mu(V_T(\xi)) \leq \mu'(V_T(\xi))$ for all $\xi \in V_T$.

Let $\|\mu\|$ denote the collection of all measures $\mu' \in M(V_T)$ such that $\mu$ dominates $\mu'$. Note that if $\mu' \in \|\mu\|$, then $\mu'$ can be obtained from $\mu$ by pushing mass “down the tree” (recall Remark 2.5). It is helpful to observe that if $\mu$ is supported on $L_T$, then $\|\mu\| = \{\mu\}$ (in intuitive terms, mass supported on $L_T$ cannot be “pushed down” any further).

Lemma 2.4 motivates the following notion of cost in which fusions are “free.” Consider an $F$-adapted sequence of measures:

$$\mu = \{\mu_t \in M_k(V_T) : t > 0\}.$$  

Let us define the reduced cost

$$\text{cost}^F_{**}(\mu) := \sum_{t \geq 0} \inf \{W^1_{**}(\mu, \mu_{t+1}) : \mu \in \|\phi_t \# \mu_t\|, \phi_t \text{ a fusion map}\}.$$  

One can think of the preceding definition as follows: When moving from $\mu_t$ to $\mu_{t+1}$, without incurring movement cost, we are allowed to first apply a fusion map, and then “push down” the resulting measure. The next result is proved in Section 5.1.

**Theorem 2.6 (Online rounding under fusions).** There exists a random integral $F$-adapted sequence $\nu = \{\nu_t \in M_k(X) : t \geq 0\}$ such that

$$\mu_t(\ell) \geq 1 \implies \nu_t(\beta(\ell)) \geq 1 \quad \forall \ell \in L_T, t \geq 1,$$

and

$$\mathbb{E} [\text{cost}_X(\nu)] \leq O(1) \cdot \text{cost}^F_{**}(\mu).$$

One might wonder why we defined a stochastic HST embedding using mappings $\alpha_t : X \to L_T$ when we can generate the request sequence knowing only the values $\alpha_t(\sigma_t)$. This is motivated by the following definition. Say that $\alpha$ has reduced distortion at most $D$ if, for every offline integral $k$-server algorithm $\nu = \{\nu_t \in M_k(X) : t > 0\}$ that services $\sigma$, it holds that

$$\mathbb{E} [\text{cost}^F_{**}(\nu(\alpha_0 \# v_0, \alpha_1 \# v_1, \alpha_2 \# v_2, \ldots))] \leq D \cdot \text{cost}_X(\nu) + O(1).$$

Suppose also that we have a $k$-server algorithm $\mu$ of the form (2.4) that services $\alpha(\sigma)$, and that $\mu$ is $C$-competitive in the sense that for every request sequence $\sigma$ and offline $k$-server algorithm $\nu$ for $\sigma$:

$$\mathbb{E} [\text{cost}^F_{**}(\mu)] \leq C \cdot \mathbb{E} [\text{cost}^F_{**}(\nu(\alpha_0 \# v_0, \alpha_1 \# v_1, \alpha_2 \# v_2, \ldots))] + O(1).$$

Then by taking $\nu = \nu^*$ to be an optimal offline algorithm for $\sigma$, Theorem 2.6 yields an $O(CD)$-competitive $k$-server algorithm for $(X, d_X)$.

Unfortunately, we will not be able to separate the analysis quite so nicely into the two pieces (2.5) and (2.6), but these inequalities represent the overarching philosophy of our approach. In particular, we compare our reduced cost to that of the $k$-server algorithm $\{\alpha_t \# v_t^* : t > 0\}$ for request sequence $\alpha(\sigma)$, where $v^*$ is the optimal offline algorithm for $\sigma$ in $(X, d_X)$. In order to conclude that some variant of (2.6) holds, we now assume the existence of an HST algorithm that satisfies a certain set of axioms.
2.3 The potential axioms

We will assume we have a fractional $k$-server algorithm that operates in the following way. It has a configuration space $\Gamma$ and a transition function $\gamma : \Gamma \times \mathcal{L}_T^0 \to \Gamma$. Every configuration $\chi \in \Gamma$ has a corresponding fractional server measure $\mu^{\chi} \in \mathcal{M}_k(V_T)$. Upon receiving a request $\sigma \in \mathcal{L}_T^0$, the algorithm updates its configuration to $\chi' := \gamma(\chi, \sigma)$ such that $\mu^{\chi'}(\sigma) \geq 1$. Moreover, there is a potential function $\Phi : \mathcal{M}_k(L_T) \times \Gamma \to \mathbb{R}$ that satisfies the following assumptions.

(A0) **Bounded $\Phi$-diameter.** For every $\chi \in \Gamma$ and $\theta \in \mathcal{M}_k(L_T)$:

$$|\Phi(\theta; \chi)| \leq K_\Phi < \infty$$

for some number $K_\Phi > 0$.

(A1) **Movement of the “optimum” cannot increase the potential too much.** For any states $\theta, \theta' \in \mathcal{M}_k(L_T)$ and configuration $\chi \in \Gamma$:

$$|\Phi(\theta; \chi) - \Phi(\theta'; \chi)| \leq f_1(k)W_T^1(\theta, \theta').$$

In other words, $\Phi$ is $f_1(k)$-Lipschitz in its first coordinate.

(A2) **Movement of the algorithm decreases the potential.** For every $\sigma \in \mathcal{L}_T$ and $\theta \in \mathcal{M}_k(L_T)$ satisfying $\theta(\sigma) \geq 1$, the following holds. Denoting $\chi' := \gamma(\chi, \sigma)$, we have

$$\Phi(\theta; \chi') - \Phi(\theta; \chi) \leq -\frac{W_T^1(\mu^{\chi}, \mu^{\chi'})}{f_2(k)}.$$

(A3) **Fusion is free.** For any fusion map $\varphi$ and configuration $\chi \in \Gamma$, there is a configuration $\chi' \in \Gamma$ such that $\mu^{\chi'} \in \|\varphi\#\mu^{\chi}\|$, and moreover

$$\Phi(\varphi \# \theta; \chi') \leq \Phi(\theta; \chi) \quad \forall \theta \in \mathcal{M}_k(L_T).$$

(2.7)

If one thinks of $\Phi(\theta; \chi)$ as the “discrepancy” between $\theta$ and $\mu^{\chi}$, then fusion corresponds to coarsening the discrepancy measure, which should make them appear more similar (hence the inequality in (2.7)).

(A4) **Fission is charged locally.** Consider $\xi^0 \in V_T^j$ and a child $\xi^1 \in V_T^{j+1}$. Let $F : \mathcal{L}_T \to \mathcal{L}_T$ be any mapping that satisfies $F(\xi) = \xi$ for $\xi \notin \mathcal{L}_T(\xi^1)$ and $F(\mathcal{L}_T(\xi^1)) \subseteq \mathcal{L}_T(\xi^0)$. Then for any $\chi \in \Gamma$ and $\theta \in \mathcal{M}_k(L_T)$, it holds that

$$\Phi(F\# \theta; \chi) - \Phi(\theta; \chi) \leq f_3(k)\tau^{-j}\mu^{\chi}(V_T(\xi^1)).$$

This says that moving the $\theta$-mass on $\xi^1$ arbitrarily underneath $\xi^0$ affects the potential by a controlled amount. Note that (1) would give $f_1(k)\tau^{-j} |\Phi(L_T(\xi^1))|$ on the RHS since $diam(\mathcal{L}_T(\xi)) \leq \tau^{-j}$, but this control is in terms of $\mu^{\chi}(V_T(\xi^1))$.

The algorithm of [BCL+17] achieves these with $f_1(k), f_2(k), f_3(k) \leq O(\log k)$ and $K_\Phi \leq O(k \log k)$. Axioms (A0), (A1), and (A2) arise naturally in their analysis; see Theorem 5.5. Axioms (A3) and (A4) are verified in Section 5.2.

**Theorem 2.7.** If there exists a transition function $\gamma : \Gamma \times \mathcal{L}_T^0 \to \Gamma$ and a potential $\Phi$ satisfying Axioms (A0)–(A4) for some functions $f_1(k), f_2(k), f_3(k) \leq (\log k)^{O(1)}$, then there is an $O((\log^2 k)f_1(k)f_2(k)^2f_3(k))$-competitive algorithm for the $k$-server problem on $(X, d_X)$.

**Corollary 2.8.** There is an $O((\log k)^6)$-competitive algorithm for the $k$-server problem on any bounded metric space.

We refer to Section 5.2.1 for a straightforward reduction from bounded metric spaces to the general case.
Sensible algorithms. Say that a fractional $k$-server algorithm $\mu = \{\mu_t : t \geq 0\}$ is sensible if $\text{supp}(\mu_t) \subseteq V^0_t$ for all $t \geq 0$. Since all our requests will occur in $L^0_t$ (by (2.3)), we will assume that (A0)–(A4) are satisfied by a sensible algorithm:

$$\text{supp}(\mu^t(x,\sigma)) \subseteq V^0_t \quad \forall x \in \Gamma, \sigma \in L^0_t.$$  

(2.8)

This property is true of the [BCL+17] algorithm (see Theorem 5.5), but it is also without loss of generality.

3 Construction of the embedding

We will now describe a stochastic HST embedding $\alpha = \{\alpha_t : X \rightarrow L_t \mid t \geq 0\}$ and a random fractional $k$-server algorithm $\mu = \{\mu_t \in M_k(V_t) : t \geq 0\}$ for $\alpha(\sigma)$. We define these inductively and denote, for $t \geq 0$, the coarse projections:

$$v_t := \Pi_X \mu_t \quad \bar{v}_t := E[v_t].$$

Let $\nu^*$ denote an optimal offline integral $k$-server algorithm for $\sigma$ in $(X, d_X)$. Denote by $\mu^* = \{\mu^*_t : t \geq 0\}$ the pushforward $\mu^*_t := \alpha_t \# v^*_t$. Our goal is to eventually prove the following.

Theorem 3.1. Under the assumptions of Theorem 2.7, for some number $c = c(\rho_0)$ depending possibly on the initial configuration $\rho_0$, and all request sequences $\sigma$, it holds that:

$$E[\text{cost}^t_\mu(\mu)] \leq O((\log k)^2) f_1(k) f_2(k) f_3(k) \text{cost}_X(\nu^*) + c.$$  

Combined with Theorem 2.6, this yields Theorem 2.7.

Remark 3.2. We remark that the additive constant $c(\rho_0)$ in Theorem 3.1 is $O(K_\rho + k \cdot \text{diam}(\rho_0))$ where $\text{diam}(\rho_0)$ is the maximum distance between two servers in $\rho_0$. Plugging in the algorithm of [BCL+17] gives $c(\rho_0) \leq O(k \log k) \cdot \text{diam}(\rho_0)$.

3.1 Embedding components

We first describe some primitives that will be used in the construction of the stochastic HST embedding $\alpha$.

3.1.1 Carving out semi-partitions

A semi-partition $P$ of $X$ is a collection of pairwise disjoint subsets of $X$. For such a semi-partition, denote

$$\Delta_P(x, y) := \sum_{S \in P} |1_S(x) - 1_S(y)|.$$  

Define $[P] \subseteq X$ by $[P] := \bigcup_{S \in P} S$. We will sometimes think of $P$ as a function that takes $x \in [P]$ to the unique set $P(x) \in P$ containing $x$. If $x \notin [P]$, we take $P(x) := \emptyset$. If $P, P'$ are two semi-partitions, say that $P$ is a refinement of $P'$ if for every $S \in P$, there is an $\hat{S} \in P'$ such that $S \subseteq \hat{S}$. Say that $\hat{P}$ is $\Delta$-bounded if $S \subseteq \hat{P} \implies \text{diam}_X(S) \leq \Delta$.

Consider a triple $(C, R, \pi)$ where $C \subseteq X$ is a finite set, $R : C \rightarrow \mathbb{R}_+$, and $\pi : |C| \rightarrow C$ is a bijection. This defines a semi-partition into at most $|C|$ sets by iteratively carving out balls:

$$\hat{P}(C, R, \pi) := \left\{ B_X(\pi(i), R(\pi(i))) \setminus \bigcup_{h \neq i} B_X(\pi(h), R(\pi(h))) : i = 1, 2, \ldots, |C| \right\}.$$  

By construction, $\hat{P}(C, R, \pi)$ is $(2 \max_{x \in C} R(x))$-bounded.
3.1.2 Heavy nets

Let $\lambda := \max(36, \tau)^2$ and $0 < \delta < 1/2$. We will choose $\delta$ later so that $\delta \approx (f_2(k)f_3(k))^{-1}$. If $B = B_X(x, r)$ is a ball in $X$, we denote $\lambda B := B_X(x, \lambda r)$. We assume that a ball $B$ comes equipped implicitly with both a center $x \in X$ and a radius $r \geq 0$; we will denote the latter quantity by $\text{rad}_X(B)$.

Say that a ball $B \subseteq X$ is $t$-heavy if

$$
\bar{\nu}_t(B) \geq (1 - \delta)\bar{\nu}_t(\lambda B).
$$

(3.1)

A set $\Lambda \subseteq X$ is called a $t$-heavy $r$-net if it satisfies

$$
B \text{ is } t\text{-heavy and } \text{rad}_X(B) \leq r \implies d_X(B, \Lambda) \leq r \sqrt{\lambda},
$$

(3.2)

and

$$
x \neq y \in \Lambda \implies d_X(x, y) > 3r.
$$

(3.3)

3.1.3 Cluster fusion

Given a semi-partition $\hat{P}$, a finite set of representatives $\Lambda \subseteq X$, and a radius $r > 0$, we now define the $r$-fusion of $\hat{P}$ along $\Lambda$ as follows. For $x \in \Lambda$, define

$$
U_x := B_X(x, r) \cup \bigcup_{S \in \hat{P} : B_X(x, r) \cap S \neq \emptyset} S.
$$

(3.4)

See Figure 2.

Define the set of fused clusters:

$$
\mathcal{H}(\hat{P}, \Lambda, r) := \{U_x : x \in \Lambda\},
$$

(3.5)

and the semi-partition (cf. Lemma 3.3) of fused and unfused clusters:

$$
\hat{Q}(\hat{P}, \Lambda, r) := \mathcal{H}(\hat{P}, \Lambda, r) \cup \{S \in \hat{P} : \exists S \cap \mathcal{H}(\hat{P}, \Lambda, r) = \emptyset\}.
$$

The idea here is that in passing from $\hat{P}$ to $\hat{Q}$, all the sets $S \in \hat{P}$ that intersect some ball $B_X(x, r)$ for $x \in \Lambda$ are “fused” into a single set $U_x$. (For technical reasons—see Lemma 4.3 below—the ball itself is also fused in.)

**Lemma 3.3.** If $\hat{P}$ is $\Delta$-bounded and $\Lambda$ is $(r + \Delta)$-separated, then $\hat{Q}(\hat{P}, \Lambda, r)$ is a $2(r + \Delta)$-bounded semi-partition.

**Proof.** Observe that $\text{diam}_X(U_x) \leq 2(r + \Delta)$. Moreover, every $y \in U_x$ satisfies $d_X(x, y) \leq r + \Delta$, hence if $\Lambda$ is $(r + \Delta)$-separated, then the sets $\{U_x : x \in \Lambda\}$ are pairwise disjoint. \(\square\)
3.1.4 Refinement and HST embeddings

Consider now a sequence $\hat{Q} = \{\hat{Q}^j : j \in \mathbb{Z}_+\}$ of semi-partitions of $X$ such that $\hat{Q}^0 = \{X\}$ and $\hat{Q}^j$ is $\tau^{-j}$-bounded for all $j \geq 1$. (3.6)

We use these to define a sequence $Q = \{Q^j : j \in \mathbb{Z}_+\}$ of successively refined full partitions of $X$ as follows.

First, we complete each semi-partition to a full partition $\bar{Q}^j$ by adding singleton clusters:

$\bar{Q}^j := \hat{Q}^j \cup \{\{x\} : x \in X \setminus [\hat{Q}^j]\}$ $\forall j \in \mathbb{Z}_+$.

Now we inductively define $Q^0 := \bar{Q}^0$ and for $j \geq 1$:

$Q^j := \{S \cap S' : S \in \bar{Q}^j, S' \in Q^{j-1}\}$.

This ensures that for each $j \in \mathbb{Z}_+$, $Q^{j+1}$ is a refinement of $Q^j$.

For $x \in X$, define

$\text{rank}^{\hat{Q}}(x) := \max \left\{ j \in \mathbb{Z}_+ : x \in \bigcap_{i < j} [\hat{Q}^i] \right\}$.

We can now define an embedding $\alpha^{\hat{Q}} : X \to \mathcal{L}_T$ by

$\alpha^{\hat{Q}}(x) := \left\langle (Q^0(x); 0), (Q^1(x); 0), \ldots, (Q^r(x); 0), (Q^{r+1}(x); 1), (Q^{r+2}(x); 1), \ldots \right\rangle$, (3.7)

where $r = \text{rank}^{\hat{Q}}(x)$.

One should verify that the latter sequence is indeed a decorated leaf chain by construction and (3.6). This is the only place that we make use of decorated chains in the proof of Theorem 2.7. The particular form of (3.7) will be employed to prove Lemma 4.15 which asserts that the $\Phi_t$ potential does not increase under insertions (essentially because we have assumed our algorithm is sensible, thus it does not place mass in subtrees with a non-zero decoration).

Later, we will use the following basic fact.

**Lemma 3.4.** For every sequence $\hat{Q}$ of semi-partitions:

$$\text{dist}_T(\alpha^{\hat{Q}}(x), \alpha^{\hat{Q}}(y)) \leq 2\tau \sum_{j \geq 1} \tau^{-j} \Delta^{\hat{Q}}(x, y) \quad \forall x, y \in X.$$  

**Proof.** Consider $x, y \in X$ and suppose that $\text{dist}_T(\alpha^{\hat{Q}}(x), \alpha^{\hat{Q}}(y)) = \tau^{-\ell}$ for some $\ell > 0$. It is straightforward to check that $\ell + 1 = \min \left\{ j : \Delta^{\hat{Q}}(x, y) > 0 \right\}$. \qed

3.1.5 Truncated exponential radii

For every $j \in \mathbb{Z}$, consider the probability distribution $\gamma_j$ with density:

$$d\gamma_j(r) := \frac{K\tau^j \log K}{K - 1} \exp(-r\tau^j \log K) 1_{[0, \tau^{-j}]}(r).$$

This is simply an exponential distribution truncated at $\tau^{-j}$. Bartal [Bar96] showed that such distributions are extremely useful in the construction of random HST embeddings.
Lemma 3.5. Consider a finite set $C \subseteq X$ and a permutation $\pi : |C| \to C$. Choose $\hat{R} : C \to \mathbb{R}_+$ so that \{\hat{R}(x) : x \in C\} are independent random variables with law $\gamma_j$, and define $R(x) := \hat{R}(x) + \tau^{-j}$. Then $\hat{P} := \hat{P}(C, R, \pi)$ is a $4\tau^{-j}$-bounded semi-partition with probability one, and moreover for every $x, y \in X$:

$$
\mathbb{P}[\Delta_{\hat{P}}(x, y) > 0] \leq O(\log(|C| + 1)) \, d_X(x, y) \tau^j.
$$

If $\Lambda \subseteq X$ is any $6\tau^{-j}$-separated set, then the $2\tau^{-j}$-fusion of $\hat{P}$ along $\Lambda$:

$$
\hat{Q} := \hat{Q}(\hat{P}, \Lambda, 2\tau^{-j})
$$

is a $12\tau^{-j}$-bounded semi-partition of $X$. If $d_X(x, C \cup \Lambda) \leq \tau^{-j}$ and $y \in X$, then:

$$
\mathbb{P}[\Delta_{\hat{Q}}(x, y) > 0] \leq O(\log(|C| + 1)) \, d_X(x, y) \tau^j.
$$

Proof. The fact that $\hat{P}$ is a $4\tau^{-j}$ semi-partition follows immediately from the fact that $\gamma_j$ is supported on $[0, \tau^{-j}]$. Moreover, (3.8) is a standard calculation (see, e.g., [BCL+17, Lem 4.8]).

That $\hat{Q}$ is a $12\tau^{-j}$-bounded semi-partition follows from Lemma 3.3. Let us now verify (3.9). We may assume that $d_X(x, \Lambda) \leq \tau^{-j}$, else the claim is vacuous. If $d_X(x, \Lambda) \leq \tau^{-j}$, then $x, y \in B_X(z, 2\tau^{-j})$ for some $z \in \Lambda$, hence $x, y \in [\hat{Q}]$ and $\hat{Q}(x) = \hat{Q}(y)$ because $x, y \in U_z$ (recall (3.4)).

Now assume that $d_X(x, C) \leq \tau^{-j}$. Observe that in this case, $x \in [\hat{P}]$ with probability one and by construction of the fusion, $\Delta_{\hat{Q}}(x, y) \leq \Delta_{\hat{P}}(x, y)$, meaning that (3.9) follows from (3.8). \qed

3.2 The embedding algorithm

For $j, t \in \mathbb{Z}_+$, we will maintain several random $\mathcal{F}$-adapted sequences: Centers $C^j_t \subseteq X$, along with radii $R^j_t : C^j_t \to \mathbb{R}_+$, permutations $\pi^j_t : |C^j_t| \to C^j_t$, and $t$-heavy $\tau^{-j}$-nets $\Lambda^j_t$. These give rise to semi-partitions $\hat{P}^j_t := \hat{P}(C^j_t, R^j_t, \pi^j_t)$ and fusions $\hat{Q}^j_t := \hat{Q}(\hat{P}^j_t, \Lambda^j_{t-1}, 2\tau^{-j-1})$, along with embeddings

$$
\alpha_t := \alpha^\hat{Q}_t,
$$

where $\hat{Q}_t := \langle \hat{Q}^j_t : j \in \mathbb{Z}_+ \rangle$.

We will also maintain a sequence $\langle \chi_t : \Gamma : t \geq 0 \rangle$ of configurations. These yield a sequence $\mu = \langle \mu_t \in \mathcal{M}_k(V_t) : t \geq 0 \rangle$ of induced fractional $k$-server measures: $\mu_t := \mu^\chi_t$.

Initialization. For all $j \geq 1$: $C^j_0 := \emptyset$, $\hat{P}^j_0 := \emptyset$, $\Lambda^j_0 := \emptyset$. We also set $C^j_t := \emptyset$, $\Lambda^j_t := \emptyset$, and $\hat{P}^j_t := X$ for all $t \geq -1$. Let $\chi_0 \in \Gamma$ denote some initial configuration.

Request. Suppose we receive a request $\sigma_t \in X$ for some $t \geq 1$. For $j \geq 1$, denote

$$
I^j_t := \begin{cases} 1 & d_X(\sigma_t, C^j_{t-1} \cup \Lambda^j_{t-1}) > \tau^{-j-1} \\ 0 & \text{otherwise.} \end{cases}
$$

Deletions. In the next definition, $K \geq 1$ is a parameter that will be chosen later (our choice will satisfy $K \leq k^{O(1)}$). For every $j \geq 1$:

$$
C^j_{t, \text{del}} := \begin{cases} C^j_{t-1} & I^j_t = 0 \text{ or } |C^j_{t-1}| < K \\ C^j_{t-1} \setminus \{z^j_t\} & \text{otherwise}, \end{cases}
$$

where $z^j_t \in C^j_{t-1}$ is chosen uniformly at random. Denote $\hat{P}^j_{t, \text{del}} := \hat{P}(C^j_{t, \text{del}}, R^j_{t-1}, \pi^j_{t-1})$. 18
Fission. Denote
\[ \hat{Q}_{t,\text{fis}}^j := \hat{Q} \left( \hat{p}_{t,\text{del}}^j, \Lambda_{t-1}^j \cap \Lambda_{t-2}^j, 2\tau^{-j-1} \right). \]
This is the semi-partition \( \hat{p}_{t,\text{del}}^j \) fused only along the centers that survive from time \( t - 2 \) to \( t - 1 \).

Insertions. For every \( j \geq 1 \), if \( I_j^t = 1 \), we define:
\[
\begin{align*}
C_j^t &:= C_{t,\text{del}}^j \cup \{ \sigma_t \}, \\
\pi_j^t(C_j^t) &:= \sigma_t, \\
R_j^t(\sigma_t) &:= \tau^{-j-1} + Z_j^t,
\end{align*}
\]
where \( Z_j^t \) is sampled independently with law \( \gamma_{j+1} \). If \( I_j^t = 0 \), then \( C_j^t := C_{t-1}^j \).

In either case, we define \( \pi_j^t \) so that it induces the same ordering on \( C_j^t \setminus \{ \sigma_t \} \) as \( \pi_{j-1}^t \), and \( R_j^t \) so that \( R_j^t|_{C_j^t \setminus \{ \sigma_t \}} = R_{j-1}^t|_{C_{j-1}^t \setminus \{ \sigma_t \}} \).

Fusion. Consider the semi-partition \( \hat{Q}_j^t = \hat{Q}(\hat{p}_j^t, \Lambda_{t-1}^j, 2\tau^{-j-1}) \) and its prefused version:
\[
\hat{Q}_{t,\text{pre}}^j := \hat{Q}_{t,\text{fis}}^j \cup \{ \hat{p}_j^t(\sigma_t) \} \cup \{ B_X(x, 2\tau^{-j-1}) \cup \hat{Q}_{t,\text{fis}}^j : x \in \Lambda_{t-1}^j \setminus \Lambda_{t-2}^j \}.
\]
We have \( [\hat{Q}_j^t] = [\hat{Q}_{t,\text{pre}}^j] \) and \( \hat{Q}_{t,\text{pre}}^j \) is a refinement of \( \hat{Q}_j^t \) by construction. Thus we ran realize \( \hat{Q}_t \) from \( \hat{Q}_{t,\text{pre}}^j \) via an iterative merging of pairs of siblings. Note that this can be expressed as a composition of canonical injections; to merge siblings \( \xi, \xi' \in V_j^t \) with \( \text{diam}_X(b(\xi) \cup b(\xi')) \leq \tau^{-j} \), we fuse \( \xi \) and \( \xi' \) into their common 0-decorated sibling \( b(\xi) \cup b(\xi'), 0 \in V_j^t \). Let \( \varphi_t \) denote the corresponding fusion map (recall that a fusion map is a composition of canonical injections). Using Axiom (A3), this yields a configuration \( \tilde{x}_{t-1} \) such that \( \mu^{x_{t-1}} \in \| \varphi_t \| \mu^{\xi_{t-1}} \) and (2.7) is satisfied.

HST evolution. We update the configuration:
\[
\chi_t := \gamma(\tilde{x}_{t-1}, 0, (\sigma_t)).
\]

Heavy net maintenance. Now we specify how to update \( \Lambda_{t-1}^j \) to \( \Lambda_t^j \).

For \( j = 1, 2, \ldots \), do the following:

1. Set \( \tilde{\Lambda}_t^j := \Lambda_{t-1}^j \).
2. While there is some \( t \)-heavy ball \( B \) with \( \text{rad}_X(B) \leq \frac{\tau^{-j}}{2\sqrt{A}} \) and \( d_X(B, \tilde{\Lambda}_t^j) > \frac{\tau^{-j}}{\sqrt{A}} \):
   (a) Remove from \( \tilde{\Lambda}_t^j \) all \( y \in X \) such that \( d_X(y, B) < \frac{\sqrt{\tau^{-j}}}{2} \).
   (b) If \( x \) is the center of \( B \), add it: \( \tilde{\Lambda}_t^j := \tilde{\Lambda}_t^j \cup \{ x \} \).
3. Set \( \Lambda_t^j := \tilde{\Lambda}_t^j \).
4 Analysis

Let us first verify a few basic properties of the embedding algorithm from Section 3.2.

Lemma 4.1. Assume that \( \tau \geq 12 \) and \( \lambda \geq 36 \). Then for each \( j \geq 1 \) and \( t \geq 1 \), it holds that

1. \( \Lambda_j^t \) is a \( t \)-heavy \( \tau^{-j} \)-net.
2. \( \hat{Q}_j^t \) is a \( \tau^{-j} \)-bounded semi-partition.

Proof. \( \Lambda_j^t \) is explicitly constructed to satisfy (3.2) and (3.3) with \( r = \tau^{-j} \) as long as \( \lambda \geq 36 \). We need to verify that the construction is well-defined, i.e., that the loop defining \( \Lambda_j^t \) always terminates.

To prove this, it suffices to show that if \( y \in X \) is removed in step 2(a), then \( B' = B_X(y, \frac{r}{2}\sqrt{\lambda}) \) is not \( t \)-heavy. To that end, it suffices to show that there cannot be two \( t \)-heavy balls \( B, B' \) in \( X \) satisfying

\[
\frac{\sqrt{\lambda}}{3} \tau^{-j} \geq d_X(B, B') > \frac{\tau^{-j}}{\sqrt{\lambda}} \text{ and } \text{rad}_X(B), \text{rad}_X(B') \leq \frac{\tau^{-j}}{2\sqrt{\lambda}}.
\]

Note that under these assumptions, \( B \cap B' = \emptyset \), but \( \lambda B \supseteq B' \) and \( \lambda B' \supseteq B \). Therefore it cannot be that both \( B \) and \( B' \) are \( t \)-heavy as long as \( \delta < 1/2 \) (recall (3.1)).

Now the fact that \( \hat{Q}_j^t \) is a \( \tau^{-j} \)-bounded semi-partition follows from Lemma 3.5. \( \square \)

We want to distinguish two types of randomness used in the algorithm. There is the probability space underlying the choice of elements \( z_j^t \) in the deletion step which we denote by \( \Omega^{\text{del}} \). All other randomness is denoted by \( \Omega^{\text{hst}} \).

Fact 4.2. The random variables \( C_j^t \) and \( \Lambda_j^t \) are independent of \( \Omega^{\text{hst}} \). Note that \( \Lambda_j^t \) depends on the average coarse projection \( \bar{v}_t \), but this is constructed by averaging over \( \Omega^{\text{hst}} \).

4.1 Interesting scales

Define the isolation radius of \( x \in X \) (at time \( t \)) by

\[
\rho_t(x) := \sup \{ r : \bar{v}_t(B_X(x, r)) < 1/2 \}.
\]

Say that a point \( x \in X \) is \((j, t)\)-heavy if \( d_X(x, \Lambda_j^{t-1}) \leq \tau^{-j-1} \). If \( d_X(x, \Lambda_j^{t-1}) \geq \frac{1}{2} \tau^{-j-1} \), we say that \( x \) is \((j, t)\)-light. (A point can be both heavy and light.) We record a fact that follows from our construction of \( \hat{Q}_t \) (cf. (3.4)).

Lemma 4.3. If \( x \in X \) is \((j, t)\)-heavy, then \( B_X(x, \tau^{-j-1}) \subseteq \hat{Q}_j^t(x) \).

Denote \( \eta := (32f_1(k)f_2(k))^{-1} \); we may assume that \( \eta \geq (\log k)^{-O(1)} \). We now define a subset \( J_t(x) \subseteq \mathbb{Z}_+ \) of “interesting” scales for a given \( x \in X \):

\[
\mathbb{L}_t(x) := \{ j \in \mathbb{Z}_+ : \text{ is } (j, t)\text{-light} \},
\]

\[
J_t(x) := \{ j \in \mathbb{Z}_+ : \tau^{-j} > \eta \rho_t(x) \} \cap \mathbb{L}_t(x).
\]

The next lemma is an essential component of all our arguments: For every \( x \in X \), there are only \( O(\frac{1}{\delta} \log k) \) interesting scales.

Lemma 4.4. For every \( x \in X \) and \( t \geq 0 \),

\[
|J_t(x)| \leq O\left(\frac{\log k}{\delta} + \frac{\log \frac{1}{\eta}}{\delta} \right) \leq O\left(\frac{\log k}{\delta} \right).
\]

20
Proof. If \( x \) is \((j, t)\)-light, it means that \( B_X\left(x, \frac{\tau^j}{2\sqrt{\lambda}}\right) \) is not \( t \)-heavy, which means that
\[
\bar{v}_t\left(B_X\left(x, \frac{\tau^{-j}}{2\sqrt{\lambda}}\right)\right) < (1 - \delta)\bar{v}_t\left(B_X\left(x, \frac{\sqrt{\lambda}}{2} \tau^{-j}\right)\right) \leq (1 - \delta)\bar{v}_t\left(B_X\left(x, \tau^{-j}\right)\right).
\]
Since \( \bar{v}_t(x) = k \) and \( \bar{v}_t\left(B_X(x, \rho_t(x))\right) \geq 1/2 \), the result follows using \( \lambda, \tau \leq O(1) \) and the fact that there are only \( O(\log \frac{1}{\eta}) \) additional scales between \( \eta \rho_t(x) \) and \( \rho_t(x) \).

\[\square\]

### 4.2 Auxiliary potential functions

We need two additional potential functions. The “accuracy potential” \( \Psi_t^A \) will help us track the cost of insertions and deletions. It measures how accurately the tree structure induced by the partitions represents the fractional server measure. One could effectively ignore \( \Psi_t^A \) upon a first reading; using a cruder bound, one loses an \( O(\log \log \mathcal{A}_X) \) factor in the competitive ratio. The “fission potential”—which is central to our approach—will allow us to track the cost of breaking previously fused clusters.

For \( \xi = (\xi_0, \xi_1, \ldots) \in \mathcal{V}_T \) and \( j \in \mathbb{Z}_+ \), define
\[
\mathcal{b}_j(\xi) = \begin{cases} 
0 & \text{len}(\xi) < j \\
\mathcal{b}\left((\xi_1, \ldots, \xi_j)\right) & \text{otherwise.}
\end{cases}
\]

For \( j \in \mathbb{Z}_+ \) and \( x, y \in X \), denote
\[
d^j_X(x, y) := \min\left(\tau^{-j}, d_X(x, y)\right).
\]

**Lemma 4.5.** For every subset \( S \subseteq X \) and \( j \in \mathbb{Z}_+ \), the map \( \xi \mapsto d^j_X(\mathcal{b}_j(\xi), S) \) is \( \tau^{-2} \)-Lipschitz on \((\mathcal{V}_T, \text{dist}_T)\).

Proof. Consider \( \xi, \xi' \in \mathcal{V}_T \). If \( \mathcal{b}(\xi) \neq \mathcal{b}(\xi') \), then \( \text{dist}_T(\xi, \xi') \geq \tau^{-j-2} \), completing the proof. \[\square\]

#### The accuracy potential

For \( \mu \in \mathcal{M}(\mathcal{V}_T) \), and sequences of finite subsets \( C = \langle C^j \subseteq X : j \geq 1 \rangle \) and \( \Lambda = \langle \Lambda^j \subseteq X : j \geq 1 \rangle \), we define:
\[
\Psi_t^A(\mu; C, \Lambda) := \sum_{\xi \in \mathcal{V}_T} \mu(\xi) \sum_{j \geq 1} \left( d^j_X(\mathcal{b}_{j+2}(\xi), C^j) \cdot \tau^j \left( d^j_X(\mathcal{b}_{j+2}(\xi), \Lambda^j) - \eta \max_{x \in \mathcal{b}_{j+2}(\xi)} \rho_{t-1}(x) - \frac{1}{2} \tau^{-j-1} \right) \right) + \tag{4.1}
\]
\[
\Psi_t^A := \Psi_t^A(\mu_t; C_t, \Lambda_{t-1}),
\]
where \( C_t = \langle C^j_t : j \geq 1 \rangle \) and \( \Lambda_t = \langle \Lambda^j_t : j \geq 1 \rangle \).

**Lemma 4.6.** If \( \mu' \in \mathcal{M}[\mu] \), then
\[
\Psi^A(\mu'; C, \Lambda) \leq \Psi^A(\mu; C, \Lambda).
\]
If \( \varphi \) is a fusion map, then
\[
\Psi^A(\varphi \# \mu; C, \Lambda) \leq \Psi^A(\mu; C, \Lambda).
\]

Proof. For the first inequality, note that if mass is pushed down the tree from \( \xi \) to a descendant \( \xi' \), then \( \mathcal{b}_j(\xi) \subseteq \mathcal{b}_j(\xi') \) for all \( j \in \mathbb{Z}_+ \), and thus the corresponding terms in (4.1) decrease. For the second, note that if \( \xi \) is fused into \( \xi' \), then by definition, we have \( \mathcal{b}_j(\xi) \subseteq \mathcal{b}_j(\xi') \) for all \( j \in \mathbb{Z}_+ \) as well. \[\square\]
Remark 4.7 (Accuracy potential). Recall that an insertion occurs at level $j$ when $d_X(\sigma_t, C^j_t \cup \Lambda^j_{t-1}) > \tau^{-j-1}$. Such an insertion does not increase the potential $\Phi$ (see Lemma 4.15), but it triggers a level-$j$ deletion which might adversely increase $\Phi$. The potential $\Psi^A_t$ measures how accurately the sets $C^j_t \cup \Lambda^j_{t-1}$ approximate the coarse projection $\Pi_X \mu_t$.

We know that the underlying $k$-server algorithm satisfies $\mu_1(\sigma_1(\sigma_t)) \geq 1$, and therefore it should be that either the HST algorithm moves substantially in response to a level-$j$ insertion or the accuracy improves (because $\sigma_t \in C^j_t$), yielding a lower $\Psi^A_t$ value. This gain is used to charge the adverse effects of deletion against the movement of the HST algorithm.

Lemma 4.8. For every $t \geq 1$ and sequence $C$, the map $\mu \mapsto \psi^A_t(\mu; C, \Lambda_{t-1})$ is $O(\frac{1}{k} \log k)$-Lipschitz on $(\mathcal{M}(V_t), W^1_t)$.

Proof. Define

$$
\psi_t(\xi) := d_X(\xi, C^j_t \cup \Lambda^j_{t-1}) \cdot \tau^j \left( d_X(\xi, C^j_t \cup \Lambda^j_{t-1}) - \eta \max_{x \in b_{j+2}(\xi)} \rho_{t-1}(x) - \frac{1}{2} \tau^{-j-1} \right) .
$$

Consider any $\xi, \xi' \in V_t$ with $\operatorname{len}(\xi) \leq \operatorname{len}(\xi')$ and let $v' = s(1_\xi - 1_{\xi'})$ for some $s > 0$. Also fix $x \in b(\xi)$ and $y \in b(\xi')$ arbitrarily. Then:

$$
\frac{1}{s} |\psi^A(v + v'; C, \Lambda_{t-1}) - \psi^A(v'; C, \Lambda_{t-1})| = \sum_{j = \operatorname{len}(\xi) + 2}^{\operatorname{len}(\xi')} \tau^{-j} + \sum_{j \geq \operatorname{len}(\xi')} |\psi_j(\xi) - \psi_j(\xi')| \\
\leq 2\tau^{-2} \operatorname{dist}_t(\xi, \xi') + \sum_{j \in J_{t-1}(x) \cup J_{t-1}(y)} |\psi_j(\xi) - \psi_j(\xi')| \\
\leq 2\tau^{-2} \operatorname{dist}_t(\xi, \xi') + O\left(\frac{1}{k} \log k\right) \sup_{y \geq 1} |\psi_j(\xi) - \psi_j(\xi')| .
$$

where in the second inequality we have used the definition of $J_{t-1}$ and in the last inequality, Lemma 4.4.

Finally, note that for any $j \geq 1$, Lemma 4.5 implies that $\psi_j : V_t \rightarrow \mathbb{R}$ is the product of a $\tau^{-2}$-Lipschitz function and a $2\tau^{-j-2}$-Lipschitz function, yielding:

$$
|\psi_j(\xi) - \psi_j(\xi')| \leq 2\tau^{-2} \operatorname{dist}_t(\xi, \xi') \tau^{-j} + \tau^{-2} \operatorname{dist}_t(\xi, \xi') \leq 3\tau^{-2} \operatorname{dist}_t(\xi, \xi') .
$$

\[ \square \]

The fission potential. Recall (3.5) and denote

$$
\mathcal{H}^j_t := \mathcal{H}(\hat{P}^j_t, \hat{\Lambda}^j_{t-1}, 2\tau^{-j-1}) .
$$

Observe that $[\mathcal{H}^j_t] \subseteq [\hat{Q}^j_t]$ is the subset of points that participate in a fused cluster in $\hat{Q}^j_t$.

Given $\mu \in \mathcal{M}(V_t)$, a sequence $\hat{P} = \{\hat{P}^j : j \geq 1\}$ of semi-partitions of $X$, and $\Lambda = \{\Lambda^j \subseteq X : j \geq 1\}$ a sequence of finite subsets, define:

$$
\psi^F_t(\mu; \hat{P}, \Lambda) := - \sum_{\xi \in V_t} \mu(\xi) \sum_{j \geq 1} \tau^{-j} \left\{ |\mathcal{H}(\hat{P}^j_t, \Lambda^j \cup 2\tau^{-j-1})| \cap b_j(\xi) \neq \emptyset \right\} ,
$$

$$
\Psi^F_t := \psi^F_t(\mu; \hat{P}_t, \Lambda_{t-1}) = - \sum_{\xi \in V_t} \mu(\xi) \sum_{j \geq 1} \tau^{-j} \left\{ |\mathcal{H}^j_t \cap b_j(\xi) \neq \emptyset \right\} .
$$
Remark 4.9 (Fission potential). The $\Psi^F_t$ potential rewards us for fusing a cluster that contains significant $\mu_t$ mass (in a suitable coarse sense). This will pay for the adverse effects of fission on the $\Phi$ potential as long as when we unfuse clusters, we are always doing it in order to fuse new clusters with much greater mass. This is why we fuse near the centers of heavy balls (which triggers a fission in the “light” annuli around the heavy ball).

Since “heavy” is defined in terms of the average coarse projections $\tilde{v}_t = E[\Pi_x \mu_t]$, such a fusion will decrease $E[\Psi^F_t]$, but not necessarily $\Psi^F_t$ (for many choices of the underlying randomness $\Omega^{\text{hst}}$).

The proof of the next lemma uses the same argument as Lemma 4.6.

Lemma 4.10. If $\mu' \in \mathcal{M}$, then

$$\psi^F(\mu'; \hat{\mathcal{P}}, \Lambda) < \psi^F(\mu; \hat{\mathcal{P}}, \Lambda).$$

If $\phi$ is a fusion map, then

$$\psi^F(\phi\#\mu; \hat{\mathcal{P}}, \Lambda) < \psi^F(\mu; \hat{\mathcal{P}}, \Lambda).$$

Lemma 4.11. The map $\mu \mapsto \psi^F_t(\mu; \hat{\mathcal{P}}, \Lambda)$ is 2-Lipschitz on $(\mathcal{M}(\mathcal{V}_t), W_1)$.

Proof. Similar to Lemma 4.5, observe that if $b_j(\xi) \neq b_j(\xi')$ then $\text{dist}_t(\xi, \xi') \geq \tau^{-j}$. □

The HST potential. Define

$$\Phi_t := \Phi(\mu'_t; \hat{\chi}_t).$$

4.3 Potential calculus

If $F_t$ is some quantity depending on $t$, we write $\Delta_t F_t := F_t - F_{t-1}$. Our analysis will proceed by decomposing:

$$\approx \Delta_t = \Delta_t^{\text{del}} + \Delta_t^{\text{ins}} + \Delta_t^{\text{fus}} + \Delta_t^{\text{fis}} + \Delta_t^{\text{hst}} + \Delta_t^{\text{opt}},$$

where each operator is associated to the movement $v_{t-1} \mapsto v_t$. To interpret (4.2) formally, we need to describe these operators separately for each quantity under consideration.

Syntactically:

$$\Delta_t^{\text{opt}} \Psi^F_t = \Delta_t^{\text{opt}} \Psi^A_t = 0.$$

We will use the expressions:

$$\Delta_t \text{cost}_{\text{f}}(\mu) := \Delta_t^{\text{hst}} \text{cost}_{\text{T}}(\mu) := W_1^t(\mu_t, \mu_{t-1})$$

$$\Delta_t \text{cost}_{\text{X}}(\nu^*) := W_1^t(v_{t-1}^*, v_t^*)$$

$$\Delta_t \text{cost}_{\text{X}}(\nu) := W_1^t(v_t, v_{t-1})$$

$$\Delta_t^{\text{opt}} \text{cost}_{\text{T}}(\mu^*) := W_1^t(\mu_t, \mu_{t-1}).$$

The next set of definitions may appear unwieldy. Since there are a number of moving pieces, we have written formally every change that occurs in the argument. In the next section, we discuss a few guiding principles that control the majority of these quantities. The most cumbersome part of the analysis will be tracking the changes to $\mu_t = \alpha_t \# v_t$ under the changes to the embedding $\alpha_t$ which occur because of changes to the underlying semi-partitions $\hat{Q}_t = \langle \hat{Q}_t^j : j \geq 1 \rangle$. To this end, make the definitions:

$$\hat{Q}_{t, \text{del}} := \hat{Q}(\hat{p}_{t, \text{del}}, \Lambda_{t-1}^j, 2\tau^{-j-1})$$

$$\hat{Q}_{t, \text{del}} := \langle \hat{Q}_{t, \text{del}}^j : j \geq 1 \rangle$$

23
where.

Furthermore, define:

\[ \Delta_t^{\text{del}} \Phi_t := \Phi(\alpha_t^{\text{del}} v_{t-1}^*; \tau_{t-1}) - \Phi(\alpha_{t-1}^{\text{del}} v_{t-1}^*; \tau_{t-1}) \]
\[ \Delta_t^{\text{fis}} \Phi_t := \Phi(\alpha_t^{\text{fis}} v_{t-1}^*; \tau_{t-1}) - \Phi(\alpha_{t-1}^{\text{fis}} v_{t-1}^*; \tau_{t-1}) \]
\[ \Delta_t^{\text{ins}} \Phi_t := \Phi(\alpha_t^{\text{ins}} v_{t-1}^*; \tau_{t-1}) - \Phi(\alpha_{t-1}^{\text{ins}} v_{t-1}^*; \tau_{t-1}) \]
\[ \Delta_t^{\text{fus}} \Phi_t := \Phi(\alpha_t v_{t-1}^*; \tilde{\tau}_{t-1}) - \Phi(\alpha_{t-1} v_{t-1}^*; \tilde{\tau}_{t-1}) \]
\[ \Delta_t^{\text{opt}} \Phi_t := \Phi(\alpha_t v_{t-1}^*; \tau_t) - \Phi(\alpha_{t-1} v_{t-1}^*; \tau_t) \]

Furthermore, define:

\[ \Delta_t^{\text{del}} \Psi_t := \psi_t(\mu_{t-1}; \hat{\Phi}_{\text{del}, t}, \Lambda_{t-1}) - \psi_t(\mu_{t-1}; \hat{\Phi}_{\text{del}, t}, \Lambda_{t-2}) \]
\[ \Delta_t^{\text{fis}} \Psi_t := \psi_t(\mu_{t-1}; \hat{\Phi}, \Lambda_{t-1}) - \psi_t(\mu_{t-1}; \hat{\Phi}, \Lambda_{t-2}) \]
\[ \Delta_t^{\text{ins}} \Psi_t := \psi_t(\mu_{t-1}; \hat{\Phi}, \Lambda_{t-1}) - \psi_t(\mu_{t-1}; \hat{\Phi}, \Lambda_{t-2}) \]
\[ \Delta_t^{\text{fus}} \Psi_t := \psi_t(\mu_{t-1}; \hat{\Phi}, \Lambda_{t-1}) - \psi_t(\mu_{t-1}; \hat{\Phi}, \Lambda_{t-1}) \]
\[ \Delta_t^{\text{opt}} \Psi_t := \psi_t(\mu_{t-1}; \hat{\Phi}, \Lambda_{t-1}) - \psi_t(\mu_{t-1}; \hat{\Phi}, \Lambda_{t-1}) \]

Finally, for \( \Psi_t^A \) we define:

\[ \Delta_t^{\text{del}} \Psi_t^A := \psi_t^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-2}) - \psi_t^A(\mu_{t-1}; C_{t-1}, \Lambda_{t-2}) \]
\[ \Delta_t^{\text{fis}} \Psi_t^A := \psi_t^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-1}) - \psi_t^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-2}) \]
\[ \Delta_t^{\text{ins}} \Psi_t^A := \psi_t^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-1}) - \psi_t^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-1}) \]
\[ \Delta_t^{\text{fus}} \Psi_t^A := \psi_t^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-1}) - \psi_t^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-1}) \]
\[ \Delta_t^{\text{opt}} \Psi_t^A := \psi_t^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-1}) - \psi_t^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-1}) \]

Our goal in the following sections is to establish the next lemma. Define

\[ \Theta_t := \Phi_t + \frac{\Psi_t^A}{2C_A(\frac{1}{k} \log k)f_2(k)} + \frac{\Psi_t}{8f_2(k)} \]

where \( C_A \geq 1 \) is the (universal) constant from Lemma 4.17.

**Lemma 4.12.** For some choice of \( \delta = \frac{1}{f_2(k)f_3(k)} \) and every \( t \geq 2 \), it holds that

\[ \mathbb{E} \Delta_t \Theta_t \leq \frac{- \mathbb{E} \Delta_t \text{cost}_t^A(\mu)}{4f_2(k)} + O((\log k)^2)f_1(k)f_2(k)f_3(k)\Delta_t \text{cost}_t(\nu^*) + 2\eta f_1(k)\rho_{t-1}(\sigma_t) \]

Let us see how this completes the proof. First, note that

\[ \rho_{t-1}(\sigma_t) \leq 2 \mathbb{E} \Delta_t^{\text{fus}}(\mu) \text{.} \tag{4.3} \]
This holds because one must have \( \rho_t(\sigma_t) \geq 1 \), and therefore at least half this mass must travel expected distance \( \rho_{t-1}(\sigma_t) \).

Hence recalling that \( \eta = (32 f_1(k) f_2(k))^{-1} \), Lemma 4.12 gives

\[
\mathbb{E} \Delta_t \cos^F_t(\mu) \leq O((\log k)^2) f_1(k) f_2(k)^2 f_3(k) \Delta_t \cos_X(\nu^*) - \Delta_t \Theta_t .
\]

(4.4)

Observe that:

\[
\mathbb{E} |\Theta_t| \leq \mathbb{E} |\Phi_t| + \mathbb{E} |\Psi^A_t| + \mathbb{E} |\Psi^F_t| \leq K_\Phi + O(k) .
\]

(4.5)

The inequalities \( \mathbb{E} |\Psi^A_t| + \mathbb{E} |\Psi^F_t| \leq O(k) \) are straightforward from the definitions and our assumption that \( \text{diam}(X) \leq 1 \), and \( |\Phi_t| \leq K_\Phi \) from Axiom (A0).

Therefore summing (4.4) over \( t \geq 2 \) yields

\[
\mathbb{E} \cos^F_t(\mu) \leq O((\log k)^2) f_1(k) f_2(k)^2 f_3(k) \cos_X(\nu^*) + K_\Phi + O(k) .
\]

completing the proof of Theorem 3.1.

### 4.4 Proof outline

The point of the extra potential functions is to control various auxiliary quantities. We attempt to clarify some of the guiding principles.

**Fusion is free.** Note that we have defined \( \Delta_t \cos^F_t(\mu) = W^1_t(\mu_t, \mu^{t-1}) \). This uses the fact that there is no reduced movement cost incurred in passing from \( \mu_{t-1} = \mu^{t-1} \) to \( \mu^{t-1} \in \| \varphi_t \# \mu_{t-1} \| \).

More interestingly:

\[
\Delta^f \cos^F_t(\mu) = 0 .
\]

(4.6)

This holds because \( \alpha_t \# \nu^*_{t-1} = \varphi_t \# \alpha^\infty_t \# \nu^*_{t-1} \), and \( \bar{\lambda} \) is chosen according to Axiom (A3) so that

\[
\Phi(\varphi_t \# \alpha^\infty_t \# \nu^*_{t-1}; \bar{\lambda} t-1) \leq \Phi(\alpha^\infty_t \# \nu^*_{t-1}; \bar{\lambda} t-1) .
\]

**Fusion pays for fission.** The basic idea here is that creation of a new \( (t-1) \)-heavy ball \( B \) decreases \( \Psi^F_t \) in proportion to \( \tilde{v}_{t-1}(B) \text{rad}_X(B) \) (such a heavy ball was created in the previous time step, but is only being fused into \( \tilde{P} \) in the current time step).

On the other hand, this creation might entail breaking apart previously fused clusters in \( \lambda B \). By Axiom (A4), the cost of this “fission” will be proportional to \( f_3(k) \nu^*_{t-1}(\lambda B \setminus B) \text{rad}_X(B) \). In expectation, this is \( f_3(k) \tilde{v}_{t-1}(\lambda B \setminus B) \text{rad}_X(B) \). But since \( B \) is \( (t-1) \)-heavy, we have:

\[
\tilde{v}_{t-1}(\lambda B \setminus B) \leq \delta \tilde{v}_{t-1}(B) .
\]

By choosing \( \delta \) small enough, it will be the case that fusion pays for fission. This is encapsulated in the following lemma.

**Lemma 4.13.** For every pair of positive numbers \( c, c' < 1 \), it holds that for

\[
\delta \leq \frac{c}{4(f_3(k) + c')},
\]

and every \( t \geq 2 \):

\[
\mathbb{E} \left[ \Delta^\text{fis}_t(\Phi_t + c \Psi^F_t + c' \Psi^A_t) \right] \leq 0 .
\]

(4.7)
On the other hand, we gain when HST potential connects $\mu^*$. Nevertheless, we still need to pay for these deletions, and that is the role of this is quite a strong bound. Let us now make four observations:

**Lemma 4.14.** For every $t \geq 2$, it holds that
\[
\Delta_t^{\text{opt}} \mathbb{E} \operatorname{cost}_T(\mu^*) \leq O\left(\frac{1}{\nu} \log(k) \log(K)\right) \Delta_t \operatorname{cost}_X(\nu^*) + 2\eta \rho_{t-1}(\sigma_t). \tag{4.8}
\]

**HST potential connects $\mu^*$ and $\mu$.** The next inequality follows immediately from Axiom (A1) and Lemma 4.14:
\[
\Delta_t^{\text{opt}} \Phi_t \leq f_1(k) \Delta_t^{\text{opt}} \operatorname{cost}_T(\mu^*) \leq f_1(k) \left[O\left(\frac{1}{\nu} \log(k) \log(K)\right) \Delta_t \operatorname{cost}_X(\nu^*) + 2\eta \rho_{t-1}(\sigma_t)\right]. \tag{4.9}
\]

On the other hand, we gain when $\mu$ moves: Axiom (A2) yields
\[
\Delta_t^{\text{hst}} \Phi_t \leq -\frac{\Delta_t^{\text{hst}} \operatorname{cost}_T(\mu)}{f_2(k)}. \tag{4.10}
\]

**The accuracy potential pays for deletions.** Modifying the sets of centers (and hence the underlying semi-partitions) has a non-trivial cost. By construction and Axiom (A4), it will turn out that $\Phi_t$ does not pay a cost for insertions.

**Lemma 4.15.** For every $t \geq 2$, it holds that
\[
\Delta_t^{\text{ins}} \Phi_t \leq 0. \tag{4.11}
\]

Deletions, on the other hand, will incur a cost, but since we delete a random level-$j$ center in response to a level-$j$ insertion, the cost can be readily controlled. The next lemma is proved in Section 4.6.1. Let $j^* = \min\{j \geq 1 : I^j_f = 1\}$ be the highest scale at which an insertion occurs (we take $j^*_0 = 0$ if no such $j$ exists).

**Lemma 4.16.** For every $t \geq 2$, it holds that
\[
\mathbb{E} \left| \Delta_t^{\text{del}} \Phi_t \right| \leq 2\tau f_1(k) \frac{k}{K} \tau^{-j^*_f} 1_{(j^*_f > 0)} \tag{4.12}
\]
\[
\mathbb{E} \left| \Delta_t^{\text{del}} \psi^A_t \right| + \mathbb{E} \left| \Delta_t^{\text{del}} \psi^F_t \right| \leq 4\tau \frac{k}{K} \tau^{-j^*_f} 1_{(j^*_f > 0)}. \tag{4.13}
\]

Since we can make $K$ large (the only adverse dependence on $K$ is in (4.9), where it is logarithmic), this is quite a strong bound. Nevertheless, we still need to pay for these deletions, and that is the role of $\psi^A_t$.

**Lemma 4.17.** For some constant $C_A \geq 1$ and every $t \geq 2$, it holds that
\[
(\Delta_t^{\text{hst}} + \Delta_t^{\text{ins}}) \psi^A_t \leq -\frac{1}{16} \tau^{-j^*_f} 1_{(j^*_f > 0)} + \left[C_A \left(\frac{1}{\log k}\right) - 1\right] \Delta_t^{\text{hst}} \operatorname{cost}_T(\mu) + \eta \rho_{t-1}(\sigma_t).
\]

Combining Lemma 4.17 and (4.3) gives
\[
\mathbb{E}(\Delta_t^{\text{hst}} + \Delta_t^{\text{ins}}) \psi^A_t \leq -\frac{1}{16} \mathbb{E}\left[\tau^{-j^*_f} 1_{(j^*_f > 0)}\right] + C_A \left(\frac{1}{\log k}\right) \mathbb{E}\Delta_t^{\text{hst}} \operatorname{cost}_T(\mu). \tag{4.14}
\]

Let us now make four observations:
\[
\Delta_t^{\text{fus}} \psi^A_t \leq 0, \tag{4.15}
\]
where in the last line we have chosen $c := 1/(8f_2(k))$ and $K := 32C_Af_2(k)(\frac{1}{2}\log k)(18\tau + 2f_1(k))$.

Using these bounds, let us now prove Lemma 4.12.

Proof of Lemma 4.12. Sum the inequalities (4.19), (4.9), and (4.7) with $c' = (2C_A(\frac{1}{3}\log k)f_2(k))^{-1}$, yielding

\[
\mathbb{E}\left[ (\Delta^\text{ins}_t + \Delta^\text{del}_t + \Delta^\text{hst}_t + \Delta^\text{fus}_t) \left( \Phi_t + \frac{\Psi_t^A}{2C_A(\frac{1}{3}\log k)f_2(k)} + c\Psi_t^F \right) \right] \\
\leq \left( 18\tau + 2f_1(k) \right) \frac{k}{K} - \frac{1}{32C_A(\frac{1}{3}\log k)f_2(k)} \mathbb{E}\left[ \tau^{-\frac{\delta}{2}} 1_{\{j_i > 0\}} \right] + \left( 2c - \frac{1}{2f_2(k)} \right) \mathbb{E} \Delta^\text{hst}_t\text{cost}_t(\mu) \\
= -\frac{\mathbb{E} \Delta^\text{hst}_t\text{cost}_t(\mu)}{4f_2(k)},
\]

where in the last line we have chosen $\delta \asymp \frac{c}{f_3(k)} \asymp \frac{1}{f_2(k)f_3(k)}$ and used $K \leq k^{O(1)}$. This completes the proof. $\square$

### 4.5 Stretch analysis

Let us now establish another central claim.

**Lemma 4.18.** For every $t \geq 1$ and every $x \in X$, it holds that

\[
\mathbb{E}_{\Omega^{\text{hst}}} \left[ \text{dist}_t(\alpha_t(x), \alpha_t(\sigma_t)) \right] \leq O\left( \frac{1}{3}\log(k) \log(K) \right) d_X(x, \sigma_t) + 2\eta\rho_{t-1}(\sigma_t).
\]

Proof. Let $M := \max(\eta\rho_{t-1}(\sigma_t), 2\tau d_X(x, \sigma_t))$ and $j_0 := \max\{ j \in \mathbb{Z}_+: \tau^{-j} \geq M \}$. From Lemma 3.4, it holds that

\[
\frac{1}{2\tau} \text{dist}_t(\alpha_t(x), \alpha_t(\sigma_t)) \leq \sum_{j \geq 1} \tau^{-j} \Delta_{Q_j}^t(x, \sigma_t)
\]
\[
\leq \eta \rho_{t-1}(\sigma_t) + 2\tau d_X(x, \sigma_t) + \sum_{j=1}^{j_0} \tau^{-j} \Delta_{Q_j}^t(x, \sigma_t).
\]

Note that \( j \leq j_0 \) implies \( x \in B_X(\sigma_t, \frac{1}{2} \tau^{-j-1}) \). Thus if additionally \( j \notin J_{t-1}(\sigma_t) \), then Lemma 4.3 asserts that \( \Delta_{Q_j}^t(x, \sigma_t) = 0 \). Therefore:

\[
\sum_{j=1}^{j_0} \tau^{-j} \Delta_{Q_j}^t(x, \sigma_t) \leq \sum_{j \in J_{t-1}(\sigma_t)} \tau^{-j} \Delta_{Q_j}^t(x, \sigma_t).
\]

Now Lemma 3.5 (specifically (3.9)) gives, for every \( j \geq 1 \):

\[
\mathbb{E}_{Q_{j \in \Omega}} [\tau^{-j} \Delta_{Q_j}^t(x, \sigma_t)] \leq O(\log K) d_X(x, \sigma_t).
\]

Therefore:

\[
\mathbb{E}_{Q_{j \in \Omega}} \left[ \sum_{j \in J_{t-1}(\sigma_t)} \tau^{-j} \Delta_{Q_j}^t(x, y) \right] \leq O(\log K) |J_{t-1}(\sigma_t)| d_X(x, \sigma_t) \leq O \left( \frac{1}{\delta} \log(k \log(K)) \right) d_X(x, \sigma_t),
\]

where the final inequality uses Lemma 4.4.

\[ \square \]

4.6 Tracking the algorithm

4.6.1 Deletions

If \( I^j_t = 1 \), then some cluster \( S_j \in \hat{P}_t^j \) with center \( z^j_t \) is possibly deleted. Therefore:

\[
W^1_t(\alpha_t \# v^*_t, \alpha_t \# v^*_t) \leq \sum_{j \geq 1} 1_{\{I^j_t = 1\}} v^*_t(S_j) \tau^{-j+1}.
\]

Recall that \( v^*_t(X) = k \). Since we remove a uniformly random level-\( j \) cluster and there are at least \( K \) of them (if a deletion takes place), it holds that

\[
\mathbb{E}_{\Omega_{\text{del}}} \left[ \Delta^{\text{del}}_t \text{cost}_I(\mu^*) \mid j^*_t \right] = \mathbb{E}_{\Omega_{\text{del}}} \left[ W^1_t(\alpha_t \# v^*_t, \alpha_t \# v^*_t) \mid j^*_t \right] \\
\leq k \sum_{j \geq 1} \tau^{-j+1} I^j_t \leq \frac{2\tau k}{K} \tau^{-j_t^*} 1_{\{j^*_t > 0\}}.
\]

where we take expectation only over the random choice of which cluster to delete. In particular, using Axiom (A1), this implies that

\[
\mathbb{E}_{\Omega_{\text{del}}} \left[ \Delta^{\text{del}}_t \phi^*_t \mid j^*_t \right] \leq \frac{2\tau f_1(k)k}{K} \tau^{-j_t^*} 1_{\{j^*_t > 0\}}.
\]

We have similar calculations for \( \Psi^A_t \) and \( \Psi^F_t \) using \( \mu_{t-1}(\mathcal{V}_t) = k \):

\[
\mathbb{E}_{\Omega_{\text{del}}} \left[ \Delta^{\text{del}}_t \Psi^A_t \mid j^*_t \right], \mathbb{E}_{\Omega_{\text{del}}} \left[ \Delta^{\text{del}}_t \Psi^F_t \mid j^*_t \right] \leq \frac{2k}{K} \tau^{-j_t^*} 1_{\{j^*_t > 0\}}.
\]

Together, these yield Lemma 4.16.
4.6.2 Insertions

We now analyze the effect of inserting $\sigma_t$.

**Proof of Lemma 4.15.** Recall that

$$
\Delta^\text{ins}_t \Phi_t = \Phi(\alpha^\text{ins}_i^{-1}; \chi_{t-1}) - \Phi(\alpha^\text{ins}_i^{-1}; \chi_{t-1}).
$$

We can bound this change by

$$
\Delta^\text{ins}_t \Phi_t \leq \sum_{j \geq 0} \left( \Phi(\alpha^j_i^{-1}; \chi_{t-1}) - \Phi(\alpha^{j-1}_i^{-1}; \chi_{t-1}) \right),
$$

(4.20)

where $\alpha^0 = \alpha^\text{ins}_i$ and $\alpha^j$ results from $\alpha^{j-1}$ by incorporating the possible insertion of a set $\{s_j\} = \hat{p}^j_i \setminus \hat{p}^j_{i,\text{del}}$. Thus $\alpha^j_i$ and $\alpha^{j-1}_i$ agree outside $S_j \setminus \{s_j\}$.

By definition of the embedding (recall (3.7)), this affects the embedding of servers at points $x \in \text{supp}(\nu^*_{t-1})$ with rank less than $j$, but the images of all such points lie outside $\mathcal{V}^0_i$. Since $\mu^{k-1}$ is supported on $\mathcal{V}^0_i$ (recall (2.8)), Axiom (A4) implies that each term in (4.20) is zero.

**Proof of Lemma 4.17.** Fix $j \geq 1$. Suppose that $I^j_i = 1$ and denote

$$
\psi(\xi; C) := d_X^j(b_{j+2}(\xi), C) \cdot \tau\left( d_X^j(b_{j+2}(\xi), \Lambda^j_{t-1}) - \eta \max_{x \in b_{j+2}(\xi)} \rho_{t-1}(x) - \frac{1}{2} \tau^{-j-1} \right).
$$

Consider some $\xi \in \mathcal{V}_t$ and let $\hat{x} \in C^j_{t-1}$ be such that $d_X(b_{j+2}(\xi), C^j_{t-1}) = d_X(b_{j+2}(\xi), \hat{x})$. Since $\sigma_t$ is inserted into $C^j_{t-1}$, it must hold that $d_X(\sigma_t, \hat{x}) > \tau^{-j-1}$ and $d_X(\sigma_t, \Lambda^j_{t-1}) > \tau^{-j-1}$. Therefore either:

1. $d_X(b_{j+2}(\xi), \sigma_t) \geq \frac{1}{4} \tau^{-j-1}$, and thus $\text{dist}_T(\xi, \sigma_t(\sigma_t)) \geq \frac{1}{4} \tau^{-j-1}$, or

2. $d_X(b_{j+2}(\xi), \sigma_t) \leq d_X(b_{j+2}(\xi), \hat{x}) - \frac{1}{2} \tau^{-j-1}$, and

$$
d_X(b_{j+2}(\xi), \Lambda^j_{t-1}) \geq d_X(\sigma_t, \Lambda^j_{t-1}) - d_X(b_{j+2}(\xi), \sigma_t) \geq \frac{3}{4} \tau^{-j-1}.
$$

In either case, we can conclude that for any $\xi \in \mathcal{V}_t$,

$$
\psi(\xi; C^j_t) - \psi(\xi; C^j_{t,\text{del}}) \leq -\frac{1}{8} \tau^{-j-1} + \text{dist}_T(\xi, \sigma_t(\sigma_t)) + \eta \max_{x \in b_{j+2}(\xi)} \rho_{t-1}(x)
$$

$$
\leq -\frac{1}{8} \tau^{-j-1} + \text{dist}_T(\xi, \sigma_t(\sigma_t)) + \eta \left( \rho_{t-1}(\sigma_t) + \tau^{-j-2} + \text{dist}_T(\xi, \sigma_t(\sigma_t)) \right)
$$

$$
\leq -\frac{1}{16} \tau^{-j-1} + (1 + \eta) \text{dist}_T(\xi, \sigma_t(\sigma_t)) + \eta \rho_{t-1}(\sigma_t).
$$

(4.21)

Consider now the movement of the HST algorithm in responding to the request at $\sigma_t(\sigma_t)$. If $\epsilon$ fractional mass is sent from $\xi$ to $\sigma_t(\sigma_t)$, then (4.21) gives us a corresponding potential change of at most

$$
\left( -\frac{1}{16} \tau^{-j-1} + (1 + \eta) \text{dist}_T(\xi, \sigma_t(\sigma_t)) + \eta \rho_{t-1}(\sigma_t) \right) \epsilon.
$$

Combining this with Lemma 4.8 gives

$$
(\Delta^\text{ins}_t + \Delta^\text{ins}_t) \Psi_t^A \leq -\frac{\tau^{-j-1}}{16} 1_{\{j > 0\}} + \left(1 + \eta \right) + O\left( \frac{1}{5} \log k \right) \Delta^\text{hst}_t \text{cost}_T(\mu) + \eta \rho_{t-1}(\sigma_t),
$$

completing the proof of Lemma 4.17.\qed
4.6.3 Fusion and fission

Here we analyze the effect of $\Delta_{\text{Fis}}$. Fix $t \geq 2$. Let $\mathcal{U}^j := \Lambda^j_{t-2} \setminus \Lambda^j_{t-1}$ denote the set of heavy net points that are ejected in the “heavy net maintenance” phase of time step $t − 1$. Let $\mathcal{V}^j := \Lambda^j_{t-1} \setminus \Lambda^j_{t-2}$. Every $u \in \mathcal{U}^j$ is ejected because of some newly added point $\hat{u} \in \mathcal{V}^j$ with $d_X(u, \hat{u}) \leq \frac{3}{12} \sqrt{\lambda \tau^j}$. Denote

\[
\mathcal{B}^j_{\mathcal{U}} := \left\{ B_X(u, \tau^{-j} + \tau^{-j-2}) : u \in \mathcal{U}^j \right\},
\mathcal{B}^j_{\mathcal{V}} := \left\{ B_X(v, \frac{\tau^{-j}}{2\sqrt{\lambda}}) : v \in \mathcal{V}^j \right\}.
\]

Note that from (3.3), it follows that all the balls in $\mathcal{B}^j_{\mathcal{U}}$ are pairwise disjoint, and the same holds for the balls in $\mathcal{B}^j_{\mathcal{V}}$. See Figure 3.

Let $\lambda' := \lambda - 2\sqrt{\lambda}$. Then since $\lambda \geq 36^2$, we have $B_X(u, \tau^{-j} + \tau^{-j-2}) \subseteq B_X(\hat{u}, \lambda' \frac{\tau^{-j}}{2\sqrt{\lambda}})$ for each $u \in \mathcal{U}^j$. Therefore:

\[
\bigcup_{B \in \mathcal{B}^j_{\mathcal{U}}} B \subseteq \bigcup_{B \in \mathcal{B}^j_{\mathcal{V}}} (\lambda' B \setminus B). \tag{4.22}
\]

Make the definitions

\[
S^j_{\text{out}} := \bigcup_{B \in \mathcal{B}^j_{\mathcal{U}}} (\lambda' B \setminus B),
S^j_{\text{in}} := \bigcup_{B \in \mathcal{B}^j_{\mathcal{V}}} B.
\]

Then Lemma 2.3 yields, for any $\mu \in \mathcal{M}(\mathcal{V}_T)$:

\[
\Pi_X \mu^{\geq j} (S^j_{\text{out}}) \leq \sum_{B \in \mathcal{B}^j_{\mathcal{V}}} \Pi_X \mu^{\geq j} (\lambda B \setminus B). \tag{4.23}
\]
If we denote \( v^j_{t-1} := \Pi_X \mu^j_{t-1} \), then (4.23) gives
\[
v^j_{t-1}(S_{\text{out}}) \leq \sum_{B \in B^j_{\nu}} v_{t-1}(\lambda B \setminus B).
\] (4.24)

The next three lemmas will yield the proof of Lemma 4.13.

**Lemma 4.19.** For every \( t \geq 2 \):
\[
\Delta^j_{t} f^{\text{fis}} \Psi^A \leq \sum_{j \geq 1} \tau^{-j} v^j_{t-1}(S_{\text{out}}).
\]

Proof. By definition:
\[
\Delta^j_{t} f^{\text{fis}} \Psi^A = \Psi^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-1}) - \Psi^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-2})
\]
\[
= \Psi^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-1}) - \Psi^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-2} \setminus U^j)
\]
\[
+ \Psi^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-2} \setminus U^j) - \Psi^A(\mu_{t-1}; C_{t, \text{del}}, \Lambda_{t-2}).
\]

Since the first term involves the addition of points in \( U_j \), it is non-positive. Thus we focus on the second term.

In order for the \( \xi \) term in \( \Psi^A_t \) to be affected, it must be that \( d_X(b_{j+2}(\xi), U^j) \leq \tau^{-j} \) and \( \text{len}(\xi) \geq j + 2 \geq j \). Therefore:
\[
\Delta^j_{t} f^{\text{fis}} \Psi^A \leq \sum_{j \geq 1} \tau^{-j} v^j_{t-1}(B_X(U^j, \tau^{-j} + \tau^{-j-2})) \leq \sum_{j \geq 1} \tau^{-j} v^j_{t-1} \left( \bigcup_{B \in B^j_{\nu}} (\lambda' B \setminus B) \right).
\]

The next lemma is the primary way that Axiom (A4) is employed.

**Lemma 4.20.** For every \( t \geq 2 \):
\[
\Delta^j_{t} \Phi_t \leq f_3(k) \sum_{j \geq 1} \tau^{-j} v^j_{t-1} \left( S_{\text{out}} \right).
\]

Proof. By definition
\[
\Delta^j_{t} \Phi_t := \Phi(\alpha^j_{\text{fis}} \# v^j_{t-1}; X_{t-1}) - \Phi(\alpha^j_{\text{del}} \# v^j_{t-1}; X_{t-1}).
\]

Observe that for each \( j \geq 1 \), the change from \( \hat{Q}_{t, \text{del}} \) to \( \hat{Q}_{t, \text{fis}} \) (which induces the change from \( \alpha^j_{\text{del}} \) to \( \alpha^j_{\text{fis}} \)) results from “unfusing” along the points of \( U^j \). Since \( \hat{Q}_{t, \text{del}} \) and \( \hat{Q}_{t, \text{fis}} \) induce the same semi-partition on \( X \setminus B^j_{U^j} \), Axiom (A4) in conjunction with (4.22) and Lemma 2.3 gives
\[
\Delta^j_{t} \Phi_t \leq f_3(k) \sum_{j \geq 1} \tau^{-j} v^j_{t-1} \left( S_{\text{out}} \right).
\]

The final lemma is key: The introduction of a new heavy ball yields a large decrease in potential.

**Lemma 4.21.** For every \( t \geq 2 \):
\[
\Delta^j_{t} \Psi^F_t \leq \sum_{j \geq 1} \tau^{-j} \left( v^j_{t-1}(S_{\text{out}}) - \sum_{B \in B^j_{\nu}} v_{t-1}(B) \right).
\]
Proof. Recall that
\[
\Delta_i \Phi_i^F = \psi_i^F (\mu_i; \hat{\Phi}_i, \Lambda_i) - \psi_i^F (\mu_i; \hat{\Phi}_i, \Lambda_{i-1}),
\]
where
\[
\psi_i^F (\mu_i; \hat{\Phi}_i, \Lambda) = - \sum_{j \geq 1} \tau^{-j} \sum_{\xi \in \hat{\Lambda}_i} \mu_{\Lambda_i}^1(\xi) \mathbb{1} \left( \left[ \mathcal{H}(\hat{\Phi}_i, \Lambda_i, 2\tau^{-j}) \right] \cap b_j(\xi) \neq \emptyset \right).
\]
Each \( B \in B_i^j \) contributes at most \(-\tau^{-j} \nu_{t-1}(B)\) to the potential, while we possibly gain \(\tau^{-j} \nu_{t-1}^{>j}(S_{out}^j)\) (recall again that \(b_j(\xi) = \emptyset\) unless \(\text{len}(\xi) \geq j\)).

Combining the preceding three lemmas with (4.24) gives, for any \(0 < c, c' < 1\):
\[
\Delta_i \Phi_i^F (\Phi_i + c \Psi_i^F + c' \Psi_i^A) \leq \sum_{j \geq 1} \tau^{-j} \sum_{B \in B_i^j} (c + f_3(k) + c')\nu_{t-1}(\lambda B \setminus B) - c\nu_{t-1}(B)).
\] (4.25)

Now observe that since \(B_i^j\) consists of \((t-1)\)-heavy balls, it holds that for every \(B \in B_i^j\),
\[
\nu_{t-1}(\lambda B \setminus B) \leq \delta (1 - \delta) \nu_{t-1}(B) \leq 2\delta \nu_{t-1}(B).
\]

Therefore taking expectations in (4.25) yields
\[
\mathbb{E} \left[ \Delta_i \Phi_i^F (\Phi_i + c \Psi_i^F + c' \Psi_i^A) \right] \leq \sum_{j \geq 1} \tau^{-j} \sum_{B \in B_i^j} \nu_{t-1}(B) \left[ \delta (f_3(k) + c') - (1 - 2\delta) c \right].
\] (4.26)

If we now choose
\[
\delta \leq \frac{c}{4(f_3(k) + c')},
\]
then (4.26) becomes at most zero, yielding Lemma 4.13.

5 Monotonicity under fusion

Consider a pair of siblings \(\xi^A, \xi^B \in V_i^j\) with \(b(\xi^A) \subseteq b(\xi^B)\) and the canonical injection \(\varphi_{\xi^A \leftarrow \xi^B}\). Using auxiliary labels \(\{1, 2\}\) (say), one can encode this injection by a multistep process:
\[
\begin{align*}
\langle \hat{\xi}_0, \ldots, \hat{\xi}_{j-1}, (b(\xi^A); 1), (\xi_{j+1}; 1), (\xi_{j+2}; 1), \ldots \rangle & \mapsto \langle \hat{\xi}_0, \ldots, \hat{\xi}_{j-1}, (b(\xi^B); 2), (\xi_{j+1}; 1), (\xi_{j+2}; 1), \ldots \rangle \\
& \mapsto \langle \hat{\xi}_0, \ldots, \hat{\xi}_{j-1}, (b(\xi^B); 2), (\xi_{j+1}; 2), (\xi_{j+2}; 1), \ldots \rangle \\
& \mapsto \langle \hat{\xi}_0, \ldots, \hat{\xi}_{j-1}, (b(\xi^B); 2), (\xi_{j+1}; 2), (\xi_{j+2}; 2), \ldots \rangle.
\end{align*}
\]

The idea is that only one label is changed from 1 to 2 at every step. (At the end, such atomic steps can be used to restore the original labeling.)

The advantage of this perspective is that if one is trying to prove monotonicity of some quantity under fusion maps, it suffices to establish monotonicity for canonical injections, and thus to establish it for one step of the above process. This corresponds to first “fusing” \(A\) into \(B\) but still distinguishing the children of \(B\) from those of \(A\), then recursively fusing the children of \(B\) into the children of \(A\), and so on. We will refer to such a step as a primitive fusion of \(\xi^A\) into \(\xi^B\).
5.1 Online rounding

The authors of [BBMN15, §5.2] present an online algorithm to round a fractional \( k \)-server algorithm on a \( \tau \)-HST (for \( \tau > 5 \)) to a random integral \( k \)-server algorithm in a way that the expected cost increases by at most an \( O(1) \) factor. Unfortunately, this does not quite suffice for us, as our model allows cluster fusion.

**Theorem 5.1** (HST rounding under fusions). Consider an \( \mathcal{F} \)-adapted sequence \( \mu = \langle \mu_t \in \mathbb{M}_k(V_t) : t \geq 0 \rangle \). There exists a random \( \mathcal{F} \)-adapted sequence \( \hat{\mu} = \langle \hat{\mu}_t \in \hat{\mathbb{M}}_k(V_t) : t \geq 0 \rangle \) such that for every \( \xi \in V_t \) and \( t \geq 0 \): With probability one, for every \( \xi \in V_t \),

\[
\hat{\mu}_t(V_t(\xi)) \in \{ [\mu_t(V_t(\xi))] , [\mu_t(V_t(\xi))] \} \tag{5.1}
\]

Moreover:

\[ \mathbb{E} [\text{cost}_t^\mathcal{F}(\hat{\mu})] \leq O(1) \text{cost}_t^\mathcal{F}(\mu) . \]

**Proof.** In [BBMN15, §5.2], the authors give a procedure for online rounding of a fractional \( k \)-server algorithm on HSTs to a distribution over integral algorithms that only loses an \( O(1) \) factor in the expected cost. The key property maintained is that the integral algorithm is supported on balanced configurations with respect to the fractional algorithm, i.e., that (5.1) holds for every \( \xi \in V_t \).

In order to extend this to our model, we need to give a method for the primitive fusion of two clusters while maintaining the balance property. Suppose that \( \hat{\mu} \) is a random integral \( k \)-server measure that satisfies, for two siblings \( \xi^A, \xi^B \in V_t \) with \( b(\xi^A) \subseteq b(\xi^B) \),

\[
\mathbb{E}[\hat{\mu}(V_t(\xi^A))] = \mu(V_t(\xi^B)) \\
\mathbb{E}[\hat{\mu}(V_t(\xi^A))] = \mu(V_t(\xi^B)),
\]

and with probability one, \( \hat{\mu} \) satisfies the balance conditions:

\[
\hat{\mu}(V_t(\xi^A)) \in \{ [\mu(V_t(\xi^A))] , [\mu(V_t(\xi^A))] \} \\
\hat{\mu}(V_t(\xi^B)) \in \{ [\mu(V_t(\xi^B))] , [\mu(V_t(\xi^B))] \}.
\]

For simplicity, let us denote

\[
\hat{\mu}_A := \hat{\mu}(V_t(\xi^A)) \\
\hat{\mu}_B := \hat{\mu}(V_t(\xi^B)) \\
\mu_A := \mu(V_t(\xi^A)) \\
\mu_B := \mu(V_t(\xi^B)) \\
\varepsilon_A := \mu_A - [\mu_A] \\
\varepsilon_B := \mu_B - [\mu_B].
\]

We need to produce a random variable \((k_A, k_B)\) with the following properties:

1. \( \text{supp} ((k_A, k_B)) \subseteq \text{supp} ((\hat{\mu}_A, \hat{\mu}_B)) \)
2. \( \mathbb{P}(k_A = [\mu_A]) = \mathbb{P}(\hat{\mu}_A = [\mu_A]) \)
3. \( \mathbb{P}(k_B = [\mu_B]) = \mathbb{P}(\hat{\mu}_B = [\mu_B]) \)
4. The balance condition is satisfied:

\[
\mathbb{P}(k_A + k_B \in [[\mu_A + \mu_B], [\mu_A + \mu_B]]) = 1 .
\]
We then define $\hat{\mu}$ of the fused cluster as $k_A + k_B$ and couple the distributions of the children accordingly using the conditional distributions $\hat{\mu} \mid \hat{\mu}_A = k_A$ and $\hat{\mu} \mid \hat{\mu}_B = k_B$. In this way, we preserve a balanced online rounding under a primitive fusion step. Note that we do not incur any reduced movement cost because we do not pay for the fusion (by definition of the reduced cost).

There are two cases. Note that the first case includes the situation in which one of $\mu_A$ or $\mu_B$ is an integer.

1. $\varepsilon_A + \varepsilon_B \leq 1$:
   
   $\Pr[(k_A, k_B) = ([\mu_A], [\mu_B])] = \Pr(\hat{\mu}_A = [\mu_A]) + \Pr(\hat{\mu}_B = [\mu_B]) - 1$

   $\Pr[(k_A, k_B) = ([\mu_A], [\mu_B])] = \Pr(\hat{\mu}_B = [\mu_B])1_{\{\varepsilon_B > 0\}}$

   $\Pr[(k_A, k_B) = ([\mu_A], [\mu_B])] = \Pr(\hat{\mu}_A = [\mu_A])1_{\{\varepsilon_A > 0\}}$

   $\Pr[(k_A, k_B) = ([\mu_A], [\mu_B])] = 0$.

2. $\varepsilon_A + \varepsilon_B > 1$:

   $\Pr[(k_A, k_B) = ([\mu_A], [\mu_B])] = 0$

   $\Pr[(k_A, k_B) = ([\mu_A], [\mu_B])] = \Pr(\hat{\mu}_A = [\mu_A])$

   $\Pr[(k_A, k_B) = ([\mu_A], [\mu_B])] = \Pr(\hat{\mu}_B = [\mu_B])$

   $\Pr[(k_A, k_B) = ([\mu_A], [\mu_B])] = \Pr(\hat{\mu}_A = [\mu_A]) + \Pr(\hat{\mu}_B = [\mu_B]) - 1$.  \(\square\)

Suppose now that $\hat{\mu} = \{\mu_t \in \hat{\mathcal{M}}_k(\mathcal{V}_T) : t \geq 0\}$ is an $\mathcal{F}$-adapted sequence of integral $k$-server measures such that $\mu_0 \in \hat{\mathcal{M}}_k(\mathcal{L}_T)$. In order to complete the proof of Theorem 2.6, we need to show how these can be pulled back to $(X, d_X)$ without increasing the movement cost. The next lemma follows by making the underlying sequence of measures lazy; we only move measure between leaves (this will be sufficient since requests come only at leaves).

**Lemma 5.2.** There is an $\mathcal{F}$-adapted sequence of leaf measures $\mu' = \{\mu'_t \in \hat{\mathcal{M}}_k(\mathcal{L}_T) : t \geq 0\}$ such that

$$\text{cost}_T^{\mathcal{F}}(\mu') \leq \text{cost}_T^{\mathcal{F}}(\mu),$$

and for all $t \geq 0$ and $\ell \in \mathcal{L}_T$,

$$\mu_t(\ell) \geq 1 \implies \mu'_t(\ell) \geq 1.$$

The final lemma of this section completes the proof of Theorem 2.6 in conjunction with Lemma 5.2 and Theorem 5.1.

**Lemma 5.3.** If $\mu' = \{\mu'_t \in \hat{\mathcal{M}}_k(\mathcal{L}_T) : t \geq 0\}$ is a sequence of integral measures taking values at leaves, and $\nu = \{\nu_t : t \geq 0\}$ is defined by $\nu_t := \beta \# \mu_t$, then

$$\text{cost}_X(\nu) \leq \text{cost}_T^{\mathcal{F}}(\mu').$$

**Proof.** This follows from three facts: First, $\beta$ is 1-Lipschitz (recall Claim 2.2). Secondly, if $\mu \in \mathcal{M}(\mathcal{L}_T)$, then $\# \mu = \{\mu\}$. And thirdly, if $\varphi$ is a fusion map and $\mu \in \mathcal{M}(\mathcal{L}_T)$, then $\varphi \# \mu \in \mathcal{M}(\mathcal{L}_T)$, and $\beta \# \varphi \# \mu = \beta \# \mu$.  \(\square\)
5.2 Verification of the potential axioms for \([BCL^+17]\)

Consider an element

$$x = \langle x^{\xi,i} \in [0, 1] : \xi \in V_T, i = 1, 2, \ldots \rangle \subseteq \ell^\infty(V_T \times \mathbb{Z}_+) .$$

For \(\xi \in V_T\), write \(ch(\xi)\) for the set of children of \(\xi\) in \(T\). Let \(K\) denote the closed convex set of such \(x\) that satisfy the following linear constraints for every \(\xi \in V_T\):

$$x^{\xi,i} = \begin{cases} 0 & i \in \{1, 2, \ldots, k\} \\ 1 & i > k, \end{cases}$$

$$\sum_{i < |S|} x^{\xi,i} \leq \sum_{(\xi',j) \in S} x^{\xi',j} \quad \text{for all finite } S \subseteq ch(\xi) \times \mathbb{Z}_+ . \quad (5.2)$$

Let us furthermore define \(z = z(x)\) by \(z^{\xi,i} := \frac{1}{1 - \delta} (1 - x^{\xi,i})\) and \(z^{\xi} := \sum_{i \geq 1} z^{\xi,i}\). (Note that \(x\) and \(z\) are related by an invertible linear transformation, and thus we need only specify one of them in order to define the corresponding set of values.)

For a leaf \(\ell = \langle \xi_0, \xi_1, \ldots \rangle \in \mathcal{L}_T\), we write

$$z^{\ell} := \lim_{j \to \infty} z^{\xi,j}_\ell .$$

Fix \(\delta := \frac{1}{3k}\) and let \(K_\delta \subseteq K\) denote the subset of \(x \in K\) for which the set \(\{ \ell : z^{\ell} \neq 0 \}\) is finite, as well as the sets \(\{ i : z^{\xi,i} \neq 0 \}\) for each \(\xi \in V_T\), and furthermore:

$$z^{\ell} \leq 1 \quad \forall \ell \in \mathcal{L}_T ,$$

$$\sum_{\ell \in \mathcal{L}_T} z^{\ell} = k \frac{1}{1 - \delta} = k + \varepsilon , \quad (5.4)$$

where we note that \(\varepsilon := \frac{\delta k}{1 - \delta} < 1\) for all \(k \geq 1\).

Define the measure \(\nu^x \in \mathcal{M}_{k+1} (\mathcal{L}_T)\) by

$$\nu^x(S) := \sum_{\ell \in S} z^{\ell} \quad \forall S \subseteq \mathcal{L}_T . \quad (5.5)$$

One should note that for \(x \in K_\delta\), the inequalities \((5.2)\) imply that for every \(\xi \in V_T\),

$$z^{\xi} \geq \sum_{\xi' \in ch(\xi)} z^{\xi'} , \quad (5.6)$$

and since \(z^X = k + \varepsilon\), \((5.4)\) implies that the inequality in \((5.6)\) holds with equality. In other words, for every \(\xi \in V_T\), we have

$$z^{\xi} = \nu^x (\mathcal{L}_T(\xi)) , \quad (5.7)$$

where \(\mathcal{L}_T(\xi) := V_T(\xi) \cap \mathcal{L}_T\) is the set of leaves “below” \(\xi\) in \(T\).

**Truncating the fractional measure.** Observe that the leaf measure \(\nu^x\) has total fractional server mass \(k + \varepsilon\). Thus we need to do a bit more before we produce a fractional \(k\)-server measure.

Define the rounding map \(\rho : \mathbb{R}_+ \to \mathbb{R}_+\) as follows: For \(h \in \mathbb{Z}_+\), define \(\rho [h, h + \varepsilon] = h\) and extend \(\rho\) affinely outside \(\bigcup_{h \in \mathbb{Z}_+} [h, h + \varepsilon]\). Note that \(\rho\) is \(\frac{1}{1 - \varepsilon}\)-Lipschitz. For a measure \(\mu \in \mathcal{M}(\mathcal{L}_T)\), define the measure \(\Lambda_\varepsilon \mu \in \mathcal{M}(V_T)\) by

$$\Lambda_\varepsilon \mu(\xi) := \begin{cases} \rho(\mu(\xi)) & \xi \in \mathcal{L}_T \\ \rho(\mu(\mathcal{L}_T(\xi))) - \sum_{\xi' \in ch(\xi)} \rho(\mu(\mathcal{L}_T(\xi'))) & \text{otherwise} . \end{cases}$$

35
Note that \( \rho \) is superadditive, i.e., \( \rho(y + y') \geq \rho(y) + \rho(y') \) for all \( y, y' \in \mathbb{R}_+ \), so \( \Lambda_\varepsilon \mu \) does define a measure. Moreover, by construction we have:
\[
\Lambda_\varepsilon \mu(\mathcal{V}_T(\xi)) = \rho(\mu(\mathcal{L}_T(\xi))) \quad \forall \xi \in V_T,
\]
and therefore
\[
\Lambda_\varepsilon \mu(\mathcal{V}_T) = \rho(\mu(\mathcal{L}_T)) = \rho(k + \varepsilon) = k,
\]
thus \( \Lambda_\varepsilon \mu \in \mathbb{M}_k(V_T) \).

The next lemma follows from the fact that \( \rho \) is superadditive.

**Lemma 5.4.** For any \( \nu \in \mathbb{M}_{k+r}(\mathcal{L}_T) \), and fusion map \( \varphi \), it holds that
\[
\Lambda_\varepsilon(\varphi \# \nu) \in \llbracket \varphi \# \Lambda_\varepsilon \nu \rrbracket.
\]

**The [BCL+17] algorithm.** To each \( \theta \in \hat{\mathbb{M}}_k(\mathcal{L}_T) \), we associate a representation \( \hat{x}_\theta \) as follows: For every \( \xi \in V_T \),
\[
\hat{x}_\theta^\xi := \sum_{\ell \in \mathcal{L}_T(\xi)} \hat{\theta}(\ell),
\]
and for \( \xi \in V_T \) and \( i \geq 1 \):
\[
\hat{x}_\theta^{\xi,i} = \begin{cases} 0 & \hat{x}_\theta^\xi \geq i \\ 1 & \text{otherwise.} \end{cases}
\]

Let \( \Gamma := \mathcal{K}_\delta \), and define the potential:
\[
\Phi(\theta; x) := D(\theta; x) - H(x),
\]
where
\[
D(\theta; x) := \sum_{j \geq 1} \left( \tau^{-j} \sum_{\xi \in V_T^j} \sum_{j \geq 1} \left( \hat{x}_\theta^{\xi,i} + \delta \right) \log \left( \frac{\hat{x}_\theta^{\xi,i} + \delta}{\hat{x}_\theta^{\xi,i} + \delta} \right) \right),
\]
\[
H(x) := \sum_{j \geq 1} \tau^{-j} \sum_{\xi \in V_T^j} \left[ (z^\xi + (1 + \tau^{-1})z) \log (z^\xi + \varepsilon) + z^\xi \log (z^\xi + \varepsilon) - (1 + \tau^{-1})\varepsilon \log \varepsilon \right],
\]
and \( \hat{\xi} \) denotes the parent of \( \xi \) in \( T \). One should note that this sum converges absolutely because the sets \( \{ \xi \in V_T^j : z^\xi > 0 \} \) are finite for every \( j \geq 0 \), and moreover \( z \) forms a measure of weight \( k + \varepsilon \) at every level.

The [BCL+17] algorithm can be interpreted as a mapping \( \gamma : \Gamma \times \mathcal{L}_T^0 \rightarrow \Gamma \) that satisfies axioms (A1), and (A2), where the measure associated to \( x \in \Gamma \) is \( \mu^x := \Lambda_\varepsilon \nu^x \).

**Theorem 5.5 ([BCL+17]).** There is a mapping \( \gamma : \Gamma \times \mathcal{L}_T^0 \rightarrow \Gamma \) and a constant \( C_0 \) such that following hold for every \( x \in \Gamma \).

1. For any two states \( \theta, \theta' \in \hat{\mathbb{M}}_k(\mathcal{L}_T) \):
\[
|\Phi(\theta; x) - \Phi(\theta'; x)| \leq C_0 \log(k) W^1_T(\theta, \theta').
\]
2. For every \( \sigma \in \mathcal{L}_T^0 \), we have \( v^{\gamma(x, \sigma)}(\sigma) \geq 1 \) (and therefore \( \mu^{\gamma(x, \sigma)}(\sigma) \geq 1 \) as well).
3. For every \( \sigma \in \mathcal{L}_T^0 \) and every integral measure \( \theta \in \mathbb{M}_k(\mathcal{L}_T) \) satisfying \( \theta(\sigma) \geq 1 \):
\[
\Phi(\theta; \gamma(x, \sigma)) - \Phi(\theta; x) \leq -\frac{W^1_T(\mu^x, \mu^{\gamma(x, \sigma)})}{C_0 \log k}.
\]

Moreover, the mapping \( \gamma \) and associated measures \( \{ \mu^x : x \in \mathcal{K}_\delta \} \) are sensible in the sense of (2.8).

It is straightforward that Axiom (A0) holds with \( \mathcal{K}_\phi \leq O(k \log k) \).

36
Axiom (A3). In order to demonstrate the validity of (A3), we need to give a way of updating the z-variables under a primitive fusion of $\xi^A$ into $\xi^B$, where $\xi^A, \xi^B \in V^j_T$ are siblings in $T$ with $b(A) \subseteq b(B)$. We will use $\bar{z}$ to denote the variables after the fusion.

For any descendant $\xi$ of $\xi^A$ (including $\xi^A$ itself), set $\bar{z}^{\xi,i} := 0$ for all $i \geq 1$. Let $\xi'$ denote the application of a primitive fusion step to $\xi$ (so that $\xi'$ is a descendant of $\xi^B$). If $\xi' \neq \xi^B$, we set $\bar{z}^{\xi,j} := z^{\xi,j}$ for all $i \geq 1$. We now specify how to update the variables $\{z^{\xi^B,i} : i \geq 1\}$. All other variables remain unchanged.

Define the sequence $\langle z^{\xi^B,1}, z^{\xi^B,2}, \ldots \rangle$ by sorting, in non-increasing order, the concatenation of the two sequences
\begin{equation}
\langle z^{\xi^B,i} : i \geq 1 \rangle, \quad \langle z^{\xi^A,i} : i \geq 1 \rangle.
\end{equation}

(Recall that since $x \in K_0$, each such sequence has only finitely many non-zero values.)

**Lemma 5.6.** It holds that $\bar{x} \in K_0$ and $v^{\bar{x}} = \varphi \# v^x$, where $\varphi$ denotes the corresponding primitive fusion map. Furthermore for any $\theta \in \mathbb{M}_k(\mathcal{L}_T)$,
\begin{equation}
\Phi(\varphi \# \theta; \bar{x}) \leq \Phi(\theta; x).
\end{equation}

**Proof.** The fact that $v^{\bar{x}} = \varphi \# v^x$ is immediate from the construction. And from this, it follows that \((5.4)\) holds. Thus we need only verify that $\bar{x} \in K$, and only for the third set of inequalities \((5.2)\) is this slightly non-trivial.

By construction, it is straightforward that those inequalities hold for $\bar{x}$ for any $\xi \in V_T$ except $\xi = \xi^B$. For the parent $\xi$ of $\xi^A$ and $\xi^B$, the fact that we have merged the two child lists means that the inequalities \((5.2)\) continue to hold for $\bar{\xi}$.

Thus we need only verify the inequalities for $\xi^B$. Note that one can rewrite the inequality in \((5.2)\) as
\begin{equation}
\sum_{i=1}^{|S|} z^{\xi^B,i} \geq \sum_{(\xi',i) \in S} z^{\xi',i} \quad \forall \text{ finite } S \subseteq \text{ch}(\xi^B) \times \mathbb{Z}_+.
\end{equation}

Since these inequalities hold for $z$, they also hold for $\bar{z}$ because when $\xi^A$ is fused into $\xi^B$, we sort the corresponding list of values in decreasing order.

Let us now prove \((5.9)\). The potential $\Phi(\eta; x)$ is a sum of two expressions; the first depends on $\eta$, whereas the second does not. To see that $H(\bar{x}) \geq H(x)$, apply the next lemma with $a = z^{\xi^A}, b = z^{\xi^B}, c = \tau^{-1}$.

**Lemma 5.7.** For any numbers $a, b, c \geq 0$ and $0 \leq \varepsilon \leq 1$ such that $(1+c)\varepsilon \leq 1$, it holds that
\begin{equation}
\left(a + b + (1+c)\varepsilon \right) \log(a+b+\varepsilon) \geq \left(a + (1+c)\varepsilon \right) \log(a+\varepsilon) + \left(b + (1+c)\varepsilon \right) \log(b+\varepsilon).
\end{equation}

**Proof.** Without loss of generality, we may assume that $a \geq b$. Define
\begin{equation}
f(t) := (a + t + (1+c)\varepsilon) \log(a+t+(1+c)\varepsilon) + (b - t + (1+c)\varepsilon) \log(b-t+(1+c)\varepsilon),
\end{equation}
and compute
\begin{equation}
\frac{d}{dt} \bigg|_{t=0} f(t) = \log \frac{a + (1+c)\varepsilon}{b + (1+c)\varepsilon} \geq 0.
\end{equation}

We conclude that
\begin{equation}
\left(a + b + (1+c)\varepsilon \right) \log(a+b+(1+c)\varepsilon) + (1+c)\varepsilon \log[(1+c)\varepsilon] \geq \left(a + (1+c)\varepsilon \right) \log(a+(1+c)\varepsilon) + \left(b + (1+c)\varepsilon \right) \log(b+(1+c)\varepsilon).
\end{equation}
Since \((1 + c)\varepsilon \leq 1\) by assumption, this yields
\[
\left( a + b + (1 + c)\varepsilon \right) \log \left( a + b + (1 + c)\varepsilon \right) \tag{5.10}
\]
\[
\geq \left( a + (1 + c)\varepsilon \right) \log \left( a + (1 + c)\varepsilon \right) + \left( b + (1 + c)\varepsilon \right) \log \left( b + (1 + c)\varepsilon \right).
\]

Observe also that
\[
\left( a + b + (1 + c)\varepsilon \right) \log \left( 1 - \frac{c\varepsilon}{a + b + (1 + c)\varepsilon} \right) \tag{5.11}
\]
\[
\geq \left( a + (1 + c)\varepsilon \right) \log \left( 1 - \frac{c\varepsilon}{a + (1 + c)\varepsilon} \right) + \left( b + (1 + c)\varepsilon \right) \log \left( 1 - \frac{c\varepsilon}{b + (1 + c)\varepsilon} \right).
\]

Adding (5.10) and (5.11) yields the desired result. \(\square\)

We now address \(D(\theta; x)\). The only terms that change are the ones corresponding to \(\xi^A\) and \(\xi^B\). Let \(\theta' = \varphi \# \theta\). By construction:
\[
\hat{x}_{\theta'}^{\xi^1} \leq \hat{x}_{\theta'}^{\xi^2} \leq \ldots.
\]
Since we sort both \(\hat{x}_{\theta'}\) and \(\hat{x}\) in increasing order, the value of the \(D(\theta; x)\) decreases, verifying the claim. \(\square\)

Now let \(\varphi\) denote any fusion map which, by definition, is a composition of finitely many canonical injections. Let \(x_{\varphi}\) denote the result of applying Lemma 5.6 to \(x\) along such a sequence of injections. Then \(x_{\varphi} \in K_h\) and \(v_{\varphi} = \varphi \# v\). Finally, note that \(\mu_{\varphi} = \Lambda_1 v_{\varphi} \in \llbracket \varphi \# \mu \rrbracket\) by Lemma 5.4. This completes the verification of (A3).

**Axiom (A4).** Clearly the change \(\theta \mapsto \theta'\) with \(\theta, \theta' \in \hat{M}_k(L_T)\) does not change \(H\), so we need only analyze the first part of \(\Phi\). Consider \(\xi^0 \in V_{\hat{\xi}}^{h-1}\) and a child \(\xi^1 \in V_{\hat{\xi}}^h\). If \(\theta' = F \# \theta\) where \(F(\xi) = \xi\) for \(\xi \notin V_{\hat{\xi}}(\xi^1)\) and \(F(V_{\hat{\xi}}(\xi^1)) \subseteq V_{\hat{\xi}}(\xi^0)\), then the value of \(\Phi\) can change by at most
\[
\left| \sum_{j \geq h} \tau^{-j} \sum_{i \geq 1} \sum_{\xi_2 \in V_{\hat{\xi}}^j; \xi_2 \leq \xi_1} \log \left( x_{\xi_2}^{\xi_2,i} + \delta \right) \right|, \tag{5.12}
\]
where we have used the notation \(\xi_2 \leq \xi_1\) to denote that \(\xi_2\) is a descendant of \(\xi_1\) (and we say that \(\xi_1\) is a descendant of itself). The desired conclusion follows from the next fact.

**Fact 5.8.** For every \(x \in [0, 1]\) and \(\delta \in [0, \frac{1}{2}]\):
\[
\log \frac{1 + \delta}{x + \delta} \leq \frac{1 - x}{1 - \delta} \log \frac{1}{\delta}.
\]

Using this and recalling that \(\delta = \frac{1}{x}\), (5.12) is bounded by
\[
O(\log k) \sum_{j \geq h} \tau^{-j} \sum_{i \geq 1} \sum_{\xi_2 \in V_{\hat{\xi}}^j; \xi_2 \leq \xi_1} \frac{1 - x_{\xi_2}^{\xi_2,i}}{1 - \delta} \leq O(\log k) \tau^{-h} \mu^X(V_{\hat{\xi}}(\xi_1)).
\]

where in the last inequality we used (5.7).
5.2.1 Extension to unbounded metric spaces

Let \( \rho_0 \in X^k \) denote the initial configuration of servers. First, it is straightforward to see that the algorithm described here works for any bounded metric space, with the restriction to diameter one being a matter of scaling. We may initialize our data structure as if the underlying space is \( B_X(x_0, r_0) \) where \( x_0 = (\rho_0)_1 \) and \( r_0 \) is the smallest number such that

\[
(\rho_0)_1, \ldots, (\rho_0)_k \in B_X(x_0, r_0).
\]

The algorithm proceeds until the first time \( t \) at which \( \sigma_t \not\in B_X(x_0, r_0) \). At that point, we proceed as if the underlying metric space is \( B_X(x_0, 2r_0) \), and so on. The key observation here is that in the potential functions \( \Phi \) and \( H \), we could have summed over all \( j \in \mathbb{N} \) (instead of \( j \geq 0 \)). The terms corresponding to scales \( \tau^{-j} \) can be made zero as long as one ensures that all requests seen so far are contained in \( B_X(x_0, \tau^{-j}/2) \).

It follows that the only aspect of the initial configuration entering into the cost of the algorithm is an additive term corresponding to the \( \Phi \)-value of \( \rho_0 \), which is bounded by \( O(r_0 k \log k) \).

Acknowledgements

I am grateful to my coauthors [BCL\(^+\)17] for stimulating preliminary discussions. The author is supported by NSF grants CCF-1616297 and CCF-1407779, and a Simons Investigator Award.

References


[BCL\(^+\)17] Sebastien Bubeck, Michael B. Cohen, James R. Lee, Yin-Tat Lee, and Aleksander Mądry. \( k \)-server via multiscale entropic regularization. 2017. 1, 4, 5, 8, 14, 15, 18, 35, 36, 39


