

Conformal growth rates and spectral geometry on distributional limits of graphs

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Abstract

For a unimodular random graph (G, ρ) , we consider deformations of its intrinsic path metric by a (random) weighting of its vertices. This leads to the notion of the *conformal growth exponent* of (G, ρ) , which is the best asymptotic degree of volume growth of balls that can be achieved by such a reweighting. Under moment conditions on the degree of the root, we show that the conformal growth exponent of a unimodular random graph bounds the almost sure spectral dimension. This has interesting consequences: For instance, in a companion paper, we show that distributional limits of finite graphs that are sphere-packed in \mathbb{R}^d have conformal growth exponent at most d , and thus the preceding connection to the spectral measure provides d -dimensional lower bounds on the heat kernel for such limits.

In two dimensions, one obtains more precise information. If (G, ρ) has a property we call *gauged quadratic conformal growth*, then the following holds: If the degree of the root is uniformly bounded almost surely, then G is almost surely recurrent. Since limits of finite H -minor-free graphs have gauged quadratic conformal growth, such limits are almost surely recurrent; this affirms a conjecture of Benjamini and Schramm (2001). For the special case of planar graphs, this gives a proof of the Benjamini-Schramm Recurrence Theorem that does not proceed via the analysis of circle packings.

Gurel-Gurevich and Nachmias (2013) resolved a central open problem by showing that the uniform infinite planar triangulation (UIPT) and quadrangulation (UIPQ) are almost surely recurrent. They proved that this holds for any distributional limit of planar graphs in which the degree of the root has exponential tails (which is known to hold for UIPQ and UIPQ).

We use the gauged quadratic conformal growth property to give a new proof of this result that holds for distributional limits of finite H -minor-free graphs. Our arguments yield quantitative bounds on the heat kernel in terms of the degree distribution at the root. For instance, they imply that the spectral dimension of UIPQ/UIPQ is almost surely at most 2. We also present a new approach to subdiffusivity of the random walk on UIPQ/UIPQ, using only the volume growth profile of balls in the intrinsic metric.

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1 Introduction

Motivated by the study of random surfaces in Quantum Geometry [ADJ97], Benjamini and Schramm [BS01] sought to understand the behavior of random planar triangulations. Toward this end, they introduced the notion of the *distributional limit* of a sequence of finite graphs $\{G_n\}$. This limit is a random rooted infinite graph (G, ρ) with the property that the laws of neighborhoods of a randomly chosen vertex in G_n converge, as $n \rightarrow \infty$, to the laws of neighborhoods of ρ in G . When the limit exists, it is a *unimodular random graph* in the sense of Aldous and Lyons [AL07].

An example of central importance is the *uniform infinite planar triangulation* (UIPT) of Angel and Schramm [AS03] which is obtained by taking the distributional limit of a uniform random triangulation of the 2-sphere with n vertices. For combinatorial reasons, a popular variant is the *uniform infinite planar quadrangulation* (UIPQ) constructed by Krikun [Kri08]. More recently, Benjamini and Curien [BC11] sought to extend these studies to graphs that can be sphere-packed in \mathbb{R}^d for $d \geq 3$, but noted that the higher-dimensional setting is substantially more difficult.

In general, the goal of this line of work is to understand the almost sure geometric properties of the limit object, where often interesting phenomena emerge. For instance, Angel [Ang03] has shown that almost surely balls of radius R in UIPQ have volume $R^{4+o(1)}$, but such a ball can be separated from infinity by removing only $R^{1+o(1)}$ vertices. This reflects the fractal geometry of UIPQ and leads one to suspect, for instance, that the random walk should be recurrent, and the speed of the walk should be subdiffusive.

Indeed, Benjamini and Curien [BC13] proved that the random walk in UIPQ is almost surely subdiffusive: The average distance from the starting point is at most $T^{1/3+o(1)}$ after T steps; the correct exponent is conjectured to be $1/4$, as predicted by the KPZ relations (see the discussion in [BC13]). More recently, Gurel-Gurevich and Nachmias [GN13] established that the random walk on UIPQ and UIPQ is almost surely recurrent.

While the theory developed here applies to a wide range of distributional limits, our methods yield new proofs of the preceding results in substantially more general settings, and more detailed information even for the specific models of UIPQ and UIPQ. For instance, we will see that the $T^{1/3+o(1)}$ speed bound actually holds for any unimodular random planar graph with quartic volume growth, or alternately in any unimodular random graph where there is a large enough discrepancy between the volume growth and isoperimetric profile. And we are able to strengthen almost sure recurrence for UIPQ/UIPQ to the conclusion that almost surely the number of returns to the root by time T grows asymptotically faster than $\log \log T$.

Previous work on distributional limits of planar graphs relies heavily on the analysis of circle packings, which can be thought of as ambient representations that conformally uniformize the geometry of the underlying graph. Here we take an intrinsic approach, deforming the graph geometry directly using a family of discrete graph metrics. This makes our methods significantly more flexible and applicable to a much broader family of graphs. The connection between discrete uniformization and spectral geometry of graphs is present in earlier joint works with Biswal and Rao [BLR10] and Kelner, Price, and Teng [KLPT11], where we showed how such metrics can be used to control the spectrum of the Laplacian in bounded-degree graphs.

1.1 Discrete conformal metrics and the growth exponent

Consider a locally finite, connected graph G . A *conformal metric* on G is a map $\omega : V(G) \rightarrow \mathbb{R}_+$. The metric endows G with a graph distance as follows: Give to every edge $\{u, v\} \in E(G)$ a length $\text{len}_\omega(\{u, v\}) := \frac{1}{2}(\omega(u) + \omega(v))$. This prescribes to every path $\gamma = \{v_0, v_1, v_2, \dots\}$ in G the induced

length

$$\text{len}_\omega(\gamma) := \sum_{k \geq 0} \text{len}_\omega(\{v_k, v_{k+1}\}).$$

Now for $u, v \in V(G)$, one defines the path metric $\text{dist}_\omega(u, v)$ as the infimum of the lengths of all u - v paths in G . Denote the closed ball

$$B_\omega(x, R) = \{y \in V(G) : \text{dist}_\omega(x, y) \leq R\}.$$

If (G, ρ) is a unimodular random graph, then a *conformal metric* on (G, ρ) is a (marked) unimodular random graph (G', ω, ρ') with $\omega : V(G) \rightarrow \mathbb{R}_+$ such that (G, ρ) and (G', ρ') have the same law. We say that the conformal weight is *normalized* if $\mathbb{E}[\omega(\rho)^2] = 1$. See [Section 1.6.1](#) for precise definitions.

One thinks of such a metric $\omega : V(G) \rightarrow \mathbb{R}_+$ as deforming the geometry of the underlying graph. It will turn out that normalized conformal metrics with nice geometric properties form a powerful tool in understanding the structure of (G, ρ) . A basic property one might hope for is controlled volume growth of balls: $|B_\omega(\rho, R)| \leq O(R^d)$ for some fixed $d > 0$. As we will see, the best exponent d one can achieve controls the *spectral dimension* of G from above.

Spectral dimension vs. conformal growth exponent. Consider a unimodular random graph (G, ρ) . We define the *upper and lower conformal growth exponents* of (G, ρ) , respectively, by

$$\begin{aligned} \overline{\text{dim}}_{\text{cg}}(G, \rho) &:= \inf_{\omega} \limsup_{R \rightarrow \infty} \frac{\log \|\#B_\omega(\rho, R)\|_{L^\infty}}{\log R}, \\ \underline{\text{dim}}_{\text{cg}}(G, \rho) &:= \inf_{\omega} \liminf_{R \rightarrow \infty} \frac{\log \|\#B_\omega(\rho, R)\|_{L^\infty}}{\log R}, \end{aligned}$$

and the infimum is over all normalized conformal metrics on (G, ρ) , and we use $\|X\|_{L^\infty}$ to denote the essential supremum of a random variable X , and $\#S$ to denote the cardinality of a finite set S .

When $\overline{\text{dim}}_{\text{cg}}(G, \rho) = \underline{\text{dim}}_{\text{cg}}(G, \rho)$, we define the *conformal growth exponent* by

$$\text{dim}_{\text{cg}}(G, \rho) := \overline{\text{dim}}_{\text{cg}}(G, \rho) = \underline{\text{dim}}_{\text{cg}}(G, \rho).$$

Note that the quantities $\overline{\text{dim}}_{\text{cg}}$, $\underline{\text{dim}}_{\text{cg}}$, dim_{cg} are functions of the law of (G, ρ) ; they are not defined on (fixed) rooted graphs.

As an indication that the conformal growth exponent can be bounded in interesting settings, let us state the next theorem which is proved in the companion paper [\[Lee17a\]](#). We use \Rightarrow to denote convergence in the distributional sense; see [Section 1.6.1](#). Say that a graph G is *sphere-packed in \mathbb{R}^d* if G is the tangency graph of a collection of interior-disjoint Euclidean balls in \mathbb{R}^d .

Theorem 1.1. *If $\{G_n\}$ are finite graphs such that each G_n is sphere-packed in \mathbb{R}^d , and $\{G_n\} \Rightarrow (G, \rho)$, then there is a normalized conformal metric $\omega : V(G) \rightarrow \mathbb{R}_+$ such that for all $R \geq 1$,*

$$\|\#B_\omega(\rho, R)\|_{L^\infty} \leq O(R^d (\log R)^2).$$

In particular, $\overline{\text{dim}}_{\text{cg}}(G, \rho) \leq d$.

We remark that some $(\log R)^{O(1)}$ factor is necessary ([Lemma 2.15](#)), even for the case $d = 2$ (planar graphs). [Theorem 1.1](#) is proved in somewhat greater generality: One can replace \mathbb{R}^d by any Ahlfors d -regular metric measure space and relax the notion of ‘‘packing’’ to allow bounded-multiplicity overlap of balls. We refer to [\[Lee17a\]](#) for details.

For a locally finite, connected graph G , denote the discrete-time heat kernel

$$p_T^G(x, y) := \mathbb{P}[X_T = y \mid X_0 = x], \quad x, y \in V(G),$$

where $\{X_n\}$ is the standard random walk on G . We recall the *spectral dimension* of G :

$$\dim_{\text{sp}}(G) := \lim_{n \rightarrow \infty} \frac{-2 \log p_{2n}^G(x, x)}{\log n},$$

whenever the limit exists. If the limit does exist, then it is the same for all $x \in V(G)$.

The spectral dimension is considered an important quantity in the study of quantum gravity, since it can be defined in a reparameterization-invariant way [ANR⁺98, AAJ⁺98]. It has long been conjectured that the spectral dimension of 2D quantum gravity is equal to two. Our results confirm that the almost sure spectral dimension is at most two for UIPT/UIPT.

Define also the *upper and lower spectral dimension* of G , respectively:

$$\begin{aligned} \overline{\dim}_{\text{sp}}(G) &:= \limsup_{n \rightarrow \infty} \frac{-2 \log p_{2n}^G(x, x)}{\log n}, \\ \underline{\dim}_{\text{sp}}(G) &:= \liminf_{n \rightarrow \infty} \frac{-2 \log p_{2n}^G(x, x)}{\log n}. \end{aligned}$$

It turns out that conformal growth exponent bounds the spectral dimension in somewhat general settings.

Say that a real-valued random variable X has *negligible tails* if its tails decay faster than any inverse polynomial:

$$\lim_{n \rightarrow \infty} \frac{\log n}{|\log \mathbb{P}[|X| > n]|} = 0, \quad (1.1)$$

where we take $\log(0) = -\infty$ in the preceding definition (in the case that X is essentially bounded).

For the sake of clarity in the next statement, we use (G, ρ) to denote the law μ of (G, ρ) , and (G, ρ) to denote the random variable with law μ .

Theorem 1.2. *If (G, ρ) is a unimodular random graph and $\deg_G(\rho)$ has negligible tails, then almost surely:*

$$\begin{aligned} \overline{\dim}_{\text{sp}}(G) &\leq \overline{\dim}_{\text{cg}}(G, \rho), \\ \underline{\dim}_{\text{sp}}(G) &\leq \underline{\dim}_{\text{cg}}(G, \rho). \end{aligned}$$

In conjunction with [Theorem 1.1](#), this shows that if (G, ρ) is the distributional limit of finite \mathbb{R}^d -packable graphs (and $\deg_G(\rho)$ has negligible tails), then almost surely:

$$p_{2T}^G(\rho, \rho) \geq T^{-d/2-o(1)} \quad \text{as } T \rightarrow \infty.$$

We remark that a more classical approach based on the ℓ^d theory of conformal length and d -parabolicity appears unable to achieve such bounds for $d > 2$ [BC11].

An overview of the proof of [Theorem 1.2](#) in the special case of 2-dimensional growth is given in [Section 1.3](#). The inequalities in [Theorem 1.2](#) imply that the conformal growth rate provides a lower bound on the return probabilities up to a $T^{o(1)}$ correction factor. We remark that if one makes the stronger assumption that $\deg_G(\rho)$ has exponential tails, then the implied correction factors are only polylogarithmic; see the discussion in [Section 4.2](#).

Finally, we note that the inequalities in [Theorem 1.2](#) cannot be reversed in general: There is a unimodular random graph (G, ρ) with uniformly bounded degrees such that $\dim_{\text{sp}}(G, \rho)$ is finite but $\dim_{\text{cg}}(G, \rho)$ is infinite; see [Section 4.4](#) where we review an example due to [AHNR16].

Uniformization and intrinsic dimension. Certainly circle packings of planar graphs are a powerful, elegant, and “conformally natural” [Roh11] tool. Still, it is enlightening to think of situations

where ambient representations do a poor job of emphasizing the intrinsic geometry of the underlying graph. In general, this is the case when the dimension of the graph differs from that of the ambient space.

A basic example is the planar graph $G = (V, E)$ which is the product of a triangle and a bi-infinite path: $V = \{0, 1, 2\} \times \mathbb{Z}$ and $\{x, y\} \in E$ if and only if $\|x - y\|_1 = 1$. This graph is quasi-isometric to \mathbb{Z} , and thus manifestly one-dimensional. The appropriate uniformizing conformal metric (by transitivity) is $\omega \equiv 1$. The circle packing in \mathbb{R}^2 (which is unique up to Möbius transformations) has an accumulation point in \mathbb{R}^2 , and the radii of the circles grow with geometrically increasing radii from the accumulation point to infinity.

Consider another example: the incipient infinite cluster (IIC) of critical percolation $(G_d^{\text{IIC}}, 0)$ on \mathbb{Z}^d . In their solution to the Alexander-Orbach conjecture in high dimensions, Kozma and Nachmias [KN09] show that for $d \geq 11^1$, almost surely $\dim_{\text{sp}}(G_d^{\text{IIC}}) = 4/3$. Theorem 1.2 implies that $\dim_{\text{cg}}(G_d^{\text{IIC}}) \geq 4/3$. Moreover, one can show that this is tight. For instance, G_d^{IIC} is spectrally homogeneous in the sense of (4.23) in Section 4.4 with $d = 4/3$ [Nac17]. In this case, the inequality in Theorem 1.2 can be reversed, and $\dim_{\text{cg}}(G_d^{\text{IIC}}) = 4/3$.

1.2 Dimension two: Gauged quadratic growth and recurrence

The conformal growth exponent is not precise enough to study recurrence (which depends on lower-order factors in the heat kernel $p_T^G(\rho, \rho)$). Say that a unimodular random graph (G, ρ) is (C, R) -quadratic for $C > 0$ and $R \geq 1$ if

$$\inf_{\omega} \|\#B_{\omega}(\rho, R)\|_{L^{\infty}} \leq CR^2, \quad (1.2)$$

where the infimum is over all normalized conformal metrics on (G, ρ) . We say that (G, ρ) has *gauged quadratic conformal growth* (gQCG) if there is a constant $C > 0$ such that (G, ρ) is (C, R) -quadratic for all $R \geq 1$. Note that we allow a different conformal weight ω for every choice of R , and this is necessary for distributional limits of finite planar graphs to have gQCG (see Lemma 2.15).

Theorem 1.3. *If (G, ρ) is a unimodular random graph with uniformly bounded degrees and gauged quadratic conformal growth, then G is almost surely recurrent.*

In Section 2.3, we argue that distributional limits of planar graphs, H -minor free graphs, string graphs, and other families have gauged quadratic conformal growth. Thus Theorem 1.3 generalizes the Benjamini-Schramm Recurrence Theorem [BS01] to H -minor-free graphs, confirming a conjecture stated there. After initial dissemination of a draft of this manuscript, we learned that Angel and Szegedy (personal communication) had previously discovered a proof of the H -minor-free case using a detailed analysis of the Robertson-Seymour classification [RS04].²

Remark 1.4 (String graphs). By the Koebe-Andreev-Thurston circle packing theorem, planar graphs are precisely the tangency graphs of interior-disjoint disks in the plane. *String graphs* are a significant generalization: They are the intersection graphs of a collection of arbitrary path-connected regions in the plane (with no assumption on disjointness). Such graphs can be dense, but string graphs with uniformly bounded degrees have quadratic conformal growth (see Section 2.3 and [Lee17b]).

Unbounded degrees. Recently Gurel-Gurevich and Nachmias [GN13] resolved a central open problem by showing that the uniform infinite planar triangulation (UIPT) and quadrangulation

¹One needs to use [FH15] to obtain $d \geq 11$; the original reference proves it for $d \geq 19$.

²We remark that establishing quadratic conformal growth for H -minor-free graphs does not require the Robertson-Seymour theory; see [Lee17b].

(UIPQ) are almost surely recurrent. They achieved this by extending the Recurrence Theorem of Benjamini and Schramm in a different direction: In every distributional limit of finite planar graphs where the degree of the root has exponential tails, the limit is almost surely recurrent. It was previously known that both UIP and UIPQ satisfy this hypothesis.

Let μ denote the law of (G, ρ) , and define $\bar{d}_\mu : [0, 1] \rightarrow \mathbb{R}_+$ by

$$\bar{d}_\mu(\varepsilon) := \sup \{ \mathbb{E} [\deg_G(\rho) \mid \mathcal{E}] : \mathbb{P}(\mathcal{E}) \geq \varepsilon \},$$

where the supremum is over all measurable sets \mathcal{E} with $\mathbb{P}(\mathcal{E}) \geq \varepsilon$. That $\deg_G(\rho)$ has exponential tails is equivalent to $\bar{d}_\mu(1/t) \leq O(\log t)$ as $t \rightarrow \infty$. Negligible tails as defined in (1.1) is equivalent to the assumption that

$$\bar{d}_\mu(1/t) \leq t^{o(1)} \quad \text{as } t \rightarrow \infty. \quad (1.3)$$

Assumption 1.5. Suppose (G, ρ) is a unimodular random graph with law μ satisfying the following:

1. (G, ρ) has gauged quadratic conformal growth.
2. (G, ρ) is uniformly decomposable (cf. Section 1.6.2).
3. $\mathbb{E}[\deg_G(\rho)^2] < \infty$.

Theorem 1.6. Under Assumption 1.5, if additionally

$$\sum_{t \geq 1} \frac{1}{t \bar{d}_\mu(1/t)} = \infty,$$

then G is almost surely recurrent.

It was previously known (see Section 1.6.2) that many families of finite graphs—planar graphs, H -minor-free graphs, and string graphs—are uniformly decomposable. This property passes to distributional limits, hence Theorem 1.6 generalizes the result of [GN13]. Note that we allow slightly heavier tails: For instance, $\bar{d}_\mu(1/t) \leq O(\log t \log \log t)$ is still enough to guarantee recurrence.

Moreover, Theorem 1.6 is tight in the following sense: For any monotonically non-decreasing sequence $\{d_t : t = 1, 2, \dots\}$ such that $\sum_{t \geq 1} \frac{1}{t d_t} < \infty$, there is a unimodular random planar graph satisfying Assumption 1.5 that is almost surely transient, and such that $\bar{d}_\mu(1/t) \leq d_t$ for all t sufficiently large; see Section 4.3.1.

1.3 Estimates on the spectral measure and the heat kernel

Let us now describe some of the elements of the proof of Theorem 1.6, along with more detailed information about the random walk. In Section 4.1, we argue that $\underline{\dim}_{\text{cg}}(G, \rho) < \infty$ implies that (G, ρ) is invariantly amenable, and thus it is a distributional limit of finite graphs: $\{G_n\} \Rightarrow (G, \rho)$.

Thus for simplicity, let us consider a finite planar graph G_n and a root $\rho_n \in V(G_n)$ chosen uniformly at random. Without loss, we may assume that $n = |V(G_n)|$. Define

$$\Delta_{G_n}(k) := \max_{S \subseteq V(G_n): |S| \leq k} \sum_{x \in S} \deg_{G_n}(x)$$

to be the sum of the k largest vertex degrees in G_n . In Section 3, we establish the bound

$$\lambda_k(G_n) \leq c \frac{\Delta_{G_n}(k)}{n}, \quad (1.4)$$

where c is a universal constant and $\{1 - \lambda_k(G_n) : k = 0, 1, \dots, n-1\}$ are the eigenvalues of the random walk operator on G_n . In [KLPT11] a weaker bound was proved, with $k \cdot \Delta_{G_n}(1)$ in place of $\Delta_{G_n}(k)$.

Such a bound provides average estimates for the diagonal of the heat kernel: Let P denote the random walk operator on G_n . Then for an integer $T \geq 0$,

$$\mathbb{E}[p_T^{G_n}(\rho_n, \rho_n)] = \frac{1}{n} \sum_{x \in V(G_n)} \langle \mathbb{1}_x, P^T \mathbb{1}_x \rangle = \frac{\text{tr}(P^T)}{n} = \frac{1}{n} \sum_{k=0}^{n-1} (1 - \lambda_k(G_n))^T \geq \frac{\#\{k : \lambda_k(G_n) \leq 1/T\}}{4n}, \quad (1.5)$$

where the last inequality holds for $T \geq 2$.

Thus if the vertex degrees are uniformly bounded along the sequence $\{G_n\}$, then we have $\Delta_{G_n}(k) \leq O(k)$, and combining (1.4) and (1.5) yields

$$\mathbb{E}[p_T^G(\rho, \rho)] \geq \liminf_{n \rightarrow \infty} \mathbb{E}[p_T^{G_n}(\rho_n, \rho_n)] \gtrsim \frac{1}{T}.$$

If $\{G_n\} \Rightarrow (G, \rho)$ and we impose only the weaker assumption that $\deg_G(\rho)$ has exponential tails, then it must hold that for n sufficiently large, $\Delta_{G_n}(\frac{n}{T}) \leq O(\frac{n}{T} \log T)$, and one obtains

$$\mathbb{E}[p_T^G(\rho, \rho)] \geq \liminf_{n \rightarrow \infty} \mathbb{E}[p_T^{G_n}(\rho_n, \rho_n)] \gtrsim \frac{1}{T \log T}. \quad (1.6)$$

In this way, the degree-modified two-dimensional Weyl law in (1.4) predicts recurrence when the degree of the root has exponential tails, since $\sum_{T \geq 1} \frac{1}{T \log T} = \infty$. But a significant obstacle is that the annealed estimate (1.6) does not necessarily imply anything for the distributional limit. The issue is that the lower bound in (1.6) could come entirely from a small set of vertices (and such small sets could be negligible in the distributional limit). Indeed, one could add to G_n only εn isolated vertices to achieve $\frac{1}{n} \sum_{x \in V(G_n)} p_T^{G_n}(x, x) \geq \varepsilon$. (See Section 4.4 for a family of examples where the annealed return probabilities are dominated by a small measure of roots.) Thus even to obtain almost sure recurrence, we need an estimate stronger than (1.5). We state now the following strengthening of Theorem 1.6.

Theorem 1.7. *Under Assumption 1.5, the following holds. There is a constant $C = C(\mu)$ such that for every $\delta > 0$ and all $T \geq C/\delta^2$,*

$$\mathbb{P} \left[p_{2T}^G(\rho, \rho) < \frac{\delta}{T \bar{d}_\mu(1/T^3)} \right] \leq C \delta^{1/8}.$$

Note that even for the special case of UIPT/UIPQ, this estimate yields $\mathbb{E} p_{2T}^G(\rho, \rho) \gtrsim \frac{1}{T \log T}$ for every $T \geq 2$, improving over the bound $\mathbb{E} p_{2T}^G(\rho, \rho) \gtrsim \frac{1}{T^{4/3}(\log T)^{O(1)}}$ from [BC13]. As a consequence, one obtains a bound on the rate of divergence of the Green function: Define

$$g_\mu(T) := \sum_{t=1}^T \frac{1}{t \bar{d}_\mu(1/t)}.$$

For instance, for UIPT/UIPQ, one has $g_\mu(T) \asymp \log \log T$.

Theorem 1.8. *Under Assumption 1.5, the following holds. If $g_\mu(T) \rightarrow \infty$, then G is almost surely recurrent. Moreover, almost surely:*

$$\limsup_{T \rightarrow \infty} \frac{\sum_{t=1}^T p_t^G(\rho, \rho)}{g_\mu(T)} > 0.$$

Exhaustion by capacitors. For a finite graph G and $\varphi : V(G) \rightarrow \mathbb{R}$, define the normalized Dirichlet energy

$$\mathcal{E}_G(\varphi) := \frac{1}{|E(G)|} \sum_{\{x,y\} \in E(G)} |\varphi(x) - \varphi(y)|^2.$$

Recall that for subsets $A \subseteq \Omega \subseteq V(G)$, one defines the *capacity* of (A, Ω) by

$$\text{cap}_\Omega^G(A) := \inf_{\varphi: V(G) \rightarrow [0,1]} \{ \mathcal{E}_G(\varphi) : \varphi|_A \equiv 1, \text{supp } \varphi \subseteq \Omega \}.$$

One refers to the pair (A, Ω) as a *capacitor*.

[Theorem 1.2](#), [Theorem 1.7](#), and [Theorem 1.8](#) are proved as follows. One uses an appropriate conformal weight on a finite graph G to locate capacitors $(A_1, \Omega_1), \dots, (A_k, \Omega_k)$ such that the sets $\{\Omega_i\}$ are pairwise disjoint. Using reversibility of the random walk, such a collection of capacitors yields a lower bound on typical return probabilities (see [Theorem 3.5](#)): For every $T \geq 1$ and $\varepsilon > 0$,

$$\pi \left(\left\{ x \in V(G) : p_{2T}^G(x, x) \geq \frac{\varepsilon^2}{4M} \right\} \right) \geq -3\varepsilon \bar{d}_G(\varepsilon) + \sum_{i=1}^k \pi(A_i) - 2T \sum_{i=1}^k \text{cap}_{\Omega_i}^G(A_i),$$

where $M = \max\{|\Omega_i| : i = 1, \dots, k\}$, and $\bar{d}_G(\varepsilon) := \frac{\Delta_G(\varepsilon n)}{\varepsilon n}$ (this is equal to $\bar{d}_\mu(\varepsilon)$ when μ is the law of (G, ρ) with G a fixed finite graph and $\rho \in V(G)$ chosen uniformly at random).

1.4 Intrinsic volume growth, Markov type, and subdiffusivity

Consider a connected, infinite, locally finite graph G . For $x \in V(G)$ and $r \geq 0$, let

$$B_G(x, r) = \{y \in V(G) : \text{dist}_G(x, y) \leq r\},$$

where dist_G denotes the (unweighted) graph distance in G . Suppose that G has nearly uniform d -dimensional volume growth in the sense that for r sufficiently large,

$$r^{d-o(1)} \leq |B_G(x, r)| \leq r^{d+o(1)} \tag{1.7}$$

holds uniformly for all $x \in V(G)$.

When G is planar and $d > 2$, one suspects that the structure of G should be fractal. Indeed, Itai Benjamini has put forth a number of conjectures to this effect. For instance, in [\[BP11\]](#) it is conjectured that if G is planar and [\(1.7\)](#) is satisfied, then the random walk on G should be *subdiffusive* with the natural speed estimate:

$$\mathbb{E}[\text{dist}_G(X_0, X_T)] \leq T^{1/d+o(1)}. \tag{1.8}$$

There is some recent evidence that the validity of the conjecture is a subtle issue; see [Remark 1.10](#).

Subdiffusivity was confirmed specifically for UIPQ: In [\[BC13\]](#), it is shown that

$$\mathbb{E}[\text{dist}_G(X_0, X_T) \mid X_0 = \rho] \leq T^{1/3}(\log T)^{O(1)}. \tag{1.9}$$

For UIPT [\[Ang03\]](#) and UIPQ [\[BC13\]](#), an almost sure asymptotic variant of [\(1.7\)](#) is satisfied with $d = 4$. Thus the estimate [\(1.9\)](#), while non-trivial, does not meet the conjectured exponent of $1/4$.

In establishing [\(1.9\)](#), the authors undertook a detailed study of the geometry of UIPQ. We show that the phenomenon is somewhat more general: To obtain subdiffusivity for a unimodular random planar graph, one need only assume asymptotic d -dimensional volume growth for some $d > 3$.

Theorem 1.9. *Suppose that (G, ρ) is a unimodular random planar graph and for some $d > 3$, there is a function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $h(r) \leq r^{o(1)}$ and such that almost surely,*

$$\frac{r^d}{h(r)} \leq |B_G(\rho, r)| \leq h(r)r^d \quad \forall r \geq 1. \quad (1.10)$$

Then the random walk on (G, ρ) is strictly subdiffusive:

$$\mathbb{E} \left[\text{dist}_G(X_0, X_T)^{d-1} \mid X_0 = \rho \right] \leq T^{1+o(1)} \quad \text{as } T \rightarrow \infty. \quad (1.11)$$

Remark 1.10 (Sharpness of the assumption $d > 3$). After initial circulation of this manuscript, Omer Angel and Asaf Nachmias constructed, for every $\varepsilon > 0$ sufficiently small, a unimodular random planar graph (G, ρ) with uniformly bounded degrees and such that almost surely the random walk is diffusive and yet

$$\lim_{r \rightarrow \infty} \frac{\log |B_G(\rho, r)|}{\log r} = 3 - \varepsilon. \quad (1.12)$$

This suggests that the assumption $d > 3$ in [Theorem 1.9](#) may be tight, except possibly at the critical value $d = 3$. Note that [Theorem 1.9](#) does not apply directly because (1.12) only entails an almost sure asymptotic bound, while (1.10) demands a uniform bound over the choice of (G, ρ) .

Their examples *do show* that the assumption $d > 3$ in [Theorem 1.16](#) below is sharp, as they satisfy Assumption (A) with $d = 3 - \varepsilon$ for every $q > 1$ and $\alpha > 0$.

The key geometric fact about planar graphs used to prove [Theorem 1.9](#) is due to Benjamini and Papasoglu [[BP11](#)]. They show that in any planar graph satisfying (1.7), for any $x \in V(G)$ and r sufficiently large, there is a set of vertices of size at most $r^{1+o(1)}$ that separates $B_G(x, r)$ from $V(G) \setminus B_G(x, 2r)$ (see [Figure 1\(a\)](#)). In fact, the existence of such separators together with (1.10) is enough to obtain the estimate (1.11), without requiring that the graph be planar.

For a connected, locally-finite graph G , a node $x \in V(G)$, and two radii $r' > r > 0$, let $\kappa_G(x; r, r')$ denote the cardinality of the smallest set

$$U \subseteq B_G(x, r') \setminus B_G(x, r)$$

such that U separates x and $V(G) \setminus B_G(x, r')$ in G .

Lemma 1.11 ([\[BP11\]](#)). *If (G, ρ) is a unimodular random planar graph satisfying (1.7), then there is a constant $c > 1$ such that*

$$\kappa_G(\rho; r, 2r) \leq c \frac{h(4r)}{h(r/4)} \quad \forall r \geq 1.$$

Using [Lemma 1.11](#), the next theorem generalizes [Theorem 1.9](#).

Theorem 1.12. *Suppose that (G, ρ) is a unimodular random graph and for some numbers $d \geq k + 1 \geq 2$, there is a function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $h(r) \leq r^{o(1)}$ and such that almost surely (1.10) holds, and moreover,*

$$\kappa_G(r, h(r)r) \leq h(r)r^{k-1} \quad \forall r \geq 1. \quad (1.13)$$

Then one has the estimate:

$$\mathbb{E} \left[\text{dist}_G(X_0, X_T)^{d-k+1} \mid X_0 = \rho \right] \leq T^{1+o(1)} \quad \text{as } T \rightarrow \infty.$$

In particular, when $d > k + 1$, the random walk is strictly subdiffusive.

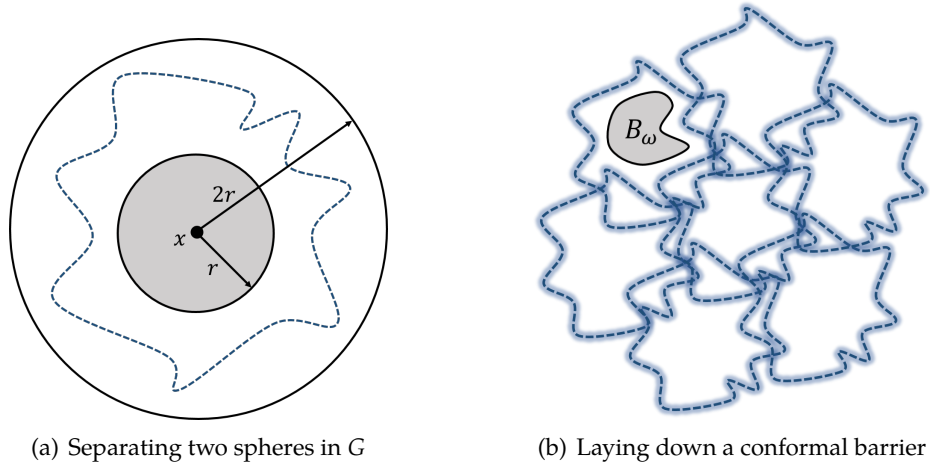


Figure 1: Establishing subdiffusivity

One can view this result as saying that when a graph has d -dimensional volume growth, but k -dimensional isoperimetry and $d > k + 1$, then this discrepancy necessitates subdiffusivity. We now elaborate on the proof of [Theorem 1.12](#) in the special case when G is almost surely planar. In particular, one will find in the argument below the relatively simple construction of a useful discrete conformal metric.

Markov type of normalized conformal metrics. A central tool is K. Ball’s notion of Markov type [[Bal92](#)].

Definition 1.13 (Markov type). A metric space (X, d) is said to have *Markov type* $p \in [1, \infty)$ if there is a constant $M > 0$ such that for every $n \in \mathbb{N}$, the following holds. For every reversible Markov chain $\{Z_t\}_{t=0}^\infty$ on $\{1, \dots, n\}$, every mapping $f : \{1, \dots, n\} \rightarrow X$, and every time $t \in \mathbb{N}$,

$$\mathbb{E} [d(f(Z_t), f(Z_0))^p] \leq M^p t \mathbb{E} [d(f(Z_0), f(Z_1))^p], \quad (1.14)$$

where Z_0 is distributed according to the stationary measure of the chain. One denotes by $M_p(X, d)$ the infimal constant M such that the inequality holds.

In [Section 5.1](#), basic Markov type theory is used to prove the following.

Theorem 1.14. *Suppose that (G, ρ) is a unimodular random graph that almost surely satisfies*

$$|B_G(\rho, r)| \leq Cr^q \quad \forall r \geq 1$$

for some numbers $C, q \geq 1$. Then for any normalized conformal metric ω on (G, ρ) , the following holds: For any $q' \geq 1$ and $T \geq 2$,

$$\mathbb{E} \left[\min\{T^{q'}, \text{dist}_\omega(X_0, X_T)\}^2 \mid X_0 = \rho \right] \leq C'T(\log T)^2,$$

where $C' = C'(C, q, q')$.

Thus in order to prove a subdiffusive estimate for the speed in dist_G , it suffices to construct a normalized conformal metric on (G, ρ) with a suitable relationship between dist_ω and dist_G .

Conformal metrics from separators. First let us fix a radius $r \geq 1$. By iteratively “cutting out” separators guaranteed by [\(1.13\)](#) in a unimodular way (see [Figure 1\(b\)](#)), a generalization of the following fact is established in [Section 5.2](#) (see [Lemma 5.8](#)).

Lemma 1.15. *Suppose the assumptions of [Theorem 1.12](#) hold. For every $r > 0$, there is a triple (G, ρ, W_r) that is unimodular as a marked network, and such that the following holds:*

1. $\mathbb{P}[\rho \in W_r] \leq r^{k-1-d+o(1)}$,
2. *Almost surely, every connected component of $G[V(G) \setminus W_r]$ has diameter at most r (in the metric dist_G).*

Heuristically, these parameters make sense: The separator occupies $r^{k-1+o(1)}$ nodes out of the $r^{d+o(1)}$ nodes in the r -ball. Now we define, for every $j \in \mathbb{N}$, the normalized conformal weight:

$$\omega_j := \frac{\mathbb{1}_{W_{2^j}}}{\sqrt{\mathbb{P}[\rho \in W_{2^j}]}}$$

and we note that from [Lemma 1.15\(1\)](#), we have

$$\omega_j \geq 2^{j(d-k+1)/2-o(1)} \mathbb{1}_{W_{2^j}}.$$

In particular, combined with [Lemma 1.15\(2\)](#), this implies that

$$\text{dist}_G(x, y) > 2^j \implies \text{dist}_{\omega_j}(x, y) \geq 2^{j(d-k+1)/2-o(1)}. \quad (1.15)$$

Finally, consider the normalized metric

$$\hat{\omega}_T := \sqrt{\frac{1}{\lceil \log_2 T \rceil} \sum_{j=0}^{\lceil \log_2 T \rceil} \omega_j^2}$$

From [\(1.15\)](#), we have almost surely for every $x, y \in V(G)$ with $\text{dist}_G(x, y) \leq T$:

$$\text{dist}_G(x, y) \leq \text{dist}_{\hat{\omega}_T}(x, y)^{2/(d-k+1)} T^{o(1)} \quad \text{as } T \rightarrow \infty.$$

In particular, for every $T \geq 1$:

$$\begin{aligned} \mathbb{E} \left[\text{dist}_G(X_0, X_T)^{d-k+1} \right] &= \mathbb{E} \left[\min\{T^{d-k+1}, \text{dist}_G(X_0, X_T)^{d-k+1}\} \right] \\ &\leq T^{o(1)} \mathbb{E} \left[\min\{T^{d-k+1}, \text{dist}_{\hat{\omega}_T}(X_0, X_T)\}^2 \right] \\ &\leq T^{1+o(1)}, \end{aligned}$$

where the final inequality follows from [Theorem 1.14](#). This yields [Theorem 1.12](#).

[Theorem 1.9](#) is not strong enough to reproduce [\(1.9\)](#) because even the vertex degrees in UIPQ are unbounded, and thus no uniform estimate of the form [\(1.10\)](#) can hold. The next result remedies this. It shows that if we have a $(1+\delta)$ -moment bound on the size of balls and a stretched exponential lower tail, then our methods can still be applied.

Theorem 1.16. *Suppose that (G, ρ) is a unimodular random graph that satisfies the following conditions for some numbers $C > 0$ and $d > 3$:*

1. *For every $r \geq 2$: $\mathbb{E}|B_G(\rho, r)| \leq r^C$.*
2. *The degree of the root has exponential tails:*

$$\mathbb{P}[\text{deg}_G(\rho) > \lambda] \leq e^{-\lambda/C} \quad \forall \lambda \geq 1.$$

Assume, additionally, that one of the following two conditions holds for some $\alpha > 0$:

(A) **Planar; volume statistics.**

- (i) G is almost surely planar, and
- (ii) For some $q > 1$ and for every $r \geq 2$:

$$\begin{aligned} (\mathbb{E} |B_G(\rho, 6r)|^q)^{1/q} &\leq Cr^d, \\ \mathbb{P} \left[|B_G(\rho, r)| < \frac{\varepsilon}{C} r^d \right] &\leq \exp(-(1/\varepsilon)^{2/\alpha}) \quad \forall \varepsilon > 0. \end{aligned}$$

(B) **Unimodular shattering.** For every $r \geq 2$, there is a random subset $W_r \subseteq V(G)$ such that (G, ρ, W_r) is unimodular (as a marked network), and

- (a) $\mathbb{P}[\rho \in W_r] \leq Cr^{1-d}(\log r)^\alpha$
- (b) Almost surely, every component of $G[V(G) \setminus W_r]$ has diameter at most r in dist_G .

Then the random walk on (G, ρ) is almost surely subdiffusive: There is a constant $C' \geq 1$ such that for all $T \geq 2$,

$$\mathbb{E} \left[\text{dist}_G(X_0, X_T)^{d-1} \mid X_0 = \rho \right] \leq C'T(\log T)^{\alpha(d-1)+5}.$$

Application to UIPT and UIPQ. It is well-known that UIPT and UIPQ satisfy conditions (1) and (2). See [AS03] for UIPT and [BC13] and for UIPQ. The fact that (A) is satisfied with $\alpha = 4$ for UIPT/UIPQ is somewhat more delicate, and is discussed in Section 5.3. In Section 5.2, it is shown that (A) \implies (B), but we include (A) to demonstrate that for unimodular random planar graphs, sufficient volume statistics are enough to yield subdiffusive behavior of the random walk.

1.5 Related work

Anomalous diffusion on fractals. The topics studied here draw from a number of areas. We are interested largely in examples where various notions of dimension fail to coincide, e.g., the topological dimension, exponent of volume growth (in the intrinsic metric), speed exponent of the random walk (cf. (1.8)), and the exponent of the isoperimetric profile (cf. Theorem 1.12). This phenomenon is characteristic of “anomalous diffusion” on fractals. We refer to [Bar04] for a discussion of such exponents and their possible relationships.

While there are conditions under which these exponents are related in a natural way (see, e.g., [BCK05]), those settings generally involve a detailed relationship between the graph distance and the effective resistance metric. In contrast, we are only able to obtain such estimates with respect to the conformal metric dist_ω (and, even then, only for most choices of the root). In this sense, the approach taken here is related to that of [ABGGN16], where the authors obtain similar estimates in a planar graph under the metric pulled back from a circle packing (under the assumption of uniformly bounded degrees).

Geometric analysis on manifolds. The eigenvalue estimate (1.4) is closely related to one established much earlier by Korevaar [Kor93] in addressing a question of S. T. Yau on the spectrum of the Laplace-Beltrami operator on orientable surfaces. There it is shown that if Ω is a subdomain in a complete d -dimensional Riemannian manifold (M, g_0) with nonnegative Ricci curvature and $(M, \varphi g_0)$ is a finite-volume conformal metric, then

$$\lambda_k \leq C_d \left(\frac{k}{\text{vol}(\Omega, \varphi g_0)} \right)^{d/2} \quad k = 1, 2, \dots,$$

where C_d is a constant depending only on d , and $\{\lambda_k\}$ are the Neumann eigenvalues of the Laplacian. An immediate consequence of the uniformization theorem is the estimate

$$\lambda_k \leq C \frac{k}{\text{vol}(\mathbb{S}^2, g)} \quad k = 1, 2, \dots$$

for any Riemannian metric g on \mathbb{S}^2 .

Korevaar’s approach (see also [GNY04] where it is generalized and expounded upon at length) also proceeds by the construction of a family of disjoint capacitors (as we do Section 3.4). But at a technical level, his argument is substantially different from the one taken here: He constructs capacitors in the metric measure space $(\mathbb{S}^2, d_{\mathbb{S}^2}, \mu)$, where $d_{\mathbb{S}^2}$ is the standard (constant curvature) distance metric on the sphere, and μ is the pushforward of the volume measure of (\mathbb{S}^2, g) under a conformal map. In particular, one knows little about μ except that it is non-atomic. On the other hand, our conformal metric dist_ω is chosen to interact nicely with the counting measure on the graph G .

Quasisymmetric uniformization. The conformal growth exponent is—at least philosophically—related to Pansu’s notion of *conformal dimension* [Pan89]. For a metric space X , this is the infimal Hausdorff dimension of all metric spaces that are quasisymmetrically equivalent to X . A closely related notion occurs implicitly in a paper of Bourdon and Pajot [BP03] and is defined by explicitly by Bonk and Kleiner [BK05]: The *Ahlfors regular conformal dimension* of X is the infimal d such that X is quasisymmetric to an Ahlfors d -regular metric measure space.

The quasisymmetric uniformization problem asks when a metric space can be quasisymmetrically parameterized by a class of model spaces. We refer to the ICM lectures of Bonk [Bon06] and Kleiner [Kle06] for a survey of this area.

1.6 Preliminaries

We use the notation $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$. We also employ the asymptotic notations $A \lesssim B$ and $A \leq O(B)$ to denote that $A \leq c \cdot B$ where $c > 0$ is a positive constant that is independent of other parameters. Since the symbol \lesssim appears only finitely many times in this paper, one could in fact take $c > 0$ to be a fixed universal constant.

We sometimes write $[n] = \{1, 2, \dots, n\}$. When X is a finite set and $f : X \rightarrow \mathbb{R}$, we use the notations:

$$\|\omega\|_{\ell^2(X)} := \sqrt{\sum_{x \in X} f(x)^2},$$

$$\|\omega\|_{L^2(X)} := \sqrt{\frac{1}{|X|} \sum_{x \in X} f(x)^2}.$$

All graphs appearing in this paper are undirected and locally finite and without loops or multiple edges. If G is such a graph, we use $V(G)$ and $E(G)$ to denote the vertex and edge set of G , respectively. If $S \subseteq V(G)$, we use $G[S]$ for the induced subgraph on S . For $A, B \subseteq V(G)$, we write $E_G(A, B)$ for the set of edges with one endpoint in A and the other in B . We write dist_G for the unweighted path metric on $V(G)$, and $B_G(x, r) = \{y \in V(G) : \text{dist}_G(x, y) \leq r\}$ to denote the closed r -ball around $x \in V(G)$. Also let $\deg_G(x)$ denote the degree of a vertex $x \in V(G)$, and $d_{\max}(G) = \sup_{x \in V(G)} \deg_G(x)$. Write $G_1 \cong G_2$ to denote that G_1 and G_2 are isomorphic as graphs. If (G_1, ρ_1) and (G_2, ρ_2) are rooted graphs, we write $(G_1, \rho_1) \cong_\rho (G_2, \rho_2)$ to denote the existence of a rooted isomorphism.

Graph minors and region intersection graphs. If H and G are finite graphs, one says that H is a *minor* of G if H can be obtained from G by a sequence of edge deletions, vertex deletions, and edge contractions. If G is infinite, say that H is a minor of G if there is a finite subgraph G' of G that contains an H minor. Recall Kuratowski's theorem: Planar graphs are precisely the graphs that do not contain $K_{3,3}$ or K_5 as a minor.

A graph G is a *region intersection graph* over G_0 if the vertices of G correspond to connected subsets of G_0 and there is an edge between two vertices of G precisely when those subsets intersect. More formally, there is a family of connected subsets $\{R_u \subseteq V_0 : u \in V\}$ such that $\{u, v\} \in E \iff R_u \cap R_v \neq \emptyset$. We use $\text{rig}(G_0)$ to denote the family of all *finite* region intersection graphs over G_0 .

A prototypical family of region intersection graphs is the set of *string graphs*; these are the intersection graphs of continuous arcs in the plane. It is not difficult to see that $\text{rig}(\mathbb{Z}^2)$ is precisely the family of all finite string graphs (see [Lee16, Lem. 1.4]).

1.6.1 Unimodular random graphs and distributional limits

We begin with a discussion of unimodular random graphs and distributional limits. One may consult the extensive reference of Aldous and Lyons [AL07]. The paper [BS01] offers a concise introduction to distributional limits of finite planar graphs. We briefly review some relevant points.

Let \mathcal{G} denote the set of isomorphism classes of connected, locally finite graphs; let \mathcal{G}_\bullet denote the set of *rooted* isomorphism classes of *rooted*, connected, locally finite graphs. Define a metric on \mathcal{G}_\bullet as follows: $\mathfrak{d}_{\text{loc}}((G_1, \rho_1), (G_2, \rho_2)) = 1/(1 + \alpha)$, where

$$\alpha = \sup \{r > 0 : B_{G_1}(\rho_1, r) \cong_\rho B_{G_2}(\rho_2, r)\} .$$

$(\mathcal{G}_\bullet, \mathfrak{d}_{\text{loc}})$ is a separable, complete metric space. For probability measures $\{\mu_n\}, \mu$ on \mathcal{G}_\bullet , write $\{\mu_n\} \Rightarrow \mu$ when μ_n converges weakly to μ with respect to $\mathfrak{d}_{\text{loc}}$.

The Mass-Transport Principle. Let $\mathcal{G}_{\bullet\bullet}$ denote the set of doubly-rooted isomorphism classes of doubly-rooted, connected, locally finite graphs. A probability measure μ on \mathcal{G}_\bullet is *unimodular* if it obeys the following *Mass-Transport Principle*: For all Borel-measurable $F : \mathcal{G}_{\bullet\bullet} \rightarrow [0, \infty]$,

$$\int \sum_{x \in V(G)} F(G, \rho, x) d\mu((G, \rho)) = \int \sum_{x \in V(G)} F(G, x, \rho) d\mu((G, \rho)) . \quad (1.16)$$

If (G, ρ) is a random rooted graph with law μ , and μ is unimodular, we say that (G, ρ) is a *unimodular random graph*.

Distributional limits of finite graphs. As observed by Benjamini and Schramm [BS01], unimodular random graphs can be obtained from limits of finite graphs. Consider a sequence $\{G_n\} \subseteq \mathcal{G}$ of finite graphs, and let ρ_n denote a uniformly random element of $V(G_n)$. Then $\{(G_n, \rho_n)\}$ is a sequence of \mathcal{G}_\bullet -valued random variables, and one has the following.

Lemma 1.17. *If $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$, then (G, ρ) is unimodular.*

If $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$, we say that (G, ρ) is the *distributional limit* of the sequence $\{(G_n, \rho_n)\}$. When $\{G_n\}$ is a sequence of finite graphs, we write $\{G_n\} \Rightarrow (G, \rho)$ for $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$ where $\rho_n \in V(G_n)$ is chosen uniformly at random.

Unimodular random conformal graphs. A *conformal graph* is a pair (G, ω) , where G is a connected, locally finite graph and $\omega : V(G) \rightarrow \mathbb{R}_+$. Let \mathcal{G}^* and \mathcal{G}_\bullet^* denote the collections of isomorphism

classes of conformal graphs and conformal rooted graphs, respectively. As in [Section 1.6.1](#), one can define a metric on \mathcal{G}_\bullet^* as follows: $\mathfrak{d}_{\text{loc}}^*((G_1, \omega_1, \rho_1), (G_2, \omega_2, \rho_2)) = 1/(\alpha + 1)$, where

$$\alpha = \sup \left\{ r > 0 : B_{G_1}(\rho_1, r) \cong_\rho B_{G_2}(\rho_2, r) \text{ and } \mathfrak{d}(\omega_1|_{B_{G_1}(\rho_1, r)}, \omega_2|_{B_{G_2}(\rho_2, r)}) \leq \frac{1}{r} \right\},$$

where for two weights $\omega_1 : V(H_1) \rightarrow \mathbb{R}_+$ and $\omega_2 : V(H_2) \rightarrow \mathbb{R}_+$ on rooted-isomorphic graphs (H_1, ρ_1) and (H_2, ρ_2) , we write

$$\mathfrak{d}(\omega_1, \omega_2) := \inf_{\psi: V(H_1) \rightarrow V(H_2)} \|\omega_2 \circ \psi - \omega_1\|_{\ell^\infty},$$

where the infimum is over all graph isomorphisms from H_1 to H_2 satisfying $\psi(\rho_1) = \rho_2$.

If $\{\mu_n\}$ and μ are probability measures on \mathcal{G}_\bullet^* , we abuse notation and write $\{\mu_n\} \Rightarrow \mu$ to denote weak convergence with respect to $\mathfrak{d}_{\text{loc}}^*$. One defines unimodularity of a random rooted conformal graph (G, ω, ρ) analogously to [\(1.16\)](#): It should now hold that for all Borel-measurable $F : \mathcal{G}_{\bullet\bullet}^* \rightarrow [0, \infty]$,

$$\int \sum_{x \in V(G)} F(G, \omega, \rho, x) d\mu((G, \omega, \rho)) = \int \sum_{x \in V(G)} F(G, \omega, x, \rho) d\mu((G, \omega, \rho)).$$

Indeed, such decorated graphs are a special case of the marked networks considered in [\[AL07\]](#), and again it holds that every distributional limit of finite unimodular random conformal graphs is a unimodular random conformal graph.

Given a random conformal graph (G, ω, ρ) , we define

$$\|\omega\|_{L^2} := \sqrt{\mathbb{E} \omega(\rho)^2}.$$

Say that ω is *normalized* if $\|\omega\|_{L^2} = 1$.

Suppose that (G, ρ) is a unimodular random graph. A *conformal weight* on (G, ρ) is a unimodular random conformal graph (G', ω, ρ') such that (G, ρ) and (G', ρ') have the same law. We will speak simply of a “conformal metric ω on (G, ρ) .” Only such unimodular metrics are considered in this work.

Convergence of infinite sums of metrics. The following construction will occasionally be useful. Consider a family $\{\omega_j : j \geq 1\}$ of normalized conformal weights on (G, ρ) . Recall that formally this is a sequence $\{(G_j, \omega_j, \rho_j) : j \geq 1\}$ such that (G_j, ω_j, ρ_j) is a unimodular random conformal graph and (G_j, ρ_j) has the same law as (G, ρ) for every $j \geq 1$. Thus we may consider a joint coupling $(G, \{\omega_j : j \geq 1\}, \rho)$.

Now we would like to define a unimodular random conformal graph (G, ω, ρ) where

$$\omega := \sqrt{\sum_{j \geq 1} \frac{\omega_j^2}{j^2}}.$$

A priori, it is not clear that almost surely this sum converges for every $x \in V(G)$. So let us momentarily allow $\omega : V(G) \rightarrow [0, +\infty]$ to take extended real values. Fix $r \geq 1$ and define the transport $F(G, \omega, x, y) := \mathbb{1}_{\{\text{dist}_G(x, y) \leq r\}} \mathbb{1}_{\{\omega(y) = +\infty\}}$. Then by the Mass-Transport Principle:

$$\mathbb{P} \left[\max_{x \in B_G(\rho, r)} \omega(x) = +\infty \right] = \mathbb{E} \left[\sum_{x \in V(G)} F(G, \omega, \rho, x) \right]$$

$$= \mathbb{E} \left[\sum_{x \in V(G)} F(G, \omega, x, \rho) \right] = \mathbb{E}[|B_G(\rho, r)| \mathbb{1}_{\{\omega(\rho) = +\infty\}}] = 0,$$

where the latter equality follows from the fact that each ω_j is normalized, hence $\mathbb{E}[\omega(\rho)^2] < \infty$. Since this holds for every $r \geq 1$, we conclude that almost surely, $\sup_{x \in V(G)} \omega(x) < \infty$.

1.6.2 Analysis on metric spaces and random partitions

Consider a pseudo-metric space (X, d) (i.e., we allow for the possibility that $d(x, y) = 0$ when $x \neq y$). Throughout the paper, we will deal only with complete, separable, pseudo-metric spaces. For $x \in X$ and two subsets $S, T \subseteq X$, we use the notations $d(S, T) = \inf_{x \in S, y \in T} d(x, y)$ and $d(x, S) = d(\{x\}, S)$. Define $\text{diam}(S, d) = \sup_{x, y \in S} d(x, y)$ and for $R \geq 0$, define the closed balls

$$B_{(X, d)}(x, R) = \{y \in X : d(x, y) \leq R\}.$$

We omit the subscript (X, d) if the underlying metric space is clear from context. We now introduce a tool that will be used for analyzing the heat kernel in [Section 3](#).

Random partitions. For a partition \mathcal{P} of X , we use $\mathcal{P}(x)$ to denote the unique set in \mathcal{P} containing x . We will consider only partitions \mathcal{P} with an at most countable number of elements. Denote

$$\Delta(\mathcal{P}) := \sup \{\text{diam}(S, d) : S \in \mathcal{P}\}.$$

A random partition \mathbf{P} is (τ, α) -padded if it satisfies the following conditions:

1. Almost surely: $\Delta(\mathbf{P}) \leq \tau$.
2. For all $x \in X$ and $\delta > 0$,

$$\mathbb{P}[B(x, \delta\tau/\alpha) \subseteq \mathbf{P}(x)] \geq 1 - \delta.$$

The reader might gain some intuition from considering the case $X = \mathbb{R}^d$ equipped with the Euclidean metric. If one takes \mathbf{P} to be a randomly translated partition of \mathbb{R}^d into axis-aligned cubes of side-length L , then \mathbf{P} is $(L\sqrt{k}, k\sqrt{k})$ -padded.

Uniformly decomposable graph families. We say that a family \mathcal{F} of locally finite, connected graphs is α -decomposable if there is an $\alpha > 0$ such that for every $G \in \mathcal{F}$, every conformal weight $\omega : V(G) \rightarrow \mathbb{R}_+$, and every $\tau > 0$, the metric space $(V(G), \text{dist}_\omega)$ admits a (τ, α) -padded random partition. We say that \mathcal{F} is *uniformly decomposable* if it is α -decomposable for some $\alpha > 0$.

The next result is proved in [\[Lee16\]](#). The special case for graphs G that themselves exclude K_h as a minor was established much earlier in [\[KPR93\]](#). For such graphs, the bound $\alpha \leq O(h^2)$ was established in [\[FT03\]](#), and this was improved to $\alpha \leq O(h)$ in [\[AGG⁺14\]](#). Let $\mathcal{F}(K_h)$ denote the family of connected, locally finite graphs that exclude K_h as a minor, and denote $\text{rig}(\mathcal{F}(K_h)) := \bigcup_{G_0 \in \mathcal{F}(K_h)} \text{rig}(G_0)$.

Theorem 1.18 ([\[Lee16\]](#)). *For every $h \geq 1$, the family $\text{rig}(\mathcal{F}(K_h))$ is α -decomposable for some $\alpha \leq O(h^2)$.*

We say that a unimodular random graph (G, ρ) is *uniformly decomposable* if there is an $\alpha > 0$ such that G is almost surely α -decomposable.

We remark that often in the literature (e.g., in [\[Lee16\]](#)), one only exhibits random partitions that satisfy property (2) of a padded partition with $\delta = 1/2$. The following (unpublished) lemma of the author and A. Naor shows that this is sufficient to conclude that it holds for all $\delta \in [0, 1]$, with a small loss in parameters.

Lemma 1.19. *Suppose that a metric space (X, d) admits a random partition \mathbf{P} with $\Delta(\mathbf{P}) \leq \tau$ almost surely, and for every $x \in X$,*

$$\mathbb{P}[B(x, \tau/\alpha) \subseteq \mathbf{P}(x)] \geq \frac{1}{2}. \quad (1.17)$$

Then there is a random partition \mathbf{P}' with $\Delta(\mathbf{P}') \leq \tau$ almost surely, and such that for every $\delta > 0$ and $x \in X$,

$$\mathbb{P}[B(x, \delta\tau/\alpha) \subseteq \mathbf{P}'(x)] \geq 1 - 4\delta. \quad (1.18)$$

Proof. For a subset $S \subseteq X$ and a number $\lambda > 0$, denote

$$S_{-\lambda} := \{x \in S : B(x, \lambda) \subseteq S\}.$$

Let $\{\mathbf{P}_k\}$ be an infinite sequence of i.i.d. random partitions with the law of \mathbf{P} . Let $\{\varepsilon_k\}$ be an independent infinite sequence of i.i.d. random variables where $\varepsilon_k \in [0, 1]$ is chosen uniformly at random. We will define a sequence $\{A_k\}$ where each A_k is a collection of disjoint subsets of X and define $\mathbf{P}' = \bigcup_{k \geq 1} A_k$.

Denote $A_0 = \emptyset, X_0 = \emptyset$ and for $k \geq 1$,

$$\begin{aligned} A_k &= \{S_{-\varepsilon_k \tau/\alpha} \setminus X_{k-1} : S \in \mathbf{P}_k\} \\ X_k &= X_{k-1} \cup \bigcup_{S \in A_k} S. \end{aligned}$$

First, observe that for every $x \in X$, it holds that almost surely $x \in S \in A_k$ for some $k \in \mathbb{N}$. This is because for every $k \geq 1$,

$$\mathbb{P}[x \in (\mathbf{P}_k(x))_{-\varepsilon_k \tau/\alpha}] \geq \mathbb{P}[B(x, \tau/\alpha) \subseteq \mathbf{P}_k(x)] \geq \frac{1}{2}, \quad (1.19)$$

where the latter inequality is from (1.17). Since we deal only with separable metric spaces, to verify that \mathbf{P}' is almost surely a partition, it suffices to consider a dense countable subset of X . Similarly, we can conclude that \mathbf{P}' almost surely satisfies $\Delta(\mathbf{P}') \leq \tau$.

So now we move on to verifying (1.18). Fix $x \in X$ and $\delta \in [0, 1/4]$. Let $R = \delta\tau/\alpha$. Then,

$$\mathbb{P}[B(x, R) \not\subseteq \mathbf{P}'(x)] \leq \sum_{k \geq 1} \mathbb{P}[B(x, R) \cap X_{k-1} = \emptyset] \cdot \mathbb{P}[B(x, R) \cap X_k \neq \emptyset \wedge B(x, R) \not\subseteq (\mathbf{P}_k(x))_{-\varepsilon_k \tau/\alpha}].$$

Now observe that (1.19) implies $\mathbb{P}[B(x, R) \cap X_{k-1} = \emptyset] \leq 2^{1-k}$.

For $y \in X$, let $\eta_k(y) = \sup\{\eta \geq 0 : B(y, \eta) \subseteq \mathbf{P}_k(y)\}$. Note that, conditioned on \mathbf{P}_k , η_k is a 1-Lipschitz function. Therefore,

$$\mathbb{P}[B(x, R) \cap X_k \neq \emptyset \wedge B(x, R) \not\subseteq (\mathbf{P}_k(x))_{-\varepsilon_k \tau/\alpha}] \leq \mathbb{P}\left(\varepsilon_k \in \left[\frac{\eta_k(x)}{\tau/\alpha} - \delta, \frac{\eta_k(x)}{\tau/\alpha} + \delta\right]\right) \leq 2\delta,$$

We conclude that

$$\mathbb{P}[B(x, R) \not\subseteq \mathbf{P}'(x)] \leq 2\delta \sum_{k \geq 1} 2^{1-k} = 4\delta,$$

completing the proof. □

2 Quadratic conformal growth and recurrence

The assumption $\dim_{\text{cg}}(G, \rho) \leq 2$ is not sufficient to ensure almost sure recurrence of G . Instead, we need a more delicate way to measure quadratic growth using a family of metrics. Recall that a unimodular random graph (G, ρ) is (C, R) -quadratic for $C > 0$ and $R \geq 1$ if

$$\inf_{\omega} \|\#B_{\omega}(\rho, R)\|_{L^{\infty}} \leq CR^2, \quad (2.1)$$

where the infimum is over all normalized conformal metrics on (G, ρ) . Say that (G, ρ) has *gauged quadratic conformal growth* (gQCG) if there is a constant $C > 0$ such that (G, ρ) is (C, R) -quadratic for all $R \geq 1$. Note that we allow a different conformal weight ω for every choice of R .

A sequence $\{(G_n, \rho_n)\}$ of unimodular random graphs has *uniform gQCG* if there is a constant $C > 0$ such that for (G_n, ρ_n) is (C, R) -quadratic for all $R \geq 1$ and $n \geq 1$. A family $\mathcal{F} \subseteq \mathcal{G}$ of finite graphs has *uniform gQCG* if the family of unimodular random graphs $\{(G, \rho) : G \in \mathcal{F}\}$ has uniform gQCG, where $\rho \in V(G)$ is chosen uniformly at random.

Finally, we say that (G, ρ) has *asymptotic gQCG* if there is a constant $C > 0$ and a sequence of radii $\{R_n\}$ with $R_n \rightarrow \infty$ such that (G, ρ) is (C, R_n) -quadratic for all $n \geq 1$. We can now state the main theorem of this section. The proof appears in [Section 2.2](#).

Theorem 2.1. *If (G, ρ) has asymptotic gQCG and $\|\deg_G(\rho)\|_{L^{\infty}} < \infty$, then G is almost surely recurrent.*

Remark 2.2. Note that $(\mathbb{Z}^2, 0)$ has gQCG, and moreover, one can consider a single conformal weight $\omega \equiv 1$ for every $R \geq 1$. On the other hand, the infinite ternary tree does not have gQCG.

Remark 2.3. Note that if (G, ρ) has gQCG, then $\dim_{\text{cg}}(G, \rho) \leq 2$. To see this, consider the corresponding family of normalized conformal weights $\{\omega_{2^k} : k \geq 1\}$ arising from applying (2.1) with $R = 2^k$, and let

$$\omega := \sqrt{\frac{6}{\pi^2} \sum_{k \geq 1} \frac{\omega_{2^k}^2}{k^2}}.$$

Then $\|\omega\|_{L^2} = 1$, and moreover $\omega \geq \frac{\sqrt{6}}{k\pi} \omega_{2^k}$ for every $k \geq 1$, hence

$$\left\| \#B_{\omega} \left(\rho, \frac{\sqrt{6}}{k\pi} 2^k \right) \right\|_{L^{\infty}} \leq \left\| \#B_{\omega_{2^k}}(\rho, 2^k) \right\|_{L^{\infty}} \leq C4^k,$$

implying that $\dim_{\text{cg}}(G, \rho) \leq 2$.

Remark 2.4. In [Section 2.3](#), we will show that the family of all finite planar graphs has uniform gQCG. But for this to hold, it must be that we allow $\omega = \omega_R$ to depend on the scale R in (2.1). Indeed, let T_n denote the complete binary tree of height n , then for some constant $c > 0$, and any normalized conformal metric $\omega : V(T_n) \rightarrow \mathbb{R}_+$,

$$\max_{R \geq 0} \frac{|B_{\omega}(x, R)|}{R^2} \geq c\sqrt{n}. \quad (2.2)$$

This is proved in [Lemma 2.15](#). Let (T, ρ) denote the distributional limit of $\{T_n\}$ (this is known as the ‘‘canopy tree’’ from [\[AW06\]](#)). Then (2.2) implies that no single normalized conformal metric ω on (T, ρ) can have quadratic growth.

2.1 Comparing graph balls to conformal balls

In order to use a conformal weight $\omega : V(G) \rightarrow \mathbb{R}_+$ to establish recurrence, we will need a way of comparing the conformal metric dist_{ω} to the graph metric dist_G . Say that a conformal graph (G, ω) is *C-regulated* if it satisfies the following properties:

1. $\omega(x) \geq 1/2$ for all $x \in V(G)$.
2. If $\{u, v\} \in E(G)$, then $\omega(u) \leq C \omega(v)$.

This definition allows us to compare balls in the metrics dist_G and dist_ω .

Lemma 2.5. *If (G, ω) is C -regulated for some $C \geq 2$, then it holds that for every $x \in V(G)$ and $r \geq 0$,*

$$B_G\left(x, \frac{\log \frac{r}{2\omega(x)}}{\log C}\right) \subseteq B_\omega(x, r) \subseteq B_G(x, 2r).$$

Proof. The latter inclusion is straightforward from property (1) of C -regulated. The proof of the former inclusion is by induction. Trivially, $B_G(x, 0) \subseteq B_\omega(x, r)$. Suppose that $B_G(x, k-1) \subseteq B_\omega(x, r)$ and $v \in B_G(x, k)$. Property (2) of C -regulated yields $\omega(v) \leq \omega(x)C^k$, which implies inductively that

$$\text{dist}_\omega(x, v) \leq \omega(x) \sum_{j=0}^k C^j \leq 2\omega(x)C^k \leq r,$$

as long as $k \leq \frac{\log \frac{r}{2\omega(x)}}{\log C}$. □

Now let us see that when the degrees are uniformly bounded, one can convert any conformal weight into a C -regulated weight where C is a constant depending only on the maximum degree.

Lemma 2.6. *Let (G, ω, ρ) be a normalized unimodular random conformal graph, and suppose that $d := \|\text{deg}_G(\rho)\|_{L^\infty}$. Then there exists a normalized, $\sqrt{2d}$ -regulated unimodular random conformal graph $(G, \hat{\omega}, \rho)$ such that $\hat{\omega} \geq \frac{1}{2}\omega$.*

Proof. For a conformal pair (G, ω) , we define

$$\omega_0(x) = \sqrt{\sum_{y \in V(G)} \omega(y)^2 (2d)^{-\text{dist}_G(x, y)}}.$$

Notice that $\omega_0 \geq \omega$ pointwise, and moreover for $\{u, v\} \in E(G)$, we have

$$\omega_0(u)^2 \leq 2d \omega_0(v)^2 \tag{2.3}$$

by construction.

In order to analyze $\|\omega_0\|_{L^2}$, we define a mass transportation: For $x, y \in V(G)$,

$$F(G, \omega, x, y) = \omega(x)^2 (2d)^{-\text{dist}_G(x, y)}.$$

Note that the total flow out of x is bounded by

$$\omega(x)^2 \sum_{y \in V(G)} (2d)^{-\text{dist}_G(x, y)} \leq \omega(x)^2 \sum_{k \geq 0} 2^{-k} \leq 2\omega(x)^2.$$

Therefore by the Mass-Transport Principle, it holds that

$$2 \mathbb{E}[\omega(\rho)^2] \geq \mathbb{E}\left[\sum_{x \in V(G)} F(G, \omega, \rho, x)\right]$$

$$\begin{aligned}
&= \mathbb{E} \left[\sum_{x \in V(G)} F(G, \omega, x, \rho) \right] \\
&= \mathbb{E} [\omega_0(\rho)^2] .
\end{aligned}$$

In particular, we conclude that $\mathbb{E}[\omega_0(\rho)^2] \leq 2 \mathbb{E}[\omega(\rho)^2] = 2$.

Now define the normalized weight

$$\hat{\omega} := \frac{\sqrt{\frac{1}{4} \mathbb{1} + \frac{3}{8} \omega_0^2}}{\sqrt{\frac{1}{4} + \frac{3}{8} \mathbb{E}[\omega_0^2]}} .$$

It satisfies property (1) of C -regulated by construction, and also $\hat{\omega} \geq \frac{1}{2} \omega_0 \geq \frac{1}{2} \omega$ pointwise. Furthermore, property (2) of C -regulated is a consequence of (2.3) with $C = \sqrt{2d}$. \square

2.2 Bounding the effective resistance

In the present section, we will use the notion of the *effective resistance* $R_{\text{eff}}^G(S \leftrightarrow T)$ between two subsets $S, T \subseteq V(G)$ in a graph. For completeness, we present one definition that aligns with our use of the quantity; for more background, we refer the reader to [LP16, Ch. 2 & 9]. For $S, T \subseteq V(G)$, the Dirichlet principle asserts that

$$R_{\text{eff}}^G(S \leftrightarrow T) = \left(\inf_{f \in \mathcal{F}_{S,T}} \mathcal{E}(f) \right)^{-1} ,$$

where $\mathcal{F}_{S,T} = \{f : V(G) \rightarrow \mathbb{R} \mid f|_S = 0, f|_T = 1\}$, and we recall the (unnormalized) Dirichlet energy functional

$$\mathcal{E}(f) := \sum_{\{x,y\} \in E(G)} |f(x) - f(y)|^2 .$$

We will soon prove [Theorem 2.1](#) using the following well-known characterization; see, e.g., [LP16, Lem. 9.22].

Theorem 2.7. *A graph G is recurrent if and only if there is some vertex $x \in V(G)$ and constant $c > 0$ such that for all $R \geq 0$, there is a finite set $S_R \subseteq V(G)$ such that*

$$R_{\text{eff}}^G(B(x, R) \leftrightarrow V(G) \setminus S_R) \geq c$$

First we will need a lemma about the expected area of balls. Let us define

$$\mathcal{A}_\omega(x, R) := \sum_{y \in B_\omega(x, R)} \omega(y)^2 .$$

Lemma 2.8. *Let (G, ω, ρ) be a unimodular random conformal graph with $\mathbb{E} \omega(\rho)^2 = 1$. Then for every $R \geq 1$,*

$$\mathbb{E} [\mathcal{A}_\omega(\rho, R)] \leq \|\#B_\omega(\rho, R)\|_{L^\infty} .$$

Proof. We employ the Mass-Transport Principle: For a conformal graph (G, ω) and $x, y \in V(G)$, define the flow

$$F(G, \omega, x, y) = \omega(x)^2 \mathbb{1}_{\{\text{dist}_\omega(x, y) \leq R\}} .$$

Then,

$$\begin{aligned}\mathbb{E}[\mathcal{A}_\omega(\rho, R)] &= \mathbb{E}\left[\sum_{x \in V(G)} F(G, \omega, x, \rho)\right] = \mathbb{E}\left[\sum_{x \in V(G)} F(G, \omega, \rho, x)\right] \\ &= \mathbb{E}[\omega(\rho)^2 |B_\omega(\rho, R)|] \leq \| \#B_\omega(\rho, R) \|_{L^\infty}. \quad \square\end{aligned}$$

In order to apply [Theorem 2.7](#), we use a conformal weight to construct a test function of small energy.

Lemma 2.9. *Consider a graph G , vertex $x \in V(G)$, and scale $R \geq 0$. Then for any C -regulated conformal weight $\omega : V(G) \rightarrow \mathbb{R}_+$, it holds that*

$$R_{\text{eff}}\left(B_G\left(x, \frac{\log \frac{R}{4\omega(x)}}{\log C}\right) \leftrightarrow V(G) \setminus B_G(x, 2R)\right) \geq \frac{1}{4(1+C)^2 d_{\max}(G)} \cdot \frac{R^2}{\mathcal{A}_\omega(x, R)}.$$

Proof. Define $f : V(G) \rightarrow \mathbb{R}$ by

$$f(x) = \frac{2}{R} \min\left(\frac{R}{2}, \max\left(0, \text{dist}_\omega(x, v) - \frac{R}{2}\right)\right).$$

Note that $v \in B_\omega(x, \frac{R}{2}) \implies f(v) = 0$ and $v \notin B_\omega(x, R) \implies f(v) = 1$.

Therefore by the Dirichlet principle,

$$R_{\text{eff}}\left(B_\omega(x, \frac{R}{2}) \leftrightarrow V(G) \setminus B_\omega(x, R)\right) \geq \frac{1}{\mathcal{E}_G(f)},$$

and

$$\begin{aligned}\mathcal{E}(f) &\leq \frac{4d_{\max}(G)}{R^2}(1+C)^2 \sum_{v \in B_\omega(x, R)} \omega(v)^2 \\ &= 4d_{\max}(G)(1+C)^2 \frac{\mathcal{A}_\omega(x, R)}{R^2},\end{aligned}$$

where we have used the fact that f is $2/R$ -Lipschitz, and the fact that ω is C -regulated which, in particular, asserts that for $\{u, v\} \in E(G)$, one has

$$\text{dist}_\omega(u, v)^2 \leq \omega_k(u)^2 + \omega_k(v)^2 \leq (1+C)^2 \omega_k(v)^2.$$

Finally, we use [Lemma 2.5](#) to arrive at the desired conclusion, replacing dist_ω balls by dist_G balls. \square

Proof of Theorem 2.1. Consider a radius $R \geq 1$ and a normalized, C -regulated conformal weight $\omega : V(G) \rightarrow \mathbb{R}_+$ satisfying $\| \#B_\omega(\rho, R) \|_{L^\infty} \leq cR^2$ for some constant $c > 0$. From [Lemma 2.8](#), we have $\mathbb{E}[\mathcal{A}_\omega(\rho, R)] \leq cR^2$, hence employing Markov's inequality,

$$\mathbb{P}\left[\omega(\rho)^2 < R \text{ and } \mathcal{A}_\omega(\rho, R) < \frac{c}{\varepsilon}R^2\right] \geq 1 - \varepsilon - \frac{1}{R}.$$

Combining this with [Lemma 2.9](#), we see that

$$\mathbb{P}\left[R_{\text{eff}}\left(B_G\left(\rho, \frac{\log(R/4)}{2\log C}\right) \leftrightarrow V(G) \setminus B_G(\rho, 2R)\right) \geq c'\varepsilon\right] \geq 1 - \varepsilon - \frac{1}{R}, \quad (2.4)$$

where c' is a constant depending only on C and $\| \text{deg}_G(\rho) \|_{L^\infty}$.

By assumption, (G, ρ) has asymptotic gQCG and $\|\deg_G(\rho)\|_{L^\infty} < \infty$. Combining the definition of asymptotic gQCG with [Lemma 2.6](#) (to derive a C -regulated conformal metric) shows that [\(2.4\)](#) holds for $R = R_n$, where $\{R_n\}$ is a sequence of radii with $R_n \rightarrow \infty$.

In particular, Fatou's Lemma tells us that

$$\mathbb{P} \left[\limsup_{n \rightarrow \infty} R_{\text{eff}} \left(B_G(\rho, \frac{\log(R_n/4)}{2 \log C}) \leftrightarrow V(G) \setminus B_G(\rho, 2R_n) \right) \geq c' \varepsilon \right] \geq 1 - \varepsilon.$$

Since $\{B_G(\rho, 2R_n)\}$ is a sequence of finite sets, [Theorem 2.7](#) yields

$$\mathbb{P}[G \text{ recurrent}] \geq 1 - \varepsilon.$$

Sending $\varepsilon \rightarrow 0$ completes the proof. □

2.3 Region intersection graphs and energy-minimizing conformal weights

The next two theorems essentially follow from prior work. We refer the reader to [\[Lee17b\]](#) for simpler proofs in a unified setting and the exact statements appearing here. Note that for the special case of planar graphs, an alternate proof of the next result based on circle packings appears in [\[Lee17a\]](#). It has the advantage that it extends suitably to graphs that are sphere-packed in any Euclidean space.

Theorem 2.10 (Uniform gQCG for H -minor-free graphs [\[KLPT11\]](#)). *For every fixed graph H , the family of finite graphs excluding H as a minor has uniform gQCG. In particular, if $H = K_h$ for some $h \geq 2$, then every such graph is (κ, R) -quadratic for all $R \geq 1$, where $\kappa \leq O(h^2 \log h)$.*

Theorem 2.11 (Uniform gQCG for region intersection graphs [\[Lee16\]](#)). *For every $\lambda > 0$ and fixed graph H , the family of finite region intersection graphs G over an H -minor-free graph with $d_{\max}(G) \leq \lambda$ has uniform gQCG. In particular, if $H = K_h$ for some $h \geq 2$, then every such graph is (κ, R) -quadratic for all $R \geq 1$, where $\kappa \leq O(\lambda h^2 \log h)$.*

Let us a remark on the proof of these results. Fix a finite graph $G = (V, E)$. Let $n = |V|$ and consider a positive number $k \leq n$. Define the set $P_k(G) \subseteq \ell^2(V)$ by

$$P_k(G) := \left\{ \omega \geq 0 : \frac{1}{k^2} \sum_{x, y \in S} \text{dist}_\omega(x, y) \geq 1 \quad \forall S \subseteq V, |S| \geq k \right\}.$$

While it may not be immediately apparent, the set $P_k(G)$ is a polytope because one can replace every such inequality indexed by a subset $S \subseteq V$ with the family of inequalities:

$$\frac{1}{k^2} \sum_{x, y \in S} \text{len}_\omega(\gamma_{xy}) \geq 1 \quad \forall \{\gamma_{xy} : x, y \in S\},$$

where in the latter quantifier, γ_{xy} ranges over all simple x - y paths in G . Since $\text{len}_\omega(\gamma)$ is a linear function in the values $\{\omega(z) : z \in V\}$, the claim follows.

Consider now the optimization problem

$$\theta_k(G) := \min \{ \|\omega\|_{L^2(V)} : \omega \in P_k(G) \}. \tag{2.5}$$

We claim that the values $\{\theta_k(G) : k = 1, 2, \dots, n\}$ dictate the quadratic conformal growth: G is (κ, R) -quadratic for some $\kappa \geq 1$ and every $R \geq 0$ if and only if $\theta_k(G) \leq C/k^2$ for some $C \geq 1$ and every $k \leq n$.

Claim 2.12. For every n -vertex graph $G = (V, E)$, the following holds.

1. For every $k \leq n$, G is (κ, R) -quadratic with $\kappa = 4k/\theta_k(G)^2$ and $R = \theta_k(G)/2$.
2. For every $\kappa, R > 0$, it holds that if G is (κ, R) -quadratic, then

$$\theta_k(G) \leq \frac{2}{R} \text{ for all } k \geq 2\kappa R^2.$$

Proof. Let us first prove (1). Consider $\omega \in P_k(G)$ with $\|\omega\|_{L^2(V)}^2 = \theta$ and let $\hat{\omega} := \omega/\|\omega\|_{L^2(V)}$. Denote $N := |B_{\hat{\omega}}(z, R)| = |B_{\hat{\omega}}(z, \theta R)|$. If $N \geq k$, let $S \subseteq B_{\hat{\omega}}(z, R)$ be any subset with $|S| \geq k$. Since $\text{diam}_{\omega}(S) \leq 2R$ and $\omega \in P_k(G)$, it holds that

$$2R > \frac{1}{k^2} \sum_{x, y \in S} \text{dist}_{\omega}(x, y) \geq 1.$$

This establishes that $|B_{\hat{\omega}}(z, \theta/2)| \leq k$ for every $z \in V$.

To prove (2), consider $\omega : V \rightarrow \mathbb{R}_+$ such that $\|\omega\|_{L^2(V)} = 1$ and $|B_{\omega}(x, R)| \leq \kappa R^2$ for all $x \in V$. Consider any $S \subseteq V$ with $|S| = \lceil 2\kappa R^2 \rceil$. Then for every $x \in S$, it holds that

$$|B_{\omega}(x, R) \cap S| \leq \frac{1}{2}|S|.$$

Hence:

$$\frac{1}{|S|^2} \sum_{x \in S} \sum_{y \in S} \text{dist}_{\omega}(x, y) \geq \frac{1}{|S|^2} \sum_{x \in S} \frac{|S|}{2} R \geq \frac{R}{2},$$

completing the proof. □

[Theorem 2.10](#) and [Theorem 2.11](#) are proved by analyzing the optimization (2.5). It entails minimizing a strongly convex function over a polytope, thus (2.5) has a unique optimal solution. The authors of [BLR10] developed a “flow crossing” theory for understanding the dual optimization problem, and that was expanded upon in the works [KLPT11, Lee16]. We refer to [Lee17b] for an overview and simplified arguments.

The following is a consequence of [Lee17a, Thm. 1.13].

Lemma 2.13. *If $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$ and $\{(G_n, \rho_n)\}$ has uniform gQCG, then (G, ρ) has gQCG. In particular, if (G, ρ) is a distributional limit of finite H -minor-free graphs, then (G, ρ) has gQCG.*

In particular, combining [Lemma 2.13](#) with the preceding two theorems and [Theorem 2.1](#) yields the following corollary. We recall that $\text{rig}(\mathcal{F}(H))$ is the set of all finite graphs that are region intersection graphs over some graph G_0 that excludes the graph H as a minor.

Corollary 2.14. *For every fixed graph H , if $\{G_n\} \subseteq \text{rig}(\mathcal{F}(H))$ is a sequence of graphs with uniformly bounded degrees and $\{G_n\} \Rightarrow (G, \rho)$, then G is almost surely recurrent.*

We end this section by observing that one cannot equip every finite planar graph with a single normalized metric that achieves uniform quadratic volume growth at all scales simultaneously (thus justifying the necessity for multiple conformal weights in the definition of gQCG).

Lemma 2.15. *There is a constant $C > 0$ such that the following holds. Let T_n be the complete binary tree of height n , and consider any normalized conformal weight ω on T_n . Then*

$$\max_{R \geq 0} \frac{|B_{\omega}(x, R)|}{R^2} \geq C\sqrt{n}.$$

Proof. Let $\omega : V(T_n) \rightarrow \mathbb{R}_+$ be a conformal weight satisfying $|B_\omega(x, R)| \leq R^2$ for all $x \in V(T_n)$ and $R \geq 1$. Consider the family \mathcal{P} of $\binom{2^n}{2}$ paths in T_n between all the leaves of T_n . There must exist a constant $c > 0$ and a subset $\mathcal{P}_n \subseteq \mathcal{P}$ of paths going through the root of T_n with $|\mathcal{P}_n| \geq c2^{2n}$ and such that every path in \mathcal{P}_n has ω -length at least $c2^{n/2}$. (Otherwise there would be some leaf that could reach $c2^n$ other leaves using paths of length $\ll 2^{n/2}$, contradicting the quadratic volume assumption.) Similarly, there exist disjoint subsets $\mathcal{P}_{n-1}^{(0)}, \mathcal{P}_{n-1}^{(1)} \subseteq \mathcal{P}$ of paths in the left and right subtrees, each containing $c2^{2(n-1)}$ paths of ω -length at least $c2^{(n-1)/2}$, and so on.

Let $\mathcal{P}_k = \bigcup_{j \in \{0,1\}^{n-k}} \mathcal{P}_k^{(j)}$ be the set of such ‘‘long’’ paths in subtrees of height k . Observe that this union is disjoint by construction. For a vertex $v \in V(T_n)$, define

$$\alpha(v) = \sum_{k=1}^n 2^{-3k/2} \#\{\gamma \in \mathcal{P}_k : v \in \gamma\}.$$

Then we have

$$\sum_{k=1}^n c^2 2^{-3k/2} 2^{n-k} 2^{2k} 2^{k/2} \leq \sum_{k=1}^n 2^{-3k/2} |\mathcal{P}_k| \min_{\gamma \in \mathcal{P}_k} \text{len}_\omega(\gamma) \leq \sum_{v \in V(T_n)} \alpha(v) \omega(v) \leq \|\alpha\|_{\ell^2(V(T_n))} \|\omega\|_{\ell^2(V(T_n))},$$

where the last inequality is Cauchy-Schwarz. The left-hand side is $c^2 n 2^n$.

Now a simple calculation yields:

$$\sum_{v \in V(T_n)} \alpha(v)^2 \leq \sum_{k=1}^n 2^{n-k} 2^{4k} 2^{-3k} = n 2^n.$$

We conclude that

$$\|\omega\|_{\ell^2(V(T_n))} \geq c^2 \sqrt{n 2^n},$$

implying that $\|\omega\|_{L^2} = 2^{-n/2} \|\omega\|_{\ell^2(V(T_n))} \geq c^2 \sqrt{n}$, and completing the argument. \square

3 Return probabilities and spectral geometry on finite graphs

We now turn to heat kernel estimates on finite graphs.

3.1 The normalized Laplacian spectrum

Let $G = (V, E)$ be a connected, finite graph with $n = |V|$. Let $\pi(x) = \frac{\deg_G(x)}{2|E|}$ denote the stationary measure. We will use $L^2(\pi)$ for the Hilbert space of functions $f : V \rightarrow \mathbb{R}$ equipped with the inner product

$$\langle f, g \rangle_\pi = \sum_{x \in V} \pi(x) f(x) g(x),$$

and denote by

$$\langle f, g \rangle = \sum_{x \in V} f(x) g(x)$$

the inner product on $\ell^2(V)$. We use $\|\cdot\| := \|\cdot\|_{\ell^2(V)}$ and $\|f\|_\pi = \sqrt{\langle f, f \rangle_\pi}$.

Define the operators $A, D, P, L, \mathcal{L} : \ell^2(V) \rightarrow \ell^2(V)$ as follows

$$Af(x) := \sum_{y: \{x,y\} \in E} f(y)$$

$$\begin{aligned}
Df(x) &:= \deg_G(x)f(x) \\
P &:= D^{-1}A \\
L &:= I - P \\
\mathcal{L} &:= I - D^{-1/2}AD^{-1/2}
\end{aligned}$$

The *normalized Laplacian* \mathcal{L} is symmetric and positive semi-definite. We denote its eigenvalues by

$$0 = \lambda_1(G) \leq \lambda_2(G) \leq \dots \leq \lambda_{n-1}(G).$$

We use $\lambda_k := \lambda_k(G)$ if the graph G is clear from context. Note that $P = I - D^{-1/2}\mathcal{L}D^{1/2}$, so if $\mathcal{L}f = \lambda f$, then $PD^{-1/2}f = (1 - \lambda)D^{-1/2}f$. Thus the spectrum of P is $\{1 - \lambda_k(G) : k = 0, 1, \dots, n - 1\}$.

Define the *Rayleigh quotient* $\mathcal{R}_G(f)$ of non-zero $f \in L^2(\pi)$ by

$$\mathcal{R}_G(f) := \frac{\langle D^{1/2}f, \mathcal{L}D^{1/2}f \rangle}{\langle D^{1/2}f, D^{1/2}f \rangle} = \frac{\langle f, Lf \rangle_\pi}{\|f\|_\pi^2} = \frac{\frac{1}{|E|} \sum_{\{x,y\} \in E} |f(x) - f(y)|^2}{\|f\|_\pi^2}.$$

Recall also the variational formula for eigenvalues:

$$\lambda_k(G) = \min_{U \subseteq L^2(\pi)} \max_{0 \neq f \in U} \mathcal{R}_G(f), \quad (3.1)$$

where the minimum is over all subspaces $U \subseteq L^2(\pi)$ with $\dim(U) = k + 1$. The preceding fact has a useful corollary.

Corollary 3.1. *Suppose that $\psi_1, \dots, \psi_r : V \rightarrow \mathbb{R}$ are disjointly supported functions with $\mathcal{R}_G(\psi_i) \leq \theta$ for $i = 1, 2, \dots, r$. Then,*

$$\lambda_{r-1}(G) \leq 2\theta.$$

Proof. Let $U = \text{span}(\psi_1, \dots, \psi_r)$, and note that $\dim(U) = r$ since $\{\psi_i\}$ are mutually orthogonal. Consider $f \in U$ and write $f = \sum_{i=1}^r \alpha_i \psi_i$.

Since the functionals have mutually disjoint supports, for any $x, y \in V$, we have

$$|f(x) - f(y)|^2 \leq 2 \sum_{i=1}^r \alpha_i^2 |\psi_i(x) - \psi_i(y)|^2.$$

Therefore,

$$\mathcal{R}_G(f) \leq \frac{2 \sum_{i=1}^r \alpha_i^2 \|\psi_i(x) - \psi_i(y)\|^2}{\sum_{i=1}^r \alpha_i^2 \|\psi_i\|_\pi^2} \leq 2\theta.$$

Now the claim follows from $\dim(U) = r$ and the variational characterization of eigenvalues. \square

Relation to return probabilities. Let $\{\phi_k\}$ be an $L^2(\pi)$ -orthonormal family of eigenfunctions for P such that $P\phi_k = (1 - \lambda_k)\phi_k$ for each $0 \leq k \leq n - 1$. The connection between return probabilities and eigenvalues is straightforward: For any $x \in V$ and $T \geq 1$,

$$p_T^G(x, x) = \frac{\langle \mathbb{1}_x, P^T \mathbb{1}_x \rangle_\pi}{\pi(x)} = \sum_{k=0}^{n-1} \pi(x) \phi_k(x)^2 (1 - \lambda_k)^T. \quad (3.2)$$

3.2 Eigenvalues and the degree distribution

Let us define $\Delta_G : \mathbb{Z}_+ \rightarrow \mathbb{N}$ by

$$\Delta_G(k) = \max \left\{ \sum_{x \in S} \deg_G(x) : S \subseteq V, |S| \leq k \right\}.$$

Momentarily, we will prove the following theorem. Say that a graph is (κ, α) -controlled if it is α -decomposable and (κ, R) -quadratic for all $R \geq 1$.

Theorem 3.2. *Suppose that a family \mathcal{F} of finite graphs has uniform gQCG and is uniformly decomposable. Then there is a constant $c > 0$ such that for every $G \in \mathcal{F}$ and $k = 0, 1, \dots, |V(G)| - 1$,*

$$\lambda_k(G) \leq c \frac{\Delta_G(k)}{|V(G)|}.$$

Quantitatively, if a finite graph G is (κ, α) -controlled, then

$$\lambda_k(G) \lesssim \alpha^2 \kappa \frac{\Delta_G(k)}{|V(G)|}.$$

We present an illustrative corollary of [Theorem 2.10](#) and [Theorem 1.18](#) in conjunction with [Theorem 3.2](#).

Corollary 3.3. *Suppose G is an n -vertex graph that excludes K_h as a minor. Then there is a constant $c_h \leq O(h^6 \log h)$ such that for every $k = 0, 1, \dots, n - 1$,*

$$\lambda_k(G) \leq c_h \frac{\Delta_G(k)}{n}.$$

Note that a weaker statement was established in [\[KLPT11\]](#) with $\Delta_G(k)$ replaced by $k \cdot d_{\max}(G)$. The proof of [Theorem 3.2](#) is immediate from [Corollary 3.1](#) and the following result.

Theorem 3.4. *Suppose G is an n -vertex graph and $\omega : V(G) \rightarrow \mathbb{R}_+$ is a normalized conformal weight such that:*

1. For all $x \in V(G)$,

$$|B_\omega(x, R_*)| \leq \kappa R_*^2,$$

$$\text{where } R_* = \sqrt{\frac{n}{16\kappa}}.$$

2. $(V(G), \text{dist}_\omega)$ admits an $(\alpha, R_*/2)$ -padded random partition.

Then for every $k \leq n$, there are disjointly supported functions $\psi_1, \dots, \psi_k : V \rightarrow \mathbb{R}$ such that

$$\mathcal{R}_G(\psi_i) \lesssim \alpha^2 \kappa \frac{\Delta_G(k)}{n}.$$

[Section 3.4](#) is devoted to the construction of these bump functions. As discussed in the introduction, the spectral bounds from [Theorem 3.2](#) are not strong enough to yield almost sure bounds on the heat kernel of a distributional limit. Consulting [\(3.2\)](#), one sees that to control the return probabilities for most vertices $x \in V(G)$ requires us to say something about the distribution of the low-frequency eigenfunctions of G .

3.3 Return probabilities and spectral delocalization

Let us now indicate now how a sufficient strengthening of [Theorem 3.2](#) will allow us to control most of the return probabilities. For our finite graph $G = (V, E)$, it will help to define for every $\varepsilon > 0$:

$$\pi_G^*(\varepsilon) := \max\{\pi(S) : |S| \leq \varepsilon|V|\}.$$

Theorem 3.5. *Let $G = (V, E)$ be an n -vertex graph. Suppose there are capacitors $(A_1, \Omega_1), \dots, (A_k, \Omega_k)$ so that $\{\Omega_i\}$ are pairwise disjoint and $|\Omega_i| \leq M$ for all $i = 1, \dots, k$. Then for all $\varepsilon > 0$ and $T \geq 1$:*

$$\pi\left(\left\{x \in V : \frac{p_{2T}^G(x, x)}{\pi(x)} \geq \frac{\varepsilon|V|}{4M}\right\}\right) \geq -2\pi_G^*(\varepsilon) + \sum_{i=1}^k \pi(A_i) - 2T \sum_{i=1}^k \text{cap}_{\Omega_i}^G(A_i). \quad (3.3)$$

In particular, for any $\beta > 0$,

$$\pi\left(\left\{x \in V : p_{2T}^G(x, x) \geq \frac{\varepsilon\beta}{4M}\right\}\right) \geq -2\pi_G^*(\varepsilon) - \beta + \sum_{i=1}^k \pi(A_i) - 2T \sum_{i=1}^k \text{cap}_{\Omega_i}^G(A_i) \quad (3.4)$$

For illustration, consider a bounded-degree graph planar graph. In [Section 3.4](#), we will show that under this assumption, for every $\varepsilon > 0$ and $M \ll n$, we can find such a family $\{(A_i, \Omega_i)\}$ satisfying

$$\sum_{i=1}^k \pi(A_i) \geq 1 - \varepsilon,$$

and for each $i = 1, \dots, k$,

$$\text{cap}_{\Omega_i}^G(A_i) \leq \frac{c(\varepsilon)}{M} \pi(A_i), \quad (3.5)$$

where $c(\varepsilon)$ is some function of ε . Since our graph has bounded degrees, we have $\pi_G^*(\varepsilon) \leq O(\varepsilon)$, so choosing $T \leq \frac{\varepsilon^2 M}{c(\varepsilon)}$ yields

$$\pi\left(\left\{x \in V : p_{2T}^G(x, x) \geq \frac{c'(\varepsilon)}{T}\right\}\right) \geq 1 - O(\varepsilon).$$

for some other function $c'(\varepsilon)$.

We will require the following two preliminary results.

Lemma 3.6. *For any capacitor (A, Ω) and $T \geq 1$,*

$$\langle \mathbb{1}_\Omega, P^T \mathbb{1}_\Omega \rangle_\pi \geq \pi(A) - T \cdot \text{cap}_\Omega^G(A).$$

Remark 3.7. We remark that the same argument gives an identical lower bound on $\langle \mathbb{1}_\Omega, (I_\Omega P I_\Omega)^T \mathbb{1}_\Omega \rangle_\pi$ where I_Ω is the multiplication operator on $L^2(\pi)$ defined by $I_\Omega f(x) = \mathbb{1}_\Omega(x) f(x)$. Then $I_\Omega P I_\Omega$ is the operator of the walk killed off Ω .

Using reversibility, a lower bound on $\langle \mathbb{1}_S, P^T \mathbb{1}_S \rangle_\pi$ will give us control on return probabilities.

Lemma 3.8. *Suppose that, for some $S \subseteq V$, we have*

$$\langle \mathbb{1}_S, P^T \mathbb{1}_S \rangle_\pi \geq (1 - \delta) \pi(S).$$

Then for any $\gamma > 0$,

$$\pi\left(\left\{x \in S : p_{2T}(x, x) \geq \frac{\pi(x)}{4\gamma|S|}\right\}\right) \geq (1 - 2\delta) \pi(S) - 2\pi(\{x \in S : \pi(x) > \gamma\}).$$

Proof of Theorem 3.5. Apply Lemma 3.6 and Lemma 3.8 to each (A_i, Ω_i) with $\delta_i = T \frac{\text{cap}_{\Omega_i}^G(A_i)}{\pi(A_i)}$ and sum over $i = 1, \dots, k$, yielding

$$\sum_{i=1}^k \pi \left(\left\{ x \in A_i : p_{2T}^G(x, x) \geq \frac{\pi(x)}{4\gamma|\Omega_i|} \right\} \right) \geq \sum_{i=1}^k (1 - 2\delta_i)\pi(A_i) - 2\pi(\{x \in V : \pi(x) > \gamma\}),$$

where we have used that the sets $\{A_i\}$ are pairwise disjoint.

The former sum is precisely

$$\sum_{i=1}^k \pi(A_i) - 2T \text{cap}_{\Omega_i}^G(A_i),$$

and $|\Omega_i| \leq M$ for all $i = 1, \dots, k$ by assumption. Conclude the proof of (3.3) by setting $\gamma = 1/(\varepsilon|V|)$ so that the second term is at least $-2\pi_G^*(\varepsilon)$. To obtain (3.4), remove all $x \in V$ with $\pi(x) < \beta/|V|$. \square

Let us now prove the lemmas.

Proof of Lemma 3.6. We need the following basic fact.

Lemma 3.9 ([MS59, BR65]). *Suppose Q is a self-adjoint operator on $L^2(\pi)$ with $\langle \mathbb{1}_u, Q\mathbb{1}_v \rangle_\pi \geq 0$ for all $u, v \in V$, and $\psi \in L^2(\pi)$ satisfies $\psi \geq 0$ and $\|\psi\|_\pi = 1$. Then for every integer $T \geq 1$:*

$$\langle \psi, Q^T \psi \rangle_\pi \geq (\langle \psi, Q\psi \rangle_\pi)^T.$$

Now let $\varphi : V \rightarrow [0, 1]$ be any function satisfying $\text{supp } \varphi \subseteq \Omega$. Define $\psi := \varphi / \|\varphi\|_\pi$. Using Lemma 3.9 and $\varphi \leq \mathbb{1}_\Omega$, we have

$$\frac{\langle \mathbb{1}_\Omega, P^T \mathbb{1}_\Omega \rangle_\pi}{\|\varphi\|_\pi^2} \geq \langle \psi, P^T \psi \rangle_\pi \geq \langle \psi, P\psi \rangle_\pi^T \geq (1 - \langle \psi, (I - P)\psi \rangle_\pi)^T = (1 - \mathcal{R}_G(\psi))^T.$$

Since $\mathcal{R}_G(\varphi) = \mathcal{R}_G(\psi)$,

$$\langle \mathbb{1}_\Omega, P^T \mathbb{1}_\Omega \rangle_\pi \geq \|\varphi\|_\pi^2 (1 - \mathcal{R}_G(\varphi))^T \geq \|\varphi\|_\pi^2 (1 - T\mathcal{R}_G(\varphi)) = \|\varphi\|_\pi^2 - T\mathcal{E}_G(\varphi)$$

Using the definition of the capacity, take now a φ that additionally satisfies $\varphi|_A \equiv 1$ and $\mathcal{E}_G(\varphi) = \text{cap}_\Omega^G(A)$, yielding

$$\langle \mathbb{1}_\Omega, P^T \mathbb{1}_\Omega \rangle_\pi \geq \pi(A) - T \cdot \text{cap}_\Omega^G(A). \quad \square$$

Proof of Lemma 3.8. Let $L_\gamma(S) = \{y \in S : \pi(y) \leq \gamma\}$. Using reversibility, write

$$\begin{aligned} p_{2T}(x, x) &\geq \sum_{y \in S} p_T(x, y) p_T(y, x) \\ &= \sum_{y \in S} p_T(x, y)^2 \frac{\pi(x)}{\pi(y)} \\ &\geq \frac{\pi(x)}{\gamma} \sum_{y \in S: \pi(y) \leq \gamma} p_T(x, y)^2 \\ &\geq \frac{\pi(x)}{\gamma|S|} \left(\sum_{y \in S: \pi(y) \leq \gamma} p_T(x, y) \right)^2 \end{aligned}$$

$$= \frac{\pi(x)}{\gamma|S|} p_T(x, L_\gamma(S))^2.$$

On the other hand, note that

$$\sum_{x \in S} \pi(x) p_T(x, S) = \sum_{x, y \in S} \langle \mathbb{1}_x, P^T \mathbb{1}_y \rangle_\pi = \langle \mathbb{1}_S, P^T \mathbb{1}_S \rangle_\pi \geq (1 - \delta) \pi(S).$$

Therefore,

$$\sum_{x \in S} \pi(x) p_T(x, L_\gamma(S)) \geq (1 - \delta) \pi(S) - \pi(S \setminus L_\gamma(S))$$

In particular, we conclude that

$$\begin{aligned} \pi \left(\left\{ x \in S : p_{2T}(x, x) \geq \frac{1}{4} \frac{\pi(x)}{\gamma|S|} \right\} \right) &\geq \pi \left(\left\{ x \in S : p_T(x, L_\gamma(S)) \geq \frac{1}{2} \right\} \right) \\ &\geq (1 - 2\delta) \pi(S) - 2\pi(S \setminus L_\gamma(S)), \end{aligned}$$

completing the proof. \square

3.4 Constructing bump functions

We will now show that, given a conformal metric $\omega : V \rightarrow \mathbb{R}_+$ with sufficiently nice properties, we can construct many disjoint bump functions with small Rayleigh quotient. Our main geometric tool will be random partitions of metric spaces (cf. [Section 1.6.2](#)).

It will be easier to first prove [Theorem 3.4](#), and then to perform the more complicated construction needed for [Theorem 3.5](#). Let us define the function $\bar{d}_G : [0, 1] \rightarrow \mathbb{N}$ by

$$\bar{d}_G(\varepsilon) := \frac{\Delta_G(\varepsilon n)}{\varepsilon n},$$

which is the average degree among the εn vertices of largest degree in G . It is useful to observe that following simple fact: For every $C \geq 1$,

$$\#\{x \in V : \deg_G(x) \geq C \bar{d}_G(\varepsilon)\} \leq \frac{\varepsilon n}{C}. \quad (3.6)$$

3.4.1 Many disjoint bumps

Suppose we have a conformal metric $\omega : V \rightarrow \mathbb{R}_+$ that satisfies the following assumptions: For some numbers $R > 0$ and $\alpha, K \geq 1$,

(A1) For all $x \in V$, it holds that $|B_\omega(x, R)| \leq K \leq n/2$.

(A2) The space (V, dist_ω) admits an $(R/2, \alpha)$ -padded random partition.

Define the quantity

$$\eta := R/(12\alpha). \quad (3.7)$$

When dealing with unbounded degrees, we have to be careful about handling vertices of large conformal weight. To this end, for $\eta > 0$, define the set

$$V_L := \{x \in V : \omega(v) \geq \eta\}.$$

For a subset $S \subseteq V$, define

$$\mathcal{A}_\omega^\eta(S) := 16 \bar{d}_G(1/K) \mathcal{A}_\omega(S) + \eta^2 \cdot |E_G(S, V_L)|.$$

Observe that \mathcal{A}_ω^η is a measure on V , and

$$\mathcal{A}_\omega^\eta(V) \leq 16 \bar{d}_G(1/K) \|\omega\|_{\ell^2(V)}^2 + \eta^2 \cdot |E_G(V, V_L)|.$$

Since $|V_L| \leq \frac{\|\omega\|_{\ell^2(V)}^2}{\eta^2}$, it holds that

$$\mathcal{A}_\omega^\eta(V) \leq \|\omega\|_{\ell^2(V)}^2 \left(16 \bar{d}_G(1/K) + \bar{d}_G \left(\frac{1}{n} \|\omega\|_{\ell^2(V)}^2 / \eta^2 \right) \right). \quad (3.8)$$

Lemma 3.10. *Under assumptions (A1) and (A2), there exist disjoint subsets $T_1, T_2, \dots, T_r \subseteq V$ such that $r \geq n/8K$, and moreover:*

1. For all $i = 1, \dots, r$, it holds that $\frac{K}{2} \leq |T_i| \leq K$, and

$$\mathcal{A}_\omega^\eta(B_\omega(T_i, R/6\alpha)) \leq \frac{3}{r} \mathcal{A}_\omega^\eta(V).$$

2. For all $i \neq j$,

$$\text{dist}_\omega(T_i, T_j) \geq \frac{R}{2\alpha}.$$

Proof. Let $\mathcal{P} = \{S_1, S_2, \dots, S_m\}$ be a partition of V such that $\text{diam}(S_i) \leq R/2$ for each i . By property (A1), it holds that

$$|S_i| \leq K. \quad (3.9)$$

For each $i = 1, \dots, m$, define

$$\hat{S}_i = \{x \in S_i : B_\omega(x, R/4\alpha) \subseteq S_i\}.$$

Observe that for $i \neq j$, we have $\text{dist}_\omega(\hat{S}_i, \hat{S}_j) \geq R/2\alpha$ by construction.

Let $N_{\mathcal{P}} = |\hat{S}_1| + \dots + |\hat{S}_m|$. Suppose now that \mathcal{P} is an $(R/2, \alpha)$ -padded random partition. From the definition and linearity of expectation, we have

$$\mathbb{E}[N_{\mathcal{P}}] \geq \frac{1}{2}.$$

So let us fix a partition \mathcal{P} satisfying $N_{\mathcal{P}} \geq \frac{1}{2}$ for the remainder of the proof.

Using (3.9), it is possible to take unions of the sets $\{\hat{S}_i : i \in I\}$ to form disjoint sets T_1, T_2, \dots, T_r with $\frac{K}{2} \leq |T_i| \leq K$ and such that for $i \neq j$, $\text{dist}_\omega(T_i, T_j) \geq R/(2\alpha)$. In this process, we discard at most $K/2$ points, thus

$$|T_1| + \dots + |T_r| \geq \frac{1}{2}|V| - \frac{K}{2} \geq \frac{1}{4}|V|,$$

using the assumption that $K \leq n/2$. In particular, we have $r \geq n/4K$.

Let us now sort the sets so that

$$\mathcal{A}_\omega^\eta(B_\omega(T_1, R/6\alpha)) \leq \mathcal{A}_\omega^\eta(B_\omega(T_2, R/6\alpha)) \leq \dots \leq \mathcal{A}_\omega^\eta(B_\omega(T_r, R/6\alpha)).$$

Then for $i \in \{1, 2, \dots, \lceil r/2 \rceil\}$, since the sets $\{B_\omega(T_j, R/6\alpha) : j \in [r]\}$ are pairwise disjoint by construction, it must be that $\mathcal{A}_\omega^\eta(B_\omega(T_i, R/6\alpha)) \leq \frac{3}{r} \mathcal{A}_\omega^\eta(V)$. Thus the statement of the lemma is satisfied by the sets $\{T_i : i < r/2 + 1\}$. \square

Next, we observe that we can remove sets that have a vertex of large degree.

Lemma 3.11. *Under assumptions (A1) and (A2), there exist disjoint subsets $T_1, T_2, \dots, T_r \subseteq V$ with $r \geq n/16K$, satisfying properties (1) and (2) of Lemma 3.10, and furthermore*

$$\max \{ \deg_G(x) : x \in B_\omega(T_i, R/6\alpha) \} \leq 16 \bar{d}_G(1/K), \quad i = 1, 2, \dots, r. \quad (3.10)$$

Proof. Recalling (3.6), there are at most $n/16K$ vertices with degree larger than $16 \bar{d}_G(1/K)$. Thus one can apply Lemma 3.10 and then remove at most $n/16K$ of the sets that contain a vertex of large degree. \square

We are now ready to construct the bump functions.

Theorem 3.12. *If $\omega : V \rightarrow \mathbb{R}_+$ is a normalized conformal metric on G satisfying assumptions (A1) and (A2), then there exist disjointly supported functions $\psi_1, \psi_2, \dots, \psi_r : V \rightarrow \mathbb{R}_+$ with $r \geq n/16K$, and such that for all $i = 1, \dots, r$,*

$$\mathcal{R}_G(\psi_i) \lesssim \frac{\alpha^2 (\bar{d}_G(1/K) + \bar{d}_G(\alpha^2/R^2))}{R^2}. \quad (3.11)$$

Proof. Let $T_1, T_2, \dots, T_r \subseteq V$ be the subsets guaranteed by Lemma 3.11. For each $i \in [r]$, define

$$\psi_i(x) = \max \{ 0, \eta - \text{dist}_\omega(x, T_i) \}.$$

By construction, $T_i \subseteq \text{supp}(\psi_i) \subseteq B_\omega(T_i, \eta)$, hence by Lemma 3.10(3), the functions $\{\psi_i : i \in [r]\}$ are disjointly supported. (Recall that $\eta = R/(12\alpha)$.)

Now use the fact that each ψ_i is 1-Lipschitz to calculate

$$\begin{aligned} \sum_{\{x,y\} \in E} |\psi_i(x) - \psi_i(y)|^2 &\leq \eta^2 |E_G(B_\omega(T_i, \eta), V_L)| + \sum_{\substack{\{x,y\} \in E: \\ \{x,y\} \subseteq B_\omega(T_i, R/6\alpha)}} \text{dist}_\omega(x, y)^2 \\ &\stackrel{(3.10)}{\leq} \eta^2 |E_G(B_\omega(T_i, \eta), V_L)| + 16 \bar{d}_G(1/K) \mathcal{A}_\omega(B_\omega(T_i, R/6\alpha)) \\ &\leq \mathcal{A}_\omega^\eta(B_\omega(T_i, R/6\alpha)). \end{aligned}$$

Combining this with Lemma 3.10(2) yields

$$\mathcal{R}_G(\psi_i) = \frac{2 \sum_{\{x,y\} \in E} |\psi_i(x) - \psi_i(y)|^2}{\sum_{x \in V} \deg_G(x) \psi_i(x)^2} \leq \frac{6 \mathcal{A}_\omega^\eta(V)}{r \eta^2 |T_i|} \leq \frac{864 \alpha^2 \mathcal{A}_\omega^\eta(V)}{R^2 |V|}.$$

To arrive at the statement of the theorem, use (3.8) and the assumption that $|V|^{-1} \|\omega\|_{\ell^2(V)}^2 = 1$. \square

Let us now use this to prove Theorem 3.4.

Proof of Theorem 3.4. Consider $R = R_* = \sqrt{n/(16\kappa \cdot k)}$. By assumption, there is a normalized conformal metric $\omega : V \rightarrow \mathbb{R}_+$ satisfying $\max_{x \in V} |B_\omega(x, R)| \leq \kappa R^2$.

We may assume that $\kappa \geq 1$. Let $K = \kappa R^2$. Again by assumption, (V, dist_ω) admits an $(R/2, \alpha)$ -padded random partition. Now apply Theorem 3.12 to find $r \geq |V|/(16\kappa R_*^2)$ disjointly supported test functions $\{\psi_i\}$, each with

$$\mathcal{R}_G(\psi_i) \lesssim \frac{\alpha^2 (\bar{d}_G(1/(\kappa R_*^2)) + \bar{d}_G(\alpha^2/R_*^2))}{R_*^2} \leq 2 \frac{\alpha^2 \bar{d}_G(k/n)}{R_*^2} \lesssim \alpha^2 \kappa \bar{d}_G(k/n) \frac{k}{n} = \alpha^2 \kappa \frac{\Delta_G(k)}{n}.$$

We may assume that $k \leq n/(16\kappa)$. (Otherwise, we can just take n functions—one supported on each vertex of the graph—since the bound we are required to prove on the Rayleigh quotient is trivial.) Note that in this case, $r \geq |V|/(16\kappa R_*^2) \geq k$, completing the proof. \square

3.4.2 Exhausting the stationary measure by capacitors

Our arguments will follow along similar lines to those of the preceding section although things will be somewhat more delicate. $G = (V, E)$ is an n -vertex, connected graph. Suppose we have a conformal metric $\omega : V \rightarrow \mathbb{R}_+$ that satisfies the following assumptions: For some numbers $R > 0$, $\alpha, K \geq 1$,

(B1) For all $x \in V$, it holds that $|B_\omega(x, R)| \leq K$.

(B2) The space (V, dist_ω) admits an $(R/2, \alpha)$ -padded random partition.

Consider a number $\delta > 0$ and define

$$\eta := \frac{\delta R}{18\alpha},$$

$$V_L := \{x \in V : \omega(v) \geq \eta\}.$$

Lemma 3.13. *For any $\delta > 0$, it holds that under assumptions (B1) and (B2), there are pairwise disjoint sets S_1, \dots, S_r satisfying $\text{diam}_\omega(S_i) \leq R/2$ for each $i = 1, \dots, r$, and such that*

$$\sum_{i=1}^r \pi(\hat{S}_i) \geq 1 - \delta - \pi_G^*(\delta), \quad (3.12)$$

where for a subset $S \subseteq V$, we denote

$$\hat{S} := \{x \in S : B_\omega(x, \delta R/6\alpha) \subseteq S\}.$$

Moreover, it holds that

$$\max\{\deg_G(x) : x \in S_1 \cup \dots \cup S_r\} \leq \bar{d}_G(\delta/K), \quad (3.13)$$

and

$$\pi(\hat{S}_i) \geq \frac{1}{2}\pi(S_i) \quad \forall i = 1, \dots, r. \quad (3.14)$$

Proof. Let \mathcal{P} denote an $(R/2, \alpha)$ -padded random partition of (V, dist_ω) . Using linearity of expectation and the definition of a padded random partition yields

$$\mathbb{E}\left[\sum_{S \in \mathcal{P}} \pi(\hat{S})\right] \geq 1 - \delta/3.$$

Let us fix a partition \mathcal{P} in the support of \mathcal{P} satisfying $\sum_{S \in \mathcal{P}} \pi(\hat{S}) \geq 1 - \delta/3$.

Let $\mathcal{P}' = \{S \in \mathcal{P} : \pi(\hat{S}) \geq \frac{1}{2}\pi(S)\}$ and note that

$$\sum_{S \in \mathcal{P}'} \pi(\hat{S}) \geq 1 - (\delta/3) - 2(\delta/3) = 1 - \delta.$$

Finally, denote

$$\{S_1, \dots, S_r\} = \{S \in \mathcal{P}' : \max\{\deg_G(x) : x \in S\} \leq \bar{d}_G(\delta/K)\}.$$

Recalling (3.6), there are at most $\frac{\delta}{K}|V|$ vertices in G with degree larger than $\bar{d}_G(\delta/K)$. Therefore,

$$\sum_{i=1}^r \pi(\hat{S}_i) \geq \sum_{S \in \mathcal{P}'} \pi(\hat{S}) - \sum_{S \in \mathcal{P}' \setminus \{S_1, \dots, S_r\}} \pi(S) \geq 1 - \delta - \pi_G^*\left(\frac{\delta}{K} \max_{S \in \mathcal{P}} |S_i|\right) \geq 1 - \delta - \pi_G^*(\delta),$$

where the final inequality uses the fact that $|S| \leq K$ for $S \in \mathcal{P}$ which follows from $\text{diam}_\omega(S) \leq R/2$ and (B1). \square

We are now ready to construct the capacitors.

Theorem 3.14. *If $\omega : V \rightarrow \mathbb{R}_+$ is a conformal metric on G satisfying assumptions (B1) and (B2), then there exist pairwise disjoint capacitors $(A_1, \Omega_1), \dots, (A_k, \Omega_k)$ with*

$$\sum_{i=1}^k \pi(A_i) \geq 1 - \delta - \pi_G^*(\delta), \quad (3.15)$$

and such that for all $i = 1, \dots, k$,

$$\text{diam}_\omega(\Omega_i) \leq R/2, \quad (3.16)$$

$$|\Omega_i| \leq K. \quad (3.17)$$

And furthermore,

$$\sum_{i=1}^k \text{cap}_{\Omega_i}^G(A_i) \leq \frac{18^2 \alpha^2}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2 \frac{\bar{d}_G\left(\frac{18^2 \alpha^2}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2\right) + \bar{d}_G(\delta/K)}{\sqrt{\bar{d}_G(1)}}. \quad (3.18)$$

Proof of Theorem 3.14. Let $S_1, S_2, \dots, S_k \subseteq V$ be the subsets guaranteed by Lemma 3.13. For each $i = 1, \dots, k$, define

$$\psi_i(x) := \frac{1}{\eta} \max\{0, \eta - \text{dist}_\omega(x, \hat{S}_i)\}.$$

We denote $A_i := \hat{S}_i$ and $\Omega_i := S_i$. By construction, we have $0 \leq \psi_i \leq 1$, as well as $\text{supp } \psi_i \subseteq \Omega_i$ and $\psi_i|_{A_i} \equiv 1$. One observes that (3.15) follows from (3.12). Similarly, (3.16) follows from Lemma 3.13, and (3.17) then follows from assumption (B1).

Use the fact that $\eta\psi_i$ is 1-Lipschitz to calculate

$$\begin{aligned} \eta^2 \sum_{\{x,y\} \in E} |\psi_i(x) - \psi_i(y)|^2 &\leq \eta^2 |E_G(B_\omega(\hat{S}_i, \eta), V_L)| + \sum_{\substack{\{x,y\} \in E: \\ \{x,y\} \subseteq B_\omega(\hat{S}_i, \delta R/6\alpha)}} \text{dist}_\omega(x, y)^2 \\ &\leq \eta^2 |E_G(S_i, V_L)| + \bar{d}_G(\delta/K) \mathcal{A}_\omega(B_\omega(\hat{S}_i, \delta R/6\alpha)) \\ &\leq \eta^2 |E_G(S_i, V_L)| + \bar{d}_G(\delta/K) \mathcal{A}_\omega(S_i). \end{aligned}$$

Therefore,

$$\sum_{i=1}^k \text{cap}_{\Omega_i}^G(A_i) \leq \sum_{i=1}^k \mathcal{E}_G(\psi_i) \leq \frac{1}{|E|} \left(|E_G(V, V_L)| + \bar{d}_G(\delta/K) \eta^{-2} \|\omega\|_{\ell^2(V)}^2 \right). \quad (3.19)$$

Note that $|V_L| \leq \eta^{-2} \|\omega\|_{\ell^2(V)}^2$, yielding

$$|E_G(V, V_L)| \leq \Delta_G \left(\eta^{-2} \|\omega\|_{\ell^2(V)}^2 \right) = \eta^{-2} \|\omega\|_{\ell^2(V)} \bar{d}_G \left(\eta^{-2} \|\omega\|_{\ell^2(V)}^2 \right).$$

Finally, note that

$$\frac{\|\omega\|_{\ell^2(V)}^2}{|E|} = \frac{\|\omega\|_{L^2(V)}^2}{\bar{d}_G(1)}.$$

Plugging these into (3.19) and using the definition of η yields

$$\sum_{i=1}^k \text{cap}_{\Omega_i}^G(A_i) \leq \frac{18^2 \alpha^2}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2 \frac{\bar{d}\left(\frac{18^2 \alpha^2}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2\right) + \bar{d}(\delta/K)}{\bar{d}_G(1)}. \quad \square$$

Combining [Theorem 3.14](#) with [Theorem 3.5](#) yields the following corollary.

Corollary 3.15. *There is a universal constant $C \geq 1$ such that if $\omega : V \rightarrow \mathbb{R}_+$ is a conformal metric satisfying assumptions (B1) and (B2), then for every $\delta, \beta > 0$ and $T \geq 1$,*

$$\pi \left(\left\{ x \in V : p_{2T}^G(x, x) < \frac{\delta\beta}{4K} \right\} \right) \leq \beta + \delta + 3\pi_G^*(\delta) + \frac{C\alpha T}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2 \frac{\bar{d} \left(\frac{\alpha^2}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2 \right) + \bar{d}(\delta/K)}{\bar{d}_G(1)}.$$

If additionally, $\|\omega\|_{L^2(V)} \geq 1/2$, then the bound simplifies to

$$\pi \left(\left\{ x \in V : p_{2T}^G(x, x) < \frac{\delta\beta}{4K} \right\} \right) \leq \beta + \delta + 3\pi_G^*(\delta) + \frac{C\alpha T}{\delta^2 R^2} \|\omega\|_{L^2(V)}^2 \bar{d}(\delta/(K + R^2)).$$

4 Conformal growth rates and random walks

We will now apply the tools of the previous section to establish our main claims on spectral dimension and the diagonal heat kernel. Toward this end, it will be convenient to start with a unimodular random graph (G, ρ) and derive from it a sequence $\{G_n\}$ of finite unimodular random graphs such that $\{G_n\} \Rightarrow (G, \rho)$.

4.1 Invariant amenability and soficity

A unimodular random graph is called *sofic* if it is the distributional limit of finite graphs. It is an open question whether *every* unimodular random graph is sofic (see, e.g., [\[AL07, §10\]](#)). But as one might expect, for the proper definition of “amenable,” it turns out that all amenable graphs are sofic.

The invariant Cheeger constant. A *percolation* on (G, ρ) is a $\{0, 1\}$ -marking $\xi : E(G) \cup V(G) \rightarrow \{0, 1\}$ of the edges and vertices such that (G, ρ, ξ) is unimodular as a marked graph. One thinks of ξ as specifying a (random) subgraph of G corresponding to all the edges and vertices with $\xi = 1$. One calls ξ a *bond percolation* if $\xi(v) = 1$ almost surely for all $v \in V(G)$. The *cluster of vertex v* is the connected component $K_\xi(v)$ of v in the ξ -percolated graph. Finally, one says that ξ is *finitary* if almost surely all its clusters are finite.

For a graph G and a finite subset $W \subseteq V(G)$, we write $\partial_G^E W$ for the *edge boundary of W* : The subset of edges $\partial_G^E W \subseteq E(G)$ that have exactly one endpoint in W . The *invariant Cheeger constant* of a unimodular random graph (G, ρ) is the quantity

$$\Phi^{\text{inv}}(G, \rho) := \inf \left\{ \mathbb{E} \left[\frac{|\partial_G^E K_\xi(\rho)|}{|K_\xi(\rho)|} \right] : \xi \text{ is a finitary percolation on } G \right\}.$$

One says that (G, ρ) is *invariantly amenable* if $\Phi^{\text{inv}}(G, \rho) = 0$. Conversely, (G, ρ) is *invariantly nonamenable* if it is not invariantly amenable.

Hyperfiniteness and Følner sequences. A unimodular random graph (G, ρ) is *hyperfinite* if there is a $\{0, 1\}^{\mathbb{N}}$ -marking $\langle \xi_i \rangle_{i \geq 1}$ such that each ξ_i is finitary, $\xi_i \subseteq \xi_{i+1}$ almost surely, and almost surely $\bigcup_{i \geq 1} \xi_i = G$. In this case, $\langle \xi_i \rangle_{i \geq 1}$ is called a *Følner sequence* for (G, ρ) . One can consult [\[AHNR16\]](#) for a proof of the following (stated without proof in [\[AL07\]](#)).

Theorem 4.1 ([AL07], Thm. 8.5). *If (G, ρ) is a unimodular random graph with $\mathbb{E}[\deg_G(\rho)] < \infty$, then (G, ρ) is invariantly amenable if and only if it is hyperfinite.*

The main point for us is that if $\langle \xi_i \rangle_{i \geq 1}$ is a Følner sequence for (G, ρ) , then one has an approximation by finite unimodular random graphs: $\{G[K_{\xi_i}(\rho)] : i \geq 1\} \Rightarrow (G, \rho)$.

Corollary 4.2. *If (G, ρ) is a hyperfinite unimodular random graph, then there is a sequence $\{(G_n, \rho_n)\}$ of finite unimodular random graphs such that $\{G_n\} \Rightarrow (G, \rho)$ and moreover:*

1. *If (G, ρ) is α -decomposable, then for each $n \geq 1$, the unimodular random graph (G_n, ρ_n) is α -decomposable.*
2. *For any $R \geq 1$ and any normalized metric ω on (G, ρ) , there is a sequence $\{\omega_n\}$ of normalized metrics on $\{G_n\}$ such that for each $n \geq 1$,*

(a) *Almost surely, $\|\omega_n\|_{L^2(V(G_n))}^2 \geq 1/2$.*

(b) *It holds that*

$$\| \#B_{\omega_n}(\rho_n, R/\sqrt{2}) \|_{L^\infty} \leq \| \#B_\omega(\rho, R) \|_{L^\infty} .$$

3. *Given any sequence $\{\varepsilon_n\}$, we may pass to a subsequence of $\{(G_n, \rho_n)\}$ such that $\bar{d}_{G_n}(\varepsilon_n) \leq 2\bar{d}_\mu(\varepsilon_n)$, where μ is the law of (G, ρ) .*

Proof. Property (1) follows from the definition of α -decomposability. Suppose G is α -decomposable; then so is $G[S]$ for every finite, connected subset $S \subseteq V(G)$, by simply extending any weight $\omega : S \rightarrow \mathbb{R}_+$ to a weight $\hat{\omega} : V(G) \rightarrow \mathbb{R}_+$ defined by $\hat{\omega}(x) = \omega(x)$ if $x \in S$ and $\hat{\omega}(x) = \text{diam}_\omega(S)$ otherwise. In this case, $\text{dist}_{\hat{\omega}}|_{S \times S} = \text{dist}_\omega$.

Denote $\hat{\omega} = \sqrt{(\omega^2 + 1)}/2$. By assumption, $\hat{\omega}$ is normalized. The Mass-Transport Principle implies that (see, e.g., [AHNR16, Lem. 3.1]) if ξ is finitary, then ρ is uniformly distributed on its component $K_\xi(\rho)$, and therefore the unimodular random conformal graph $(G[K_\xi(\rho)], \hat{\omega}|_{K_\xi(\rho)}, \rho)$ is normalized as well. Moreover,

$$\| \#B_{\omega_n}(\rho_n, R/\sqrt{2}) \|_{L^\infty} \leq \| \#B_{\hat{\omega}}(\rho, R/\sqrt{2}) \|_{L^\infty} \leq \| \#B_\omega(\rho, R) \|_{L^\infty} ,$$

verifying property (2).

For every fixed $\varepsilon > 0$, we have $\bar{d}_{G_n}(\varepsilon) \rightarrow \bar{d}_\mu(\varepsilon)$. Thus we may pass to a subsequence satisfying property (3). \square

Subexponential conformal growth and invariant amenability. In conjunction with [Corollary 4.2](#), the next result will allow to approximate a unimodular random graph with bounded conformal growth exponent by a sequence of finite unimodular random graphs.

Lemma 4.3. *If (G, ρ) is a unimodular random graph with $\mathbb{E}[\deg_G(\rho)^2] < \infty$ and $\underline{\dim}_{\text{cg}}(G, \rho) < \infty$, then (G, ρ) is invariantly amenable.*

Proof. Suppose that (G, ρ) is invariantly nonamenable and ω is a normalized conformal metric on (G, ρ) . We will show that $\|B_\omega(\rho, R)\|_{L^\infty}$ grows at least exponentially in R , implying that $\underline{\dim}_{\text{cg}}(G, \rho) = \infty$.

By replacing ω with $\sqrt{(1 + \omega^2)}/2$, we may assume that $\omega \geq 1/2$ almost surely. Let $h > 0$ be such that $\Phi^{\text{inv}}(G, \rho) \geq h$. For some $K > 0$ to be specified soon, define a bond percolation $\xi : V(G) \rightarrow \{0, 1\}$ by

$$\xi(x) = \mathbb{1}_{\{\omega(x) \leq K\}} .$$

For a vertex $x \in V(G)$, define $\deg_\xi(x) := \sum_{y:\{x,y\} \in E(G)} \xi(y)$.

From [AL07, Thm. 8.13(i)], one concludes that if

$$\mathbb{E}[\deg_\xi(\rho) \mid \xi(\rho) = 1] > \mathbb{E}[\deg_G(\rho)] - h, \quad (4.1)$$

then with positive probability, the subgraph $\{x \in V(G) : \xi(x) = 1\}$ is nonamenable. Since a non-amenable subgraph has exponential growth and $\xi(x) = 1 \implies 1/2 \leq \omega(x) \leq K$, we conclude that $\|B_\omega(\rho, R)\|_{L^\infty}$ grows (at least) exponentially as $R \rightarrow \infty$. We are thus left to verify (4.1) for some $K > 0$.

Using Chebyshev's inequality and the fact that ω is normalized:

$$\mathbb{P}[\xi(\rho) = 0] = \mathbb{P}[\omega(\rho) > K] \leq \frac{1}{K^2}. \quad (4.2)$$

Now applying the Mass-Transport principle yields

$$\begin{aligned} \mathbb{E}[\deg_G(\rho) - \deg_\xi(\rho)] &= \mathbb{E} \left[\sum_{x:\{x,\rho\} \in E(G)} (1 - \xi(x)) \right] \\ &= \mathbb{E}[\deg_G(\rho)(1 - \xi(\rho))] \\ &\leq \sqrt{\mathbb{E}[\deg_G(\rho)^2]} \sqrt{\mathbb{P}[\xi(\rho) = 0]} \\ &\leq \frac{C}{K}, \end{aligned} \quad (4.3)$$

where $C := (\mathbb{E}[\deg_G(\rho)^2])^{1/2}$. This gives

$$\begin{aligned} \mathbb{E}[\deg_\xi(\rho) \mid \xi(\rho) = 1] &\geq \mathbb{E}[\deg_\xi(\rho)\xi(\rho)] \\ &\geq \mathbb{E}[\deg_\xi(\rho)] - \mathbb{E}[\deg_G(\rho)(1 - \xi(\rho))] \\ &\stackrel{(4.3)}{\geq} \mathbb{E}[\deg_G(\rho)] - \frac{2C}{K}. \end{aligned}$$

Choosing K large enough verifies (4.1), completing the proof. \square

The preceding argument was suggested to us by Tom Hutchcroft, replacing a considerably more complicated proof (a variant of which appears in Lemma 5.5 below).

4.2 Conformal growth exponent bounds the spectral dimension

We now restate Theorem 1.2.

Theorem 4.4 (Restatement of Theorem 1.2). *If (G, ρ) is a unimodular random graph and $\deg_G(\rho)$ has negligible tails, then almost surely:*

$$\begin{aligned} \overline{\dim}_{\text{sp}}(G) &\leq \overline{\dim}_{\text{cg}}(G, \rho), \\ \underline{\dim}_{\text{sp}}(G) &\leq \underline{\dim}_{\text{cg}}(G, \rho). \end{aligned}$$

Polylogarithmic corrections. It should be noted that the upper bound asserted in Theorem 4.4 holds up to polylogarithmic correction factors as long as one makes a correspondingly strong assumption on the distribution of $\deg_G(\rho)$. For two functions $f, g : [0, \infty) \rightarrow (0, \infty)$, let us write $f \leq_L g$ to denote that for some constant $c > 0$,

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x) |\log f(x)|^c} = 0.$$

If $f \preceq_L g$ and $g \preceq_L f$, we write $f \asymp_L g$.

Define

$$h(R) := \inf_{\omega} \|\#B_{\omega}(\rho, R)\|_{L^{\infty}},$$

where the infimum is over all normalized conformal metrics on (G, ρ) . Suppose the law of $\deg_G(\rho)$ has exponential tails and moreover, that $\lim_{R \rightarrow \infty} h(R)R^{-d} = 0$ for some $d > 0$. Then almost surely there is a function $f(R) \asymp_L R^2$ such that

$$p_{f(R)}^G(\rho, \rho) \preceq_L \frac{1}{h(R)}.$$

We record first some preliminary results. The following lemma is well-known; see, e.g., [LN05, Lem. 3.11]. Let (X, dist) be a metric space and consider $R > r > 0$. Let $C(X; R, r)$ denote the largest cardinality of a set $S \subseteq X$ such that $x \neq y \in S \implies r \leq \text{dist}(x, y) \leq R$.

Lemma 4.5. *For any metric space (X, dist) and $\tau > 0$, it holds that (X, dist) admits a (τ, α) -padded random partition for some $\alpha \leq O(\log C(X; 2\tau, \tau/4))$.*

Lemma 4.6. *Suppose that (G, ρ) is a finite unimodular random graph with stationary measure π_G , and such that $E[\deg_G(\rho)] < \infty$. Then for any $\delta > 0$,*

$$\pi_G^*(\delta) = \delta \frac{\bar{d}_G(\delta)}{\bar{d}_G(1)} \leq \delta \bar{d}_G(\delta). \quad (4.4)$$

(Recall the definition of π_G^* from Section 3.3.) Moreover, for any set of vertices $U_G \subseteq V(G)$, it holds that

$$\mathbb{P}[\rho \in U_G \mid G] \leq \pi_G(U_G) \bar{d}_G(1).$$

Proof. The first fact follows directly from the definitions and $\bar{d}_G(1) \geq 1$ since G is almost surely connected. The second fact is a reformulation of $|U_G| \leq \sum_{x \in U_G} \deg_G(x)$. \square

Lemma 4.7. *Suppose that (G, ρ) is a unimodular random graph with law μ , that $\deg_G(\rho)$ has negligible tails, and that $\underline{\dim}_{\text{cg}}(G, \rho) \leq d$ or $\overline{\dim}_{\text{cg}}(G, \rho) \leq d$. Then there exists a sequence of finite, normalized unimodular random conformal graphs $\{(G_k, \omega_k, \rho_k)\}$ so that*

1. $\{(G_k, \rho_k)\} \Rightarrow (G, \rho)$,
2. $\|\omega_k\|_{L^2(V(G_k))} \geq 1/2$ almost surely,
3. There is an increasing sequence of radii $\{R_n \geq 4\}$ so that for every $k \geq 1, n \geq 1$,

$$\|\#B_{\omega_k}(\rho_k, R_n)\|_{L^{\infty}} \leq R_n^{d+o(1)} \quad \text{as } n \rightarrow \infty. \quad (4.5)$$

If $\underline{\dim}_{\text{cg}}(G, \rho) \leq d$, then $\{R_n\}$ is unbounded, and if $\overline{\dim}_{\text{cg}}(G, \rho) \leq d$ then $\mathbb{N} \setminus \{R_n\}$ is finite.

4. Almost surely over the choice of (G_k, ρ_k) : For every $n \geq 1$, $(V(G_k), \text{dist}_{\omega_k})$ admits an $(R_n/2, \alpha_n)$ -padded random partition with

$$\alpha_n \lesssim d \log R_n.$$

Proof. Combining Lemma 4.3 with Corollary 4.2, we may take a sequence of finite normalized unimodular random conformal graphs $\{(G_k, \omega_k, \rho_k)\}$ so that $\{(G_k, \rho_k)\} \Rightarrow (G, \rho)$ and (2) and (3) are satisfied. Under the assumption $\underline{\dim}_{\text{cg}}(G, \rho) \leq d$, we may assume that the sequence $\{R_n\}$ is unbounded, and under the assumption $\overline{\dim}_{\text{cg}}(G, \rho) \leq d$, we may assume that $\mathbb{N} \setminus \{R_n\}$ is finite.

Note that Lemma 4.5 implies that $(V(G_k), \text{dist}_{\omega_k})$ almost surely admits, for every $n \geq 1$, an $(R_n/2, \alpha_n)$ -padded partition with

$$\alpha_n \lesssim \log \|\#B_{\omega_k}(\rho_k, R_n)\|_{L^{\infty}} \lesssim d \log R_n,$$

where the final inequality employs (4.5). \square

Proof of Theorem 4.4. Apply Lemma 4.7 to obtain a sequence $\{(G_k, \omega_k, \rho_k)\}$ satisfying conclusions (1)–(4). Define $K_{n,k} := \|\#B_{\omega_k}(\rho_k, R_n)\|_{L^\infty}$ and $\delta_n := (\log R_n)^{-1}$ for $k, n \geq 1$. Define also

$$\varepsilon_k := \min_{n \leq k} \frac{\delta_n}{K_{n,k}}.$$

Since $(G_k, \rho_k) \Rightarrow (G, \rho)$, we may pass to a subsequence of $\{(G_k, \omega_k, \rho_k)\}$ satisfying

$$\bar{d}_{G_k}(\varepsilon_k) \leq 2\bar{d}_\mu(\varepsilon_k) \quad (4.6)$$

$$\bar{d}_{G_k}(1) \leq 2\bar{d}_\mu(1) \leq O(1), \quad (4.7)$$

where the last inequality follows from the fact that $\deg_G(\rho)$ has negligible tails.

Fix $k, n \geq 1$ and condition on (G_k, ω_k, ρ_k) . Let π_k denote the stationary measure on G_k . Apply Corollary 3.15 with $R = R_n$, $\alpha = \alpha_n$, $K = K_{n,k} = \|\#B_{\omega_k}(\rho, R_n)\|_{L^\infty}$, and $\delta = \delta_n$ to obtain for any $T \geq 1$,

$$\pi_k \left(\left\{ x \in V : p_{2T}^{G_k}(x, x) < \frac{\delta_n^2}{4K_{n,k}} \right\} \right) \lesssim \delta_n + \pi_{G_k}^*(\delta_n) + \frac{\alpha_n^2 T}{\delta_n^2 R_n^2} \|\omega_k\|_{L^2(V(G_k))}^2 \bar{d}_{G_k}(\delta_n/(K_{n,k} + R_n^2)).$$

From Lemma 4.6 and (4.7), this implies

$$\mathbb{P} \left(p_{2T}^{G_k}(\rho_k, \rho_k) < \frac{\delta_n^2}{4K_{n,k}} \mid (G_k, \omega_k, \rho_k) \right) \lesssim \delta_n + \pi_{G_k}^*(\delta_n) + \frac{\alpha_n^2 T}{\delta_n^2 R_n^2} \|\omega_k\|_{L^2(V(G_k))}^2 \bar{d}_G(\delta_n/(K_{n,k} + R_n^2)). \quad (4.8)$$

Taking expectations over (G_k, ω_k, ρ_k) and using that ω_k is normalized yields

$$\mathbb{P} \left(p_{2T}^{G_k}(\rho_k, \rho_k) < \frac{\delta_n^2}{4K_{n,k}} \right) \lesssim \delta_n + \pi_{G_k}^*(\delta_n) + \frac{\alpha_n^2 T}{\delta_n^2 R_n^2} \bar{d}_G(\delta_n/(K_{n,k} + R_n^2)). \quad (4.9)$$

Observe that for any $\beta \geq \varepsilon_k$,

$$\pi_{G_k}^*(\beta) \stackrel{(4.4)}{\leq} \beta \bar{d}_{G_k}(\beta) \stackrel{(4.6)}{\leq} 2\beta \bar{d}_\mu(\beta).$$

Using this in (4.9) yields

$$\mathbb{P} \left(p_{2T}^{G_k}(\rho_k, \rho_k) < \frac{\delta_n^2}{4K_{n,k}} \right) \lesssim \delta_n (1 + \bar{d}_\mu(\delta_n)) + \frac{\alpha_n^2 T}{\delta_n^2 R_n^2} \bar{d}_G(\delta_n/(K_{n,k} + R_n^2)).$$

Let us now take $k \rightarrow \infty$ and use the fact that $\{(G_k, \rho_k)\} \Rightarrow (G, \rho)$, along with

$$K_n := \sup_{k \geq n} K_{n,k} \leq R_n^{d+o(1)} \quad \text{as } n \rightarrow \infty \quad (\text{from (4.5)}), \quad (4.10)$$

yielding, for all $n \geq 1$,

$$\mathbb{P} \left[p_{2T}^G(\rho, \rho) < \frac{\delta_n^2}{4K_n} \right] \lesssim \delta_n (1 + \bar{d}_\mu(\delta_n)) + \frac{\alpha_n^2 T}{\delta_n^2 R_n^2} \bar{d}_\mu(\delta_n/(K_n + R_n^2)).$$

Now the assumption that $\deg_G(\rho)$ has negligible tails (recall (1.3)) yields

$$\bar{d}_\mu(\beta) \leq \beta^{-o(1)} \quad \text{as } \beta \rightarrow 0,$$

hence for all $T \geq 1$, as $n \rightarrow \infty$:

$$\mathbb{P} \left[p_{2T}^G(\rho, \rho) < \frac{\delta_n^2}{4K_n} \right] \lesssim \delta_n^{1-o(1)} + \frac{T}{R_n^2} h(n),$$

where $h(n) \leq R_n^{o(1)}$ as $n \rightarrow \infty$ and $h(n)$ is non-decreasing.

Define for $n \geq 1$:

$$T_n := \left\lceil \delta_n \frac{R_n^2}{h(n)} \right\rceil. \quad (4.11)$$

This yields

$$\mathbb{P} \left[p_{2T_n}^G(\rho, \rho) < \frac{\delta_n^2}{4K_n} \right] \leq \delta_n^{1-o(1)} \quad \text{as } n \rightarrow \infty.$$

Using this together with (4.10), there is a function $h'(n) \leq T_n^{o(1)}$ such that

$$\mathbb{P} \left[p_{2T_n}^G(\rho, \rho) \geq \frac{h'(n)}{T_n^{d/2}} \right] \geq 1 - \delta_n^{1-o(1)} \geq 1 - \frac{1}{(\log T_n)^{1-o(1)}} \quad \text{as } n \rightarrow \infty. \quad (4.12)$$

In particular, if the sequence $\{R_n\}$ is unbounded, then $\{T_n\}$ is unbounded (cf. (4.11)) and this implies that almost surely $\underline{\dim}_{\text{sp}}(G, \rho) \leq d$.

Suppose that $\mathbb{N} \setminus \{R_n\}$ is finite. In conjunction with (4.11), this implies that almost surely $\overline{\dim}_{\text{sp}}(G, \rho) \leq d$. To see the latter fact, it suffices to observe that the even return times are monotone (see, e.g., (4.30)): For all integers $t \geq 1$:

$$p_{2t}^G(\rho, \rho) \geq p_{4t}^G(\rho, \rho).$$

Thus in order to have $\limsup_{T \rightarrow \infty} \frac{-\log p_{2T}^G(\rho, \rho)}{\log T} \leq d/2$, it suffices that (4.12) holds for some unbounded sequence $\{T_n\}$ satisfying $T_{n+1} \leq 2T_n$ for n sufficiently large. \square

4.3 On-diagonal heat kernel bounds

Our goal now is to prove [Theorem 1.7](#) and [Theorem 1.8](#). We start with the former and restate it here for ease of reference.

Theorem 4.8 (Restatement of [Theorem 1.7](#)). *Suppose that (G, ρ) satisfies the the conditions:*

1. (G, ρ) has gauged quadratic conformal growth and is uniformly decomposable,
2. $\mathbb{E}[\deg_G(\rho)^2] < \infty$.

Then there is a constant $C = C(\mu)$ such that for every $\delta > 0$ and all $T \geq C/\delta^2$,

$$\mathbb{P} \left[p_{2T}^G(\rho, \rho) < \frac{\delta}{T \bar{d}_\mu(1/T^3)} \right] \leq C\delta^{1/8}. \quad (4.13)$$

Proof. Let $C_\mu = \mathbb{E}[\deg_G(\rho)^2]$. Then for $\varepsilon > 0$,

$$C_\mu \geq \varepsilon \bar{d}_\mu(\varepsilon)^2. \quad (4.14)$$

Fix $\delta > 0$ and $T \geq C/\delta^{10}$ for some constant $C = C(\mu) \geq 1$ to be chosen later.

From [Corollary 4.2](#), we can take a sequence $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$ such that (G_n, ρ_n) is a finite unimodular random graph that is almost surely:

1. α -decomposable,
2. (κ, R) -quadratic for every $R \geq 1$,

where $\alpha, \kappa > 0$ are some constants depending on μ , and such that

$$\bar{d}_{G_n}(1/T^3) \leq 2\bar{d}_\mu(1/T^3), \quad (4.15)$$

$$\bar{d}_{G_n}(1) \leq 2\bar{d}_\mu(1) \leq 2\sqrt{C_\mu}. \quad (4.16)$$

Fix $n \geq 1$. Let $R = \sqrt{\gamma T \bar{d}_\mu(1/T^3)}$ for some number $\gamma > 0$ to be chosen soon. Recall from [Corollary 4.2](#) that we may assume that $\|\omega_n\|_{L^2(V(G_n))}^2 \geq 1/2$ almost surely.

Set $K = \kappa R^2$ and apply [Corollary 3.15](#) with $\beta = \sqrt{\delta}$ to obtain, for some constant $C_1 = C_1(\alpha, \kappa)$,

$$\pi_{G_n} \left(\left\{ x \in V : p_{2T}^{G_n}(x, x) < \frac{\delta^{3/2}}{4K} \right\} \right) \lesssim \sqrt{\delta} + \pi_{G_n}^*(\delta) + \frac{C_1}{\delta^2 \gamma} \|\omega_n\|_{L^2(V_n)}^2 \frac{\bar{d}_{G_n} \left(\frac{\delta}{C_1 \gamma T \bar{d}_\mu(1/T^3)} \right)}{\bar{d}_\mu(1/T^3)}$$

Observe that

$$\bar{d}_{G_n} \left(\frac{\delta}{C_1 \gamma T \bar{d}_\mu(1/T^3)} \right) \stackrel{(4.14)}{\leq} C_1 \bar{d}_{G_n} \left(\frac{\delta}{C_1 C_\mu \gamma T^{2.5}} \right) \leq 2C_1 \bar{d}_\mu \left(\frac{\delta}{C_1 C_\mu \gamma T^{2.5}} \right), \quad (4.17)$$

where the latter inequality follows from (4.15) for T chosen large enough.

Taking expectation over (G_n, ω_n, ρ_n) and using (4.17) and [Lemma 4.6](#) (with (4.16)) yields

$$\mathbb{P} \left(p_{2T}^{G_n}(\rho_n, \rho_n) < \frac{\delta^{3/2}}{4K} \right) \lesssim \sqrt{C_\mu} \left(\sqrt{\delta} + \pi_{G_n}^*(\delta) + \frac{C_1^2}{\delta^2 \gamma} \frac{\bar{d}_\mu \left(\frac{\delta}{C_1 C_\mu \gamma T^{2.5}} \right)}{\bar{d}_\mu(1/T^3)} \right)$$

From [Lemma 4.6](#), we have

$$\pi_{G_n}^*(\delta) \leq \delta \bar{d}_\mu(\delta) \stackrel{(4.14)}{\leq} 2\delta \sqrt{\frac{C_\mu}{\delta}} = 2\sqrt{C_\mu} \delta.$$

Now set $\gamma = C_1^2 \delta^{-2.5}$ so that for $T \geq C/\delta^7$ and C chosen large enough (depending on μ), we have

$$\mathbb{P} \left(p_{2T}^{G_n}(\rho_n, \rho_n) < \frac{\delta^4}{C_2 T \bar{d}_\mu(1/T^3)} \right) \lesssim C_\mu \sqrt{\delta}.$$

where $C_2 = C_2(\mu)$, and we recall that $K = \kappa R^2 = \kappa \gamma T \bar{d}_\mu(1/T^3)$. Since $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$, we obtain the same estimate for the limit, concluding the proof of (4.13). \square

Now we move on to the proof of [Theorem 1.8](#).

Proof of Theorem 1.8. Let $d_t = \bar{d}_\mu(1/t)$ and observe that since $\{d_t\}$ is monotone increasing,

$$\sum_{t \geq 1} \frac{1}{t d_t} \geq \sum_{t \geq 1} \frac{1}{t d_{t^3}} \geq \frac{1}{8} \sum_{k \geq 0} \frac{1}{d_{2^{3k}}} \geq \frac{1}{48} \sum_{k \geq 0} \frac{1}{d_{2^k}} \geq \frac{1}{96} \sum_{t \geq 1} \frac{1}{t d_t}.$$

Define $c_t := \frac{1}{t \bar{d}_\mu(1/t^3)}$. From the preceding inequalities, it suffices to consider $\hat{g}(T) = \sum_{t=1}^T c_t$ in place of $g(T)$.

Let $C = C(\mu)$ be the constant from (4.13). Fix $\delta > 0$. For $N \geq 1$, let $T_N = \min\{T : \hat{g}(T) \geq N\}$. Choose $N(\delta)$ large enough so that for $N \geq N(\delta)$, we have $T_N \geq C/\delta^2$ and

$$N \leq \hat{g}(T_N) \leq (1 + \delta)N.$$

Define the random variable

$$Z_N = \sum_{t=1}^{T_N} \min\{p_{2t}^G(\rho, \rho), \delta c_t\}.$$

Then by definition, $Z_N \leq \delta \hat{g}(T_N)$, and (4.13) implies that $\mathbb{E}[Z_N] \geq \delta(1 - C\delta^{1/8})\hat{g}(T_N)$, hence

$$\mathbb{P}\left[Z_N \geq \frac{\delta}{2}\hat{g}(T_N)\right] \geq 1 - 2C\delta^{1/8}.$$

Define the sequence $\{Y_N\}$ by

$$Y_N := \frac{1}{N} \sum_{n \leq N} \mathbb{1}_{\{Z_n \geq \frac{\delta}{2}\hat{g}(T_n)\}}.$$

Since $0 \leq Y_N \leq 1$ almost surely, Fatou's Lemma yields

$$\mathbb{P}\left[\limsup_{N \rightarrow \infty} Y_N > 0\right] \geq \mathbb{E}\left[\limsup_{N \rightarrow \infty} Y_N\right] \geq \limsup_{N \rightarrow \infty} \mathbb{E}[Y_N] \geq 1 - 2C\delta^{1/8}. \quad (4.18)$$

By construction, this implies

$$\mathbb{P}\left[\limsup_{T \rightarrow \infty} \frac{\sum_{t=1}^T p_{2t}^G(\rho, \rho)}{g_\mu(T)} > 0\right] \geq 1 - 2C\delta^{1/8}.$$

Now send $\delta \rightarrow 0$, concluding the proof. \square

4.3.1 Fatter degree tails and transience

We generalize the example from [GN13, §1.3].

Lemma 4.9. *For every monotonically increasing sequence $\{d_t : t = 1, 2, \dots\}$ of positive integers such that $\sum_{t \geq 1} \frac{1}{td_t} < \infty$, there is a unimodular random planar graph (G, ρ) with law μ such that for all t sufficiently large,*

$$\bar{d}_\mu(1/t) \leq d_t, \quad (4.19)$$

$\mathbb{E}[\deg_G(\rho)^2] < \infty$, and G is almost surely transient.

Proof. We may replace d_t by $\min(d_t, t^{1/4})$ so that $d_t \leq t^{1/4}$. We may also assume that $d_{2t} \leq 2d_t$ for all $t \geq 1$ without affecting convergence of the sequence $\sum_{t \geq 1} \frac{1}{td_t}$.

Consider an increasing function $f : \mathbb{N} \rightarrow \mathbb{N}$. Let T_n be a complete binary tree of height n and replace each edge at height $k = 1, 2, \dots, n$ from the leaves by $f(k)$ parallel edges (at the end of the proof, we indicate how to convert the construction into a simple graph).

Let (T, ρ) be the distributional limit of $\{T_n\}$, and let μ be the law of (T, ρ) . If $f(k) \leq 2^{o(k)}$ as $k \rightarrow \infty$, then the distributional limit exists and, moreover, the unique path to infinity is the one moving away from the leaves. Thus almost surely,

$$R_{\text{eff}}^T(\rho \leftrightarrow \infty) \leq \sum_{k=1}^{\infty} \frac{1}{f(k)}. \quad (4.20)$$

Let us now define $f(k) := 2d_{2^k} - d_{2^{k+1}}$ so that

$$d_{2^k} = 2^k \sum_{j=k+1}^{\infty} (2^{-j+1}d_{2^j} - 2^{-j}d_{2^{j+1}}) = \sum_{j=1}^{\infty} f(k+j)2^{-j}, \quad (4.21)$$

where convergence of the telescopic sum follows from our assumption that $d_{2^j} \leq 2^{j/4}$. Now observe if $t > 2^{k-1}$, then

$$\bar{d}_\mu(1/t) \leq \sum_{j=1}^{\infty} f(k+j)2^{-j} = d_{2^k},$$

hence (4.19) is satisfied.

We also have:

$$\sum_{k=1}^{\infty} \frac{1}{f(k)} \stackrel{(4.21)}{\lesssim} \sum_{k=1}^{\infty} \frac{1}{d_{2^k}} \leq 2 \sum_{t=1}^{\infty} \frac{1}{td_t} < \infty.$$

From (4.20), this implies that almost surely T is transient. Finally, note that $\mathbb{E}[\deg_G(\rho)^2] \leq 2 \sum_{k \geq 1} 2^{-k} (d_{2^k})^2 < \infty$ since we assumed that $d_{2^k} \leq 2^{k/4}$.

We may replace every parallel edge by a path of length two while affecting the degree distribution only by a factor of 2 (and one can rescale f accordingly to maintain property (4.19)). \square

4.4 Spectrally heterogeneous graphs

There are unimodular random graphs (G, ρ) with $\deg_G(\rho) \leq O(1)$ and $\dim_{\text{sp}}(G) \leq O(1)$ almost surely, but $\dim_{\text{cg}}(G, \rho) = \infty$. Indeed, there exist invariantly nonamenable graphs (G, ρ) for which $\overline{\dim}_{\text{sp}}(G) \leq O(1)$ almost surely. This is asserted in [AHNR16, §9.3].

We recall the construction alluded to there. Fix a value $\alpha > 0$. Let \mathcal{T} denote the infinite 3-regular tree and fix a vertex $v_0 \in V(\mathcal{T})$. To each $v \in V(\mathcal{T})$, we attach a random path of length L_v , where the random variables $\{L_v : v \in V(\mathcal{T})\}$ are independent and satisfy, for $\ell \geq 1$:

$$\mathbb{P}(L_v = \ell) = \begin{cases} c_1(\ell+1)\ell^{-2-\alpha} & v = v_0 \\ c_2\ell^{-2-\alpha} & v \in V(\mathcal{T}) \setminus \{v_0\}, \end{cases}$$

where $c_1, c_2 > 0$ are the unique values that give rise to probability measures. Let P_0 denote the path attached to v_0 . It is not difficult to verify that (\mathcal{T}, ρ) is unimodular when $\rho \in \{v_0\} \cup V(P_0)$ is chosen uniformly at random.

An interesting feature of this random graph is that the mean return probability is dominated by a small set of vertices (of measure $\approx T^{-\alpha/2}$):

$$\mathbb{E}[p_{2T}^{\mathcal{T}}(\rho, \rho)] \approx \mathbb{P}[L_{v_0} \geq \sqrt{T}] \cdot \frac{1}{\sqrt{T}} \approx T^{-(1+\alpha)/2}.$$

It turns out that one can obtain polynomial conformal volume growth if they are willing to ignore the ‘‘spectrally insignificant’’ vertices. Moreover, if (G, ρ) is spectrally homogeneous in a strong sense ((4.24) below), one can reverse the bound of Theorem 1.2 and obtain $\overline{\dim}_{\text{cg}}(G, \rho) \leq \text{a.s.} \overline{\dim}_{\text{sp}}(G)$.

Consider a monotone non-decreasing function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $h(n) \leq n^{o(1)}$ as $n \rightarrow \infty$ and a number $d > 0$. For $T \geq 1$, define the set of vertices with d -dimensional lower bounds on the diagonal heat kernel:

$$H_G(T) := \left\{ x \in V(G) : p_{2T}^G(x, x) \geq \frac{T^{-d/2}}{h(T)} \right\}.$$

Define also, for $R \geq 0$,

$$\widehat{H}_G(R) := H_G(R^2 h(R)^4 (\log R)^4).$$

Theorem 4.10. *Let (G, ρ) be a unimodular random graph and suppose that for all $T \geq 1$,*

$$\mathbb{E}[p_{2T}^G(\rho, \rho)] \leq h(T)T^{-d/2}. \quad (4.22)$$

Then there is a normalized conformal metric $\omega : V(G) \rightarrow \mathbb{R}_+$ such that

$$\left\| \mathbb{1}_{\widehat{H}_G(R)}(\rho) \cdot \#(B_\omega(\rho, R) \cap \widehat{H}_G(R)) \right\|_{L^\infty} \leq R^{d+o(1)} \quad \text{as } R \rightarrow \infty. \quad (4.23)$$

Moreover, if

$$\mathbb{P}[\rho \notin H_G(T)] \leq \frac{h(T)}{T} \quad \text{for all } T \geq 1, \quad (4.24)$$

then $\overline{\dim}_{\text{cg}}(G, \rho) \leq d$.

In order to prove [Theorem 4.10](#), we will need to recall some background on the spectral measures of infinite graphs.

4.4.1 Spectral measures on infinite graphs

Fix a connected, locally finite graph G . We use $\ell^2(G)$ for the Hilbert space of real-valued functions $f : V(G) \rightarrow \mathbb{R}$ equipped with the inner product

$$\langle f, g \rangle_{\ell^2(G)} = \sum_{x \in V(G)} \deg_G(x) f(x) g(x).$$

For a graph G , define the averaging operator $P_G : \ell^2(G) \rightarrow \ell^2(G)$ by

$$P_G \psi(u) := \frac{1}{\deg_G(u)} \sum_{v: \{u,v\} \in E(G)} \psi(v).$$

Observe that P_G is self-adjoint:

$$\langle \varphi, P_G \psi \rangle_{\ell^2(G)} = \sum_{x \in V(G)} \deg_G(x) \varphi(x) \frac{1}{\deg_G(x)} \sum_{y: \{x,y\} \in E(G)} \psi(y) = 2 \sum_{\{x,y\} \in E(G)} \varphi(x) \psi(y).$$

Thus given a vertex $v \in V(G)$, one can define the associated *spectral measure* μ_G^v as the unique probability measure on \mathbb{R} such that

$$\deg_G(v) \int \lambda^T d\mu_G^v(\lambda) = \langle \mathbb{1}_v, P_G^T \mathbb{1}_v \rangle_{\ell^2(G)} \quad (4.25)$$

for all integers $T \geq 1$.

Let us record a few additional equalities. Fix $\rho \in V(G)$. Then by self-adjointness, for any $T \geq 1$, we have:

$$\begin{aligned} \frac{\|P_G^T \mathbb{1}_\rho\|_{\ell^2(G)}^2}{\deg_G(\rho)} - \sum_{x \sim \rho} \frac{\langle P_G^T \mathbb{1}_x, P_G^T \mathbb{1}_\rho \rangle_{\ell^2(G)}}{\deg_G(\rho) \deg_G(x)} &= \frac{\langle \mathbb{1}_\rho, (I - P_G) P_G^{2T} \mathbb{1}_\rho \rangle_{\ell^2(G)}}{\deg_G(\rho)} \\ &= \int (1 - \lambda) \lambda^{2T} d\mu_G^\rho(\lambda), \end{aligned} \quad (4.26)$$

$$\sum_{x \sim \rho} \left\| \frac{P_G^T \mathbb{1}_\rho}{\deg_G(\rho)} - \frac{P_G^T \mathbb{1}_x}{\deg_G(x)} \right\|_{\ell^2(G)}^2 = \frac{\|P_G^T \mathbb{1}_\rho\|_{\ell^2(G)}^2}{\deg_G(\rho)} + \sum_{x \sim \rho} \frac{\|P_G^T \mathbb{1}_x\|_{\ell^2(G)}^2}{\deg_G(x)^2} - 2 \sum_{x \sim \rho} \frac{\langle P_G^T \mathbb{1}_x, P_G^T \mathbb{1}_\rho \rangle_{\ell^2(G)}}{\deg_G(\rho) \deg_G(x)} \quad (4.27)$$

For any $x \in V(G)$ and integer $T \geq 0$, we have

$$\|P_G^T \mathbb{1}_x\|_{\ell^2(G)}^2 = \langle \mathbb{1}_x, P_G^{2T} \mathbb{1}_x \rangle_{\ell^2(G)} = \deg_G(x) \cdot p_{2T}^G(x, x). \quad (4.28)$$

Moreover, observe that $\{\mathbb{1}_x / \sqrt{\deg_G(x)} : x \in V(G)\}$ forms an orthonormal basis for $\ell^2(G)$, hence

$$\sum_{x \in V(G)} \frac{\langle P_G^T \mathbb{1}_x, P_G^T \mathbb{1}_\rho \rangle_{\ell^2(G)}^2}{\deg_G(x)} = \sum_{x \in V(G)} \frac{\langle \mathbb{1}_x, P_G^{2T} \mathbb{1}_\rho \rangle_{\ell^2(G)}^2}{\deg_G(x)} = \|P_G^{2T} \mathbb{1}_\rho\|_{\ell^2(G)}^2 \quad (4.29)$$

Note also that since P_G is a Markov operator, it is a contraction on $\ell^2(G)$, hence for all integers $T \geq 1$,

$$p_{2T}^G(x, x) = \deg_G(x) \|P_G^T \mathbb{1}_x\|_{\ell^2(G)}^2 \geq \deg_G(x) \|P_G^{2T} \mathbb{1}_x\|_{\ell^2(G)}^2 = p_{4T}^G(x, x). \quad (4.30)$$

The heat kernel embedding and growth rates. Suppose that G is a connected, locally finite graph, and let $d > 0$ and $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be as in [Section 4.4](#). Let us define $\Phi_T^G : V(G) \rightarrow \ell^2(G)$ by

$$\Phi_T^G(x) := \frac{P_G^T \mathbb{1}_x}{\sqrt{\deg_G(x)}}.$$

The heat-kernel embedding is closely related to the spectral embedding which can be described as follows. Let I_{P_G} be a resolution of the identity satisfying $P_G = \int \lambda dI_{P_G}(\lambda)$, whose existence and uniqueness follows from the spectral theorem for bounded, self-adjoint operators on $\ell^2(G)$. Then for $\delta > 0$, the spectral embedding $\Psi_\delta^G : V(G) \rightarrow \ell^2(G)$ is given by

$$\Psi_\delta^G(x) := \frac{I_{P_G}([1 - \delta, 1]) \mathbb{1}_x}{\sqrt{\deg_G(x)}}.$$

Clearly these two embeddings are closely related for $T \asymp \frac{1}{\delta}$. The geometry of the spectral embedding has been used in work on higher-order Cheeger inequalities [[LOT14](#)] and in connection with return probabilities [[LO15](#)].

For $x \in V(G)$, also define the set of points that are closer to $\Phi_T^G(x)$ than the origin in the heat kernel embedding:

$$C_T^G(x) := \{y \in V(G) : \|\Phi_T^G(x) - \Phi_T^G(y)\|_{\ell^2(G)} \leq \|\Phi_T^G(y)\|_{\ell^2(G)}\}.$$

The next lemma gives a relationship between return probabilities and the size of $C_T^G(x)$. This is inspired by the ‘‘mass spreading’’ property of the spectral embedding employed in [[LOT14](#)].

Lemma 4.11. *For any $x \in V(G)$, it holds that*

$$|C_T^G(x)| \leq \frac{4}{p_{2T}^G(x, x)}.$$

Proof. Employ (4.29) and $\langle x, y \rangle = \frac{1}{2} (\|x\|^2 + \|y\|^2 - \|x - y\|^2)$ to write

$$\|\Phi_{2T}^G(\rho)\|_{\ell^2(G)}^2 = \sum_{x \in V(G)} \langle \Phi_T^G(x), \Phi_T^G(\rho) \rangle_{\ell^2(G)}^2 \geq |C_T^G(\rho)| \frac{\|\Phi_T^G(\rho)\|_{\ell^2(G)}^4}{4}.$$

To finish, use the fact that P_G is a contraction on $\ell^2(G)$: $\|\Phi_{2T}^G(\rho)\|_{\ell^2(G)} \leq \|\Phi_T^G(\rho)\|_{\ell^2(G)}$, hence

$$|C_T^G(\rho)| \leq \frac{4}{\|\Phi_T^G(\rho)\|_{\ell^2(G)}^2} \stackrel{(4.28)}{=} \frac{4}{p_{2T}^G(\rho, \rho)}. \quad \square$$

4.4.2 Spectrally significant vertices in unimodular random graphs

If (G, ρ) is a random rooted graph such that P_G is almost surely self-adjoint, one defines the *spectral measure* of (G, ρ) by

$$\mu := \mathbb{E} [\mu_G^\rho].$$

Let (G, ρ) be a unimodular random graph with spectral measure μ . Taking expectations in (4.26) and (4.27) and applying the Mass-Transport Principle shows that for all $T \geq 1$,

$$\mathbb{E} \left[\sum_{x \sim \rho} \|\Phi_T^G(x) - \Phi_T^G(y)\|_{\ell^2(G)}^2 \right] = \mathbb{E} \left[\sum_{x \sim \rho} \left\| \frac{P_G^T \mathbf{1}_\rho}{\deg_G(\rho)} - \frac{P_G^T \mathbf{1}_x}{\deg_G(x)} \right\|_{\ell^2(G)}^2 \right] = 2 \int (1 - \lambda) \lambda^{2T} d\mu(\lambda). \quad (4.31)$$

Consider some $d > 0$ and split the latter integral into two pieces, depending on whether $\lambda \leq 1 - \frac{(d+1)\log T}{T}$:

$$\begin{aligned} \int (1 - \lambda) \lambda^{2T} d\mu(\lambda) &\leq T^{-d-1} + \frac{(d+1)\log T}{T} \int \lambda^{2T} d\mu(\lambda) \\ &\stackrel{(4.25)}{=} T^{-d-1} + \frac{(d+1)\log T}{T} \mathbb{E}[p_{2T}^G(\rho, \rho)]. \end{aligned} \quad (4.32)$$

Proof of Theorem 4.10. For $k \geq 1$, define the conformal metric $\omega_k : V(G) \rightarrow \mathbb{R}_+$ by

$$\omega_k(x) := \sqrt{\sum_{y: \{x, y\} \in E(G)} \|\Phi_{2^k}^G(x) - \Phi_{2^k}^G(y)\|_{\ell^2(G)}^2}$$

Under assumption (4.22), we can employ (4.32) and (4.31) to write

$$\mathbb{E} \omega_k(\rho)^2 \leq 2^{-k(d+1)} + \frac{(d+1)k}{2^k} 2^{-kd/2} h(2^k).$$

Define now

$$\omega := \sqrt{\sum_{k \geq 1} \frac{2^{k(1+d/2)}}{k^3 h(2^k)} \omega_k^2}$$

so that

$$\mathbb{E} \omega(\rho)^2 \lesssim \sum_{k \geq 1} \frac{1}{k^2} \lesssim 1.$$

By construction, we have, for every $k \geq 1$ and $x, y \in V(G)$,

$$\text{dist}_\omega(x, y)^2 \geq \frac{2^{k(1+d/2)}}{k^3 h(2^k)} \|\Phi_{2^k}^G(x) - \Phi_{2^k}^G(y)\|_{\ell^2(G)}^2. \quad (4.33)$$

Lemma 4.12. *For all $k \geq 1$, if $x \in H_G(2^k)$, then*

$$\left| B_\omega \left(x, \frac{2^{k/2}}{h(2^k)k^{3/2}} \right) \cap H_G(2^k) \right| \leq h(2^k) 2^{kd/2+2},$$

Proof. From (4.33),

$$y \in B_\omega \left(x, \frac{2^{k/2}}{h(2^k)k^{3/2}} \right) \implies \|\Phi_{2^k}^G(x) - \Phi_{2^k}^G(y)\|_{\ell^2(G)}^2 \leq \frac{2^{-kd/2}}{h(2^k)}.$$

By definition: $y \in H_G(2^k) \implies p_{2^{k+1}}^G(y, y) \geq 2^{kd/2}/h(2^k)$. Therefore:

$$\begin{aligned} y \in B_\omega \left(x, \frac{2^{k/2}}{h(2^k)k^{3/2}} \right) \cap H_G(2^k) &\implies \|\Phi_{2^k}^G(x) - \Phi_{2^k}^G(y)\|_{\ell^2(G)}^2 \leq p_{2^{k+1}}^G(y, y) \stackrel{(4.28)}{=} \|\Phi_{2^k}^G(y)\|_{\ell^2(G)}^2 \\ &\implies y \in C_{2^k}^G(x). \end{aligned}$$

Now [Lemma 4.11](#) implies the desired bound. □

Corollary 4.13. *For all R sufficiently large, if $x \in \widehat{H}_G(R)$, then*

$$|B_\omega(x, R) \cap \widehat{H}_G(R)| \leq R^{d+o(1)}.$$

This confirms [\(4.23\)](#). To verify that $\overline{\dim}_{\text{cg}}(G, \rho) \leq d$ under [\(4.24\)](#), we define

$$\hat{\omega}_k = \frac{2^k}{\sqrt{h(4^k h(2^k)^4 k^4)}} \mathbb{1}_{V(G) \setminus \widehat{H}_G(2^k)}.$$

Observe that from [\(4.24\)](#),

$$\mathbb{E} \hat{\omega}_k(\rho)^2 = \frac{4^k}{h(4^k h(2^k)^4 k^4)} \mathbb{P}[\rho \notin \widehat{H}_G(2^k)] \lesssim 1.$$

Define

$$\hat{\omega} := \sqrt{\sum_{k \geq 1} \frac{\hat{\omega}_k^2}{k^2}}$$

so that $\mathbb{E} \hat{\omega}(\rho)^2 \lesssim 1$. Finally, note that for any $k \geq 1$:

$$x \notin \widehat{H}_G(2^k) \implies B_{\hat{\omega}} \left(x, 2^k / \sqrt{h(4^k h(2^k)^4 k^4)} \right) = \{x\}.$$

Thus taking the final conformal metric $\omega_0 = \sqrt{\omega^2 + \hat{\omega}^2}$ verifies that $\overline{\dim}_{\text{cg}}(G, \rho) \leq d$. □

5 Markov type and speed of the random walk

We will now address the speed of the random walk on unimodular random graphs. Our approach is to first establish that the random walk is at most diffusive under any conformal metric, and our main tool will be the theory of Markov type. We then use separators to construct conformal metrics with Hölder-type comparisons to the graph metric, allowing us to establish subdiffusive speed in certain settings.

5.1 Diffusive bounds in the conformal metric

In this section, it will be helpful to think about *reversible* random graphs. Suppose (G, ρ) is a random rooted graph and let $\{X_t\}$ denote the random walk on G (conditioned on (G, ρ)) with $X_0 = \rho$. Then (G, ρ) is *reversible* if we have the identity of laws:

$$(G, X_0, X_1) \stackrel{\text{law}}{=} (G, X_1, X_0).$$

The following lemma is from [\[BC12, Prop. 2.5\]](#).

Lemma 5.1. *There is a correspondence between unimodular random graphs with $\mathbb{E}[\deg_G(\rho)] < \infty$ and reversible random graphs: (G, ρ) is unimodular if and only if $(\bar{G}, \bar{\rho})$ is reversible, where $(\bar{G}, \bar{\rho})$ has the law of (G, ρ) biased by $\deg_G(\rho)$.*

We first prove a general result for the case when a reversible conformal random graph (G, ω, ρ) is such that the Markov type 2 constant of the metric space $(V(G), \text{dist}_\omega)$ is essentially bounded. For instance, by [DLP13], this is true when G is almost surely planar. Then we move on to graphs of annealed polynomial growth.

Theorem 5.2. *Suppose that (G, ρ) is an invariantly amenable (cf. Section 4.1) reversible random graph. Then for any conformal metric ω on (G, ρ) , the following holds: For all $T \geq 1$,*

$$\mathbb{E}[\text{dist}_\omega(X_0, X_T)^2 \mid X_0 = \rho] \leq T \|M_2(V(G), \text{dist}_\omega)\|_{L^\infty}^2 \cdot \mathbb{E}[\omega(\rho)^2].$$

Proof. Let $\{\xi_j : j \geq 1\}$ denote a Følner sequence for (G, ρ) so that

$$\{(G[K_{\xi_j}(\rho)], \rho) : j \geq 1\} \Rightarrow (G, \rho).$$

In particular, we have

$$\{(G[K_{\xi_j}(\rho)], \omega_j, \rho_j)\} \Rightarrow (G, \omega, \rho), \quad (5.1)$$

where $\omega_j = \omega|_{K_{\xi_j}(\rho)}$ and ρ_j is distributed according to the stationary measure on $G[K_{\xi_j}(\rho)]$.

Let $\{X_t^j\}$ denote the random walk conditioned on $G[K_{\xi_j}(\rho)]$, where X_0^j has the law of the stationary measure on $G[K_{\xi_j}(\rho)]$. If we let $M := \|M_2(V(G), \text{dist}_\omega)\|_{L^\infty}$, then the definition of Markov type yields, for any $j \geq 1$:

$$\mathbb{E}[\text{dist}_{\omega_j}(X_0^j, X_T^j)^2 \mid G[K_{\xi_j}(\rho)]] \leq TM^2 \mathbb{E}[\text{dist}_{\omega_j}(X_0^j, X_1^j)^2 \mid G[K_{\xi_j}(\rho)]] .$$

Recall that if $\{x, y\} \in E(G)$, then $\text{dist}_\omega(x, y) \leq \frac{1}{2}(\omega(x)^2 + \omega(y)^2)$, hence by stationarity, the latter quantity is bounded by

$$\mathbb{E}[\text{dist}_{\omega_j}(X_0^j, X_1^j)^2 \mid G[K_{\xi_j}(\rho)]] \leq \mathbb{E}[\omega_j(X_0^j)^2 \mid G[K_{\xi_j}(\rho)]] = \mathbb{E}[\omega_j(\rho_j)^2 \mid G[K_{\xi_j}(\rho)]] .$$

Taking expectation over $G[K_{\xi_j}(\rho)]$ yields

$$\mathbb{E}[\text{dist}_{\omega_j}(X_0^j, X_T^j)^2] \leq TM^2 \mathbb{E}[\omega_j(\rho_j)^2].$$

Sending $j \rightarrow \infty$ and employing (5.1) yields $\mathbb{E}[\omega_j(\rho_j)^2] \rightarrow \mathbb{E}[\omega(\rho)^2]$ and $\mathbb{E}[\text{dist}_{\omega_j}(X_0^j, X_T^j)^2] \rightarrow \mathbb{E}[\text{dist}_\omega(X_0, X_T)^2]$, completing the proof. \square

In order to prove a similar result for reversible random graphs of polynomial growth, we need a result about the Markov type 2 constants of finite metric spaces. It is an immediate consequence of the following facts: (1) Hilbert spaces have Markov type 2 with constant 1 [Bal92], (2) Markov type is a bi-Lipschitz invariant, (3) every n -point metric space embeds into a Hilbert space with $O(\log n)$ bi-Lipschitz distortion [Bou85].

Theorem 5.3. *If (X, d) is an n -point metric space with $n \geq 2$, then $M_2(X, d) \leq O(\log n)$.*

The next lemma will allow us to choose a (unimodular) Følner sequence whose sets have controlled diameters.

Lemma 5.4. *Suppose that (G, ρ) is a unimodular random graph. Then there is a sequence of bond percolations $\langle \xi_j : j \geq 1 \rangle$ with the following properties for every $j \geq 1$:*

1. $(G, \rho, \langle \xi_j : j \geq 1 \rangle)$ is unimodular as a marked network.
2. Almost surely, $\text{diam}_G(K_{\xi_j}(\rho)) \leq 2j$.
3. For every $r > 0$, it holds that

$$\mathbb{P}[B_G(\rho, r) \not\subseteq K_{\xi_j}(\rho)] \leq \frac{1}{|B_G(\rho, j)|^2} + \frac{12r}{j} \log |B_G(\rho, 3j)|.$$

Proof. Fix a parameter $R \geq 1$ and let $\{R_x : x \in V(G)\}$ denote a sequence of independent exponential random variables where R_x has mean

$$\frac{R}{3 \log |B_G(x, 2R)|}. \quad (5.2)$$

Let $\{\beta_x \in [0, 1] : x \in V(G)\}$ denote an independent family of i.i.d. uniform random variables.

Let $\hat{R}_x := \min(R_x, R)$ and define for every $x \in V(G)$:

$$\Theta_x := \{y \in B_G(x, R) : \hat{R}_y \geq \text{dist}_G(x, y)\}.$$

Let $\theta(x) \in \Theta_x$ be the vertex $y \in \Theta_x$ with β_y minimal. By construction, $\{\theta^{-1}(y) : y \in V(G)\}$ is almost surely a partition of $V(G)$ and $\theta^{-1}(y) \subseteq B_G(y, R)$ for every $y \in V(G)$.

Define a bond percolation $\xi_R : E(G) \rightarrow \{0, 1\}$ on G by $\xi_R(\{u, v\}) = \mathbb{1}_{\{\theta(u)=\theta(v)\}}$. By construction, every cluster $K_{\xi_R}(x)$ is of the form $\theta^{-1}(y)$ for some $y \in V(G)$, and thus $\text{diam}_G(K_{\xi_R}(x)) \leq 2R$ for every $x \in V(G)$. The next lemma completes the proof.

Lemma 5.5. *For every $x \in V(G)$, the following holds:*

$$\mathbb{P}[B_G(x, r) \not\subseteq K_{\xi_R}(x)] \leq \frac{1}{|B_G(x, R)|^2} + \frac{12r}{R} \log |B_G(x, 3R)|.$$

Proof. Denote the event $\mathcal{E} = \{\exists y \in B_G(x, R) : R_y > R\}$. Note that $y \in B_G(x, R) \implies |B_G(y, 2R)| \geq |B_G(x, R)|$, and thus reviewing the mean of each R_y in (5.2), a union bound yields

$$\mathbb{P}[\mathcal{E}] \leq e^{-3 \log |B_G(x, R)|} |B_G(x, R)| \leq \frac{1}{|B_G(x, R)|^2}.$$

Let $y_* \in V(G)$ denote the vertex with β_{y_*} minimal in the set

$$\{y \in B_G(x, R) : R_y \geq \text{dist}_G(x, y) - r\}.$$

Let \mathcal{E}' denote the event $\{R_{y_*} > \text{dist}_G(x, y_*) + r\}$. Then:

$$\mathcal{E}' \wedge \neg \mathcal{E} \implies B_G(x, r) \subseteq B_G(y_*, R_{y_*}) \implies B_G(x, r) \subseteq K_{\xi_R}(x).$$

By the memoryless property of the exponential distribution:

$$\begin{aligned} \mathbb{P}[\neg \mathcal{E}' \mid R_{y_*} \geq \text{dist}_G(x, y_*) - r] &= \frac{6 \log |B_G(y_*, 2R)|}{R} \int_0^{2r} \exp\left(\frac{-6t \log |B_G(y_*, 2R)|}{R}\right) dt \\ &\leq \frac{12r}{R} \log |B_G(y_*, 2R)| \leq \frac{12r}{R} \log |B_G(x, 3R)|. \quad \square \end{aligned}$$

□

We will soon define a finite-state Markov chain on the cluster $K_\xi(\rho)$; the following definition will be helpful.

Definition 5.6 (Restricted random walk). Consider a graph $G = (V, E)$ and a finite subset $S \subseteq V$. Let

$$N_G(x) = \{y \in V : \{x, y\} \in E\}$$

denote the neighborhood of a vertex $x \in V$.

Extend \deg_G to a measure on subsets $S \subseteq V(G)$ in the obvious way: $\deg_G(S) := \sum_{x \in S} \deg_G(x)$. Define a measure π_S on S by

$$\pi_S(x) := \frac{\deg_G(x)}{\deg_G(S)}. \quad (5.3)$$

We define *the random walk restricted to S* as the following process $\{Z_t\}$: For $t \geq 0$, put

$$\mathbb{P}(Z_{t+1} = y \mid Z_t = x) = \begin{cases} \frac{|N_G(x) \cap S|}{\deg_G(x)} & y = x \\ \frac{1}{\deg_G(x)} & y \in N_G(x) \cap S \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that $\{Z_t\}$ is a reversible Markov chain on S with stationary measure π_S . If Z_0 has law π_S , we say that $\{Z_t\}$ is the *stationary random walk restricted to S* .

We now prove the following theorem; it immediately yields [Theorem 1.14](#) (since the assumption of the latter theorem implies that $\deg_G(\rho)$ is essentially bounded).

Theorem 5.7. *Suppose that (G, ρ) is a random rooted graph and for some constants $C, q \geq 1$, it holds that*

$$\mathbb{E} |B_G(\rho, r)| \leq Cr^q \quad \forall r \geq 1. \quad (5.4)$$

Then:

1. *If (G, ρ) is reversible, then for any conformal metric ω on (G, ρ) : For any $q' \geq 1$ and $T \geq 2$,*

$$\mathbb{E} \left[\min\{T^{q'}, \text{dist}_\omega(X_0, X_T)\}^2 \mid X_0 = \rho \right] \leq C'T(\log T)^2 \cdot \mathbb{E}[\omega(\rho)^2] + 1, \quad (5.5)$$

where $C' = C'(C, q, q')$ is number depending only on C, q, q' .

2. *If (G, ρ) is unimodular and*

$$\mathbb{P}[\deg_G(\rho) > \lambda] \leq e^{\lambda/C} \quad \forall \lambda \geq 1, \quad (5.6)$$

then for any conformal metric ω on (G, ρ) : For any $q' \geq 1$ and $T \geq 2$,

$$\mathbb{E} \left[\min\{T^{q'}, \text{dist}_\omega(X_0, X_T)\}^2 \mid X_0 = \rho \right] \leq C'T(\log T)^4 \cdot \mathbb{E}[\omega(\rho)^2] + 1, \quad (5.7)$$

where $C' = C'(C, q, q')$.

If, additionally, G is almost surely planar, then the bound improves to

$$\mathbb{E} \left[\min\{T^{q'}, \text{dist}_\omega(X_0, X_T)\}^2 \mid X_0 = \rho \right] \leq C'T(\log T)^2 \cdot \mathbb{E}[\omega(\rho)^2] + 1. \quad (5.8)$$

Proof. Let $\langle \xi_j : j \geq 1 \rangle$ denote the sequence of bond percolations guaranteed by [Lemma 5.4](#). Note that [Lemma 5.4](#)(1) assures that each ξ_j is almost surely finitary. Assume first that (G, ρ) is reversible. Then the (degree-biased) mass-transport principle shows that if we choose $\hat{\rho}$ according to the measure $\pi_{K_{\xi_j}(\rho)}$ (recall [\(5.3\)](#)), then (G, ρ) and $(G, \hat{\rho})$ have the same law.

Let $\{Z_t^j\}$ denote the stationary random walk restricted to $K_{\xi_j}(\rho)$ (conditioned on $(G, \rho), \xi_j$). Therefore [Theorem 5.3](#) yields, for any $T, j \geq 1$:

$$\mathbb{E} \left[\text{dist}_\omega(Z_0^j, Z_T^j)^2 \mid (G, \rho), \xi_j \right] \lesssim T(\log |K_{\xi_j}(\rho)|)^2 \mathbb{E} \left[\text{dist}_\omega(Z_0^j, Z_1^j)^2 \mid (G, \rho), \xi_j \right]. \quad (5.9)$$

Define the events

$$\begin{aligned} \mathcal{E}_T &:= \left\{ |K_{\xi_j}(\rho)| \leq 2T^{2q'} \mathbb{E} |B_G(\rho, j)| \right\}, \\ \mathcal{B}_T &:= \left\{ B_G(\hat{\rho}, T) \subseteq K_{\xi_j}(\rho) \right\}, \end{aligned}$$

and let $\{X_t\}$ denote the random walk on G . Observe now that for $j \geq 2$:

$$\begin{aligned} &\mathbb{E} \left[\text{dist}_\omega(X_0, X_T)^2 \mathbb{1}_{\mathcal{B}_T} \mathbb{1}_{\mathcal{E}_T} \mid X_0 = \hat{\rho}, (G, \rho), \xi_j \right] \\ &= \mathbb{E} \left[\text{dist}_\omega(Z_0^j, Z_T^j)^2 \mathbb{1}_{\mathcal{B}_T} \mathbb{1}_{\mathcal{E}_T} \mid Z_0^j = \hat{\rho}, (G, \rho), \xi_j \right] \\ &\leq \mathbb{E} \left[\text{dist}_\omega(Z_0^j, Z_T^j)^2 \mathbb{1}_{\mathcal{E}_T} \mid Z_0^j = \hat{\rho}, (G, \rho), \xi_j \right] \\ &\lesssim T(q' \log T + \log(\mathbb{E} |B_G(\rho, j)|))^2 \mathbb{E}[\omega(\hat{\rho})^2 \mid (G, \rho), \xi_j]. \end{aligned}$$

Taking expectations and using [\(5.4\)](#) yields, for $T \geq 2$:

$$\mathbb{E} \left[\text{dist}_\omega(X_0, X_T)^2 \mathbb{1}_{\mathcal{B}_T} \mathbb{1}_{\mathcal{E}_T} \mid X_0 = \rho \right] \lesssim CqT(q' \log T + \log j)^2 \mathbb{E}[\omega(\rho)^2].$$

Using assumptions [\(5.4\)](#) and [Lemma 5.4\(2\)](#), we can choose $j \leq T^{O(q')}$ so that

$$\mathbb{P}[\mathcal{B}_T \text{ and } \mathcal{E}_T] \geq 1 - \frac{1}{T^{2q'}},$$

verifying [\(5.5\)](#).

Let us now assume that (G, ρ) is unimodular and verify [\(5.7\)](#). In this case, if we choose $\hat{\rho} \in K_{\xi_j}(\rho)$ uniformly at random, then the mass-transport principle shows that (G, ρ) and $(G, \hat{\rho})$ have the same law.

Return momentarily to [\(5.9\)](#) and observe that we can replace the stationary measure with the uniform measure on both the left and right, losing two factors of $d_{\max}(K_{\xi_j}(\rho))$:

$$\mathbb{E} \left[\text{dist}_\omega(Z_0^j, Z_T^j)^2 \mid Z_0^j = \hat{\rho}, (G, \rho), \xi_j \right] \lesssim (d_{\max}(K_{\xi_j}(\rho)))^2 (\log |K_{\xi_j}(\rho)|)^2 \cdot T \mathbb{E}[\omega(\hat{\rho})^2 \mid (G, \rho), \xi_j].$$

Define instead

$$\mathcal{E}_T := \left\{ |K_{\xi_j}(\rho)| \leq 2T^{2q'} \mathbb{E} |B_G(\rho, j)| \wedge d_{\max}(K_{\xi_j}(\rho)) \leq c' \log T \right\},$$

Using [\(5.6\)](#), along with assumptions [\(5.4\)](#) and [Lemma 5.4\(2\)](#), we can choose $c' \leq O(q')$ and $j \leq T^{O(q')}$ so that

$$\mathbb{P}[\mathcal{B}_T \text{ and } \mathcal{E}_T] \geq 1 - \frac{1}{T^{2q'}},$$

and the proof is finished as before.

Finally, to verify [\(5.8\)](#), note that when G is planar, for any conformal metric $\omega : V(G) \rightarrow \mathbb{R}_+$, it holds that $M_2(V(G), \text{dist}_\omega) \leq O(1)$ by the results of [\[DLP13\]](#) which establishes that planar graphs metrics have Markov type 2 with a uniform constant. Appealing to this fact instead of [Theorem 5.3](#) yields the desired improvement. \square

5.2 Conformal metrics from separators

We now turn to the proofs of [Lemma 1.15](#) and [Theorem 1.16](#). For a graph G , $x \in V(G)$, and $r' > r > 0$, define

$$q_G(x; r, r') := \max \left\{ \sum_{y \in S} \frac{1}{|B_G(y, r)|} : S \subseteq B_G(x, r'), |S| = \kappa_G(\rho; r, r') \right\}.$$

Note that under the assumptions of [Theorem 1.12](#), we have almost surely:

$$q_G(\rho; r, h(r)r) \leq r^{k-1-d+o(1)} \quad \forall r \geq 1.$$

Lemma 5.8. *Suppose (G, ρ) is a unimodular random graph. Then for every $r' > r > 0$, there is a subset $W_{r,r'} \subseteq V(G)$ such that $(G, \rho, W_{r,r'})$ is unimodular as a marked network, and the following hold:*

1. $\mathbb{P}[\rho \in W_{r,r'}] \leq \mathbb{E}[q_G(\rho; r, r')]$.
2. Every connected component of $G[V(G) \setminus W_{r,r'}]$ has diameter at most $2r'$ in dist_G .

Proof. Fix $r' > r \geq 1$. For each $x \in V(G)$, let $U_x \subseteq B_G(x, r') \setminus B_G(x, r)$ denote a separator achieving $\kappa_G(x; r, r')$. Let $\{\beta_x \in [0, 1] : x \in V(G)\}$ be a collection of i.i.d. uniform random variables. Define, for every $x \in V(G)$, the set

$$\hat{U}_x := U_x \setminus \bigcup_{y: \beta_y < \beta_x} B_G(y, r),$$

and

$$W_{r,r'} := \bigcup_{x \in V(G)} \hat{U}_x.$$

To see that almost surely every connected component of $G[V(G) \setminus W_{r,r'}]$ has diameter at most $2r'$ in dist_G , consider $x, y \in V(G)$ with $\text{dist}_G(x, y) > 2r'$. Let γ be a simple path from x to y in G . We will show that almost surely $\gamma \cap W_{r,r'} \neq \emptyset$. Let $z \in V(G)$ be the vertex in the finite set $\{z : \gamma \cap B_G(z, r) \neq \emptyset\}$ with β_z minimal. Since $\text{dist}_G(x, y) > 2r'$, it cannot be that both $x, y \in B_G(z, r')$, hence it must be that $\gamma \cap U_z \neq \emptyset$. By minimality of β_z , we have $\gamma \cap \hat{U}_z \neq \emptyset$ as well.

Now note that for any $x \in V(G)$,

$$\mathbb{E}[|\hat{U}_x| \mid (G, \rho)] \leq \sum_{y \in U_x} \frac{1}{|B_G(y, r)|}.$$

If we define a flow $F(G, x, y) := \mathbb{1}_{\hat{U}_x}(y)$, then combining the preceding inequality with the mass-transport principle yields

$$\mathbb{P}[\rho \in W_{r,r'}] \leq \mathbb{E} \left[\sum_{y \in U_\rho} \frac{1}{|B_G(y, r)|} \right]. \quad \square$$

Finally, we move on to the proof of [Theorem 1.16](#).

Proof of [Theorem 1.16](#). We begin by showing that assumption (A) implies assumption (B).

Lemma 5.9 ([\[BP11\]](#)). *Let H be a planar graph. Consider $x \in V(H)$ and $\tau \geq 1$. If $B_H(x, 4\tau)$ can be covered by λ balls of radius τ , then there is a subset $W \subseteq B_H(x, 6\tau) \setminus B_H(x, \tau)$ whose removal separates $B_H(x, \tau)$ and $V(H) \setminus B_H(x, 6\tau)$, and furthermore, $|W| \leq (\lambda + 1)(2\tau + 1)$.*

Define

$$v_G(x, r) := \min \{|B_G(y, r)| : y \in B_G(x, 6r)\} .$$

By a straightforward packing/covering argument, [Lemma 5.9](#) yields the following if G is almost surely planar: For any $r \geq 2$, it holds that

$$\kappa_G(\rho; r, 6r) \lesssim r \frac{|B_G(\rho, 6r)|}{v_G(x, r)} ,$$

and thus

$$q_G(\rho; r, 6r) \lesssim r \frac{|B_G(\rho, 6r)|}{v_G(x, r)^2} .$$

Combining this with [Lemma 5.8](#) and the next lemma shows that assumption (A) implies assumption (B).

Lemma 5.10. *Suppose (G, ρ) is a unimodular random graph and the following two conditions hold for some $C \geq 1$, $q > 1$, $\beta, \gamma > 0$, and $r \geq 2$:*

$$(\mathbb{E} |B_G(\rho, 6r)|^q)^{1/q} \leq Cr^d . \quad (5.10)$$

$$\mathbb{P} \left[|B_G(\rho, r)| < \frac{\varepsilon}{C(\log r)^\gamma} r^d \right] \leq \exp(-1/\varepsilon^\beta) \quad \forall \varepsilon > 0 . \quad (5.11)$$

Then:

$$\mathbb{E} \left[\frac{|B_G(\rho, 6r)|}{v_G(\rho, r)^2} \right] \leq C' r^{-d} (\log r)^{2/\beta+2\gamma} .$$

for some $C' = C'(C, \beta, \gamma, d, q)$.

Proof. Fix $\varepsilon > 0$. Let us define a flow

$$F(G, x, y) := \mathbb{1}_{\{|B_G(x, r)| < \varepsilon/(C(\log r)^\gamma) r^d\}} \mathbb{1}_{\{\text{dist}_G(x, y) \leq 6r\}} .$$

Then we can apply Hölder's inequality with q' such that $1/q + 1/q' = 1$ to conclude that

$$\begin{aligned} \mathbb{E} \left[\sum_{x \in V(G)} F(G, \rho, x) \right] &= \mathbb{E} \left[|B_G(\rho, 6r)| \mathbb{1}_{\{|B_G(\rho, r)| < \varepsilon/(C(\log r)^\gamma) r^d\}} \right] \\ &\leq (\mathbb{E} |B_G(\rho, 6r)|^q)^{1/q} \mathbb{P}[|B_G(\rho, r)| < \varepsilon/(C(\log r)^\gamma) r^d]^{1/q'} \\ &\leq Cr^d \exp(-1/(q' \varepsilon^\beta)) . \end{aligned}$$

By the Mass-Transport Principle, this bounds $\mathbb{E}[\sum_{x \in V(G)} F(G, x, \rho)] = \mathbb{P}[v_r(\rho) < \varepsilon/(C(\log r)^\gamma) r^d]$, hence for every $\varepsilon > 0$:

$$\mathbb{P} \left[v_r(\rho) < \frac{\varepsilon}{C(\log r)^\gamma (q' d \log r)^{1/\beta}} r^d \right] \leq Cr^{d(1-\varepsilon^{-\beta})} .$$

Thus we can bound:

$$\left(\mathbb{E} \left[\frac{1}{v_r(\rho)^{2q'}} \right] \right)^{1/q'} \leq C_1 (\log r)^{2/\beta+2\gamma} r^{-2d} , \quad (5.12)$$

where $C_1 = C_1(C, q, \beta, \gamma, d)$.

Now apply Hölder's inequality to [\(5.10\)](#) and [\(5.12\)](#) yields

$$\mathbb{E} \left[\frac{|B_G(\rho, 6r)|}{v_r(\rho)^2} \right] \leq Cr^d \cdot C_1 (\log r)^{2/\beta+2\gamma} r^{-2d} ,$$

which gives the desired bound. \square

Therefore it suffices to prove the desired conclusion under assumption (B). Let us now apply (B) to obtain, for every $r \geq 2$, a marked unimodular network (G, ρ, W_r) satisfying properties (B)(i) and (B)(ii).

Define normalized conformal metrics $\{\omega_j : j \in \mathbb{N}\}$ on (G, ρ) by

$$\omega_j := \frac{\mathbb{1}_{W_{2^j}}}{\sqrt{\mathbb{P}[\rho \in W_{2^j}]}}$$

and note that from (B)(i), we have

$$\omega_j \geq j^{-\alpha/2} 2^{j(d-1)/2} \mathbb{1}_{W_{2^j}} \quad \forall j \geq 1.$$

Combining this with (B)(ii) gives for every $x, y \in V(G)$:

$$\text{dist}_G(x, y) \geq 6 \cdot 2^j \implies \text{dist}_{\omega_j}(x, y) \geq j^{-\alpha/2} 2^{j(d-1)/2}. \quad (5.13)$$

Consider now the unnormalized metric

$$\omega^{(T)} := \sqrt{\sum_{j=1}^{\lceil \log_2 T \rceil} \omega_j^2},$$

and observe the bound: For $T \geq 2$,

$$\mathbb{E}[\omega^{(T)}(\rho)^2] \lesssim \log T. \quad (5.14)$$

Finally, we apply [Theorem 5.7\(5.7\)](#) to give: For every $T \geq 2$,

$$\begin{aligned} \mathbb{E} \left[\text{dist}_G(X_0, X_T)^{d-1} \right] &= \mathbb{E} \left[\min\{T^{d-1}, \text{dist}_G(X_0, X_T)^{d-1}\} \right] \\ &\stackrel{(5.13)}{\lesssim} (\log T)^{\alpha(d-1)} \mathbb{E} \left[\min\{T^{(d-1)/2}, \text{dist}_{\omega^{(T)}}(X_0, X_T)\}^2 \right] \\ &\lesssim T(\log T)^{\alpha(d-1)+5}, \end{aligned}$$

where our application of [Theorem 5.7\(5.7\)](#) in the final inequality uses (5.14). \square

5.3 Appendix: Growth bounds for UIPT/UIPQ

We now sketch a justification for why [Theorem 1.16](#) applies to UIPT/UIPQ with $d = 4$.

Upper tail (5.10). It is known that q -th moment bounds of the form (5.10) hold for the size of the “ r -hull” (which contains $B_G(\rho, r)$) for $d = 4$ and every $q < 3/2$. See [\[Kri08\]](#) for UIPQ and [\[Mén16\]](#) for UIPT.

Lower tail (5.11). The stretched exponential lower tail is more delicate. We sketch here the verification for UIPQ. One can use Schaeffer’s correspondence to establish that the law of $|B_G(\rho, r)|$ is the law of the number of nodes N_r of label at most r in a random labeled tree (see [\[CD06, §5\]](#) and [\[LGM10, §2\]](#)). The spine of the tree is governed by a Markov process $\{X_n \in \mathbb{N} : n \geq 0\}$, and conditioned on $\{X_n\}$, one has

$$N_r = \sum_{n: X_n \leq r} Y_{X_n}^{(n)},$$

where for each $i \geq 1$, $\{Y_i^{(n)} : n \geq 0\}$ are independent copies of a nonnegative random variable Y_i .

The law of Y_i is sampled as follows: Let T_i be a critical Galton-Watson tree where the number of offspring is geometric with parameter $1/2$. Inductively label the vertices of T_i by a map $\ell : V(T_i) \rightarrow \mathbb{Z}$ as follows: The root is labeled i . If a node has label ℓ , the labels of its offspring are independent and uniform in the set $\{\ell - 1, \ell, \ell + 1\}$. Let \hat{T}_i have the law of T_i conditioned on the event $\{\ell(x) > 0 \forall x \in V(T_i)\}$. Then Y_i has the law of $|\{x \in V(\hat{T}_i) : \ell(x) \leq r\}|$. See, e.g., [LGM10, §2.3].

The tree T_i satisfies, for all $N \geq 1$:

$$\mathbb{P}(|V(T_i)| = N) \gtrsim N^{-3/2}$$

Let $W(h)$ denote the number of nodes in T_i of height h . Then there is a constant $c_1 > 0$ such that

$$\mathbb{P}\left[\sum_{h=1}^N W(h) \geq c_1 N^2 \mid |V(T_i)| \geq N^2\right] \geq c_1.$$

See, for instance, [Drm09].

From this, it is elementary to establish that if $i \in [r/4, r/2]$, then Y_i satisfies

$$\mathbb{P}[Y_i \geq \varepsilon r^4] \gtrsim c_2 \frac{r^{-2}}{\sqrt{\varepsilon}} \quad \forall \varepsilon > 0, r \geq \varepsilon^{-4}. \quad (5.15)$$

Therefore if $Z = \#\{n \geq 1 : X_n \in [r/4, r/2]\}$, then we have

$$\mathbb{P}[N_r < \varepsilon r^4 \mid Z] \leq \left(1 - c_2 \frac{r^{-2}}{\sqrt{\varepsilon}}\right)^Z \leq \exp\left(-c_2 \frac{Z r^{-2}}{\sqrt{\varepsilon}}\right). \quad (5.16)$$

The Markov process $\{X_n\}$ is a birth and death chain whose transition probabilities converge to a limiting distribution as $X_n \rightarrow \infty$. When scaled appropriately, $\{X_n\}$ converges to a Bessel process [Mén10, Prop. 5], and satisfies, for some $c > 0$ and r sufficiently large:

$$\mathbb{P}[Z < c r^2] \lesssim e^{-c r^2}.$$

Combining this with (5.16) yields

$$\mathbb{P}[N_r < \varepsilon r^4] \leq \mathbb{P}[Z_n < c r^2] + \exp\left(-c_2 \frac{c}{\sqrt{\varepsilon}}\right) \lesssim e^{-c r^2} + \exp\left(-c_2 \frac{c}{\sqrt{\varepsilon}}\right).$$

This verifies (5.11) for $\beta = 1/2$ and $\gamma = 0$.

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