Discrete uniformizing metrics on distributional limits of sphere packings

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Abstract

Suppose that $\{G_n\}$ is a sequence of finite graphs such that each G_n is the tangency graph of a sphere packing in $\mathbb{R}^d$. Let ρ_n be a uniformly random vertex of G_n and suppose that (G, ρ) is the distributional limit of $\{(G_n, \rho_n)\}$ in the sense of Benjamini and Schramm. Then the conformal growth exponent of (G, ρ) is at most d. In other words, there exists a unimodular "unit volume" weighting of the graph metric on (G, ρ) such that the volume growth of balls in the weighted path metric is bounded by a polynomial of degree d. This assertion generalizes to limits of graphs that can be "quasi-packed" in an Ahlfors d-regular metric measure space.

It implies that, under moment conditions on the degree of the root ρ, the almost sure spectral dimension of G is at most d. This fact was known previously only for graphs packed in $\mathbb{R}^2$ (planar graphs), and the case $d > 2$ eluded approaches based on extremal length. In the process of bounding the spectral dimension, we establish that the spectral measure of (G, ρ) is dominated by a variant of the d-dimensional Weyl law.

Contents

1 Introduction

The theory of random planar graphs has been an active area of study in the last twenty years (see, e.g., [Ben10]), inspired partially by the connection to two-dimensional quantum gravity [ADJ97]. As noted by Benjamini and Curien [BC11], an analogous theory in higher dimensions has proved elusive, in part based on the difficulty of enumeration for higher-dimenisonal simplicial complexes (see [BZ11] and the references therein).

To address this discrepancy, the authors of [BC11] explored the extension of analytic and probabilistic methods based on potential theory. A graph G is said to be *sphere-packed in* $\mathbb{R}^d$ if G is the tangency graph of a collection of interior-disjoint spheres in $\mathbb{R}^d$. Benjamini and Curien proved that if a family of finite graphs can be sphere-packed in $\mathbb{R}^d$ with spheres of bounded aspect ratio (so that the ratio of the radii of tangent spheres is $O(1)$), then a distributional limit of such graphs is *d-parabolic.*

Roughly speaking, d-parabolicity means that the ℓ_d extremal length from a fixed vertex to ∞ is infinite, where the ℓ_d extremal length is a natural analog Cannon's vertex extremal length [Can94] (the case $d = 2$); see also [Duf62] and Section 1.3. It is well-known that the special case of 2-parabolicity carries strong probabilistic significance; for instance, for graphs with uniformly bounded degrees, 2-parabolicity is equivalent to recurrence of the random walk (see [Duf62, DS84]). For $d > 2$, the theory of ℓ_d extremal length seems somewhat less powerful, and is not known to yield such control on the random walk.

In this work, we study a related notion that one might refer to as the "extremal growth rate ." For graphs that can be sphere-packed in $\mathbb{R}^d$, we show that it is possible to construct metrics that uniformize their underlying geometry so that the *counting measure* has d-dimensional volume growth. Employing the results of [Lee17], one does obtain substantial probabilistic consequences, including d-dimensional lower bounds on the diagonal heat kernel (see Theorem 1.6 below). Moreover, our results hold in considerable generality; they require no assumption on the ratio of radii of adjacent balls in the packing, and they extend to graphs that can be "quasi-packed" in an Ahlfors regular metric measure space, as we now describe.

Quasi-packings and the spectral dimension. Consider a metric space (X, dist). A *τ-quasi-ball in X* is a Borel set $S \subseteq X$ that is sandwiched between two closed balls: $B(x,r) \subseteq S \subseteq B(x, \tau r)$ for some $x \in X, r > 0$. Let $\mathcal{B}_\tau$ denote the collection of τ-quasi-balls in X. Say that a graph G is *(τ, M)-quasi-packed in* (X, dist) if there is a mapping $\Phi : V(G) \to \mathcal{B}_\tau$ that satisfies:

1. **Quasi-tangency:**
$$\{u, v\} \in E(G) \implies \mathsf{dist}(\Phi(u), \Phi(v)) \leqslant \tau \min\{\mathsf{diam}(\Phi(u)), \mathsf{diam}(\Phi(v))\}\,. \tag{1.1}$$

2. **Quasi-multiplicity:** For every $x \in X$ and $r \geqslant 0$:
$$\#\left\{v \in V(G) : B(x,r) \cap \Phi(v) \neq \emptyset \text{ and } \mathsf{diam}(\Phi(v)) \geqslant \tau r\right\} \leqslant M\,. \tag{1.2}$$

Say that a graph G *quasi-packs in* (X, dist) if G is (τ, M)-quasi-packed in (X, dist) for some numbers $M, \tau \geqslant 1$. A family $\{G_n\}$ of graphs *uniformly quasi-packs in* (X, dist) if there are $M, \tau \geqslant 1$ such that each G_n is (τ, M)-quasi-packed in (X, dist). Of course, the collection $\{\Phi(v) : v \in V(G)\}$ is only a genuine packing for $M = 1$. We now state a representative theorem.

Theorem 1.1. *Consider a sequence of random rooted finite graphs $\{(G_n, \rho_n)\}$ with $\rho_n \in V(G_n)$ chosen uniformly at random. Suppose the family $\{G_n\}$ has uniformly bounded degrees and is uniformly quasi-packed in an Ahlfors d-regular metric measure space. If (G, ρ) is the distributional limit of this sequence, then almost surely $\overline{\dim}_{\mathsf{sp}}(G) \leqslant d$. Moreover, if $d = 2$, then G is almost surely recurrent.*

Here, "distributional limit" refers to convergence in the Benjamini-Schramm sense (i.e., in the weak local topology): $\{(G_n, \rho_n)\} \to (G, \rho)$ means that the laws of neighborhoods of ρ_n in G_n converge to the law of neigborhoods of ρ in G, where neighborhoods are considered up to rooted isomorphism. See Section 1.6 for precise definitions.

And we use $\overline{\dim}_{\mathsf{sp}}$ to denote the *upper spectral dimension:*

$$\overline{\dim}_{\mathsf{sp}}(G) := \limsup_{n\to\infty} \frac{-2\log p^G_{2n}(v,v)}{\log n},$$

where $p^G_t(v,v) = \mathbb{P}[X_t = v \mid X_0 = v]$ and $\{X_t\}$ is the standard random walk on G. (The value does not depend on the choice of $v \in V(G)$.)

Remark 1.2 (Coarse packings)**.** It is not hard to check that if two metric spaces X and Y are bi-Lipschitz equivalent, then G quasi-packs in X if and only if G quasi-packs in Y, making the notion a bi-Lipschitz invariant. More generally, it is a quasisymmetric invariant when X is uniformly perfect. See Section 2.2.

To relate quasi-packings to more standard notions, it is helpful to consider a simpler set of assumptions. Say that a graph G *coarsely packs in* X if there are numbers $M, \tau \geqslant 1$ and a map $\Phi : V(G) \to \mathcal{B}_1$ so that (1.1) is satisfied, as well as

$$\#\{v \in V(G) : x \in \Phi(v)\} \leqslant M \quad \forall x \in X. \tag{1.3}$$

Note that this is simply (1.2) for $r = 0$ and $\mathcal{B}_1$ is precisely the collection of closed balls in X. If (X, dist) is an Ahlfors d-regular length space (cf. Section 1.7) and G coarsely packs in X, then it quasi-packs in X. This is proved in Section 2.1.

This implies that if G is the tangency graph of interior-disjoint spheres in $\mathbb{R}^d$, then it is automatically (τ, M)-quasi-packed in $\mathbb{R}^d$ for some $M, \tau \geqslant 1$ depending only on d. For a non-Euclidean example, consider that the same is true of the tangency graphs of interior-disjoint balls in the Heisenberg groups equipped with their Carnot-Carathéodory metrics. See Section 2.1 for a detailed discussion. In general, the reader will suffer no great conceptual loss by thinking only of classical sphere packings in $\mathbb{R}^d$.

1.1 Discrete conformal metrics on sphere-packed graphs

Consider a locally finite, connected graph G. A *conformal metric* (or *conformal weight*) on G is a map $\omega : V(G) \to \mathbb{R}_+$. This endows G with a graph distance as follows: Give to every edge $\{u, v\} \in E(G)$ a length $\mathrm{len}_\omega(\{u,v\}) := \frac{1}{2}(\omega(u) + \omega(v))$. This prescribes to every path $\gamma = \{v_0, v_1, v_2, \ldots\}$ in G the induced length

$$\mathrm{len}_\omega(\gamma) := \sum_{k\geqslant 0} \mathrm{len}_\omega(\{v_k, v_{k+1}\}).$$

Now for $u, v \in V(G)$, one defines the path metric $\mathsf{dist}_\omega(u,v)$ as the infimum of the lengths of all u-v paths in G. Denote the closed ball

$$B_\omega(x, R) := \left\{y \in V(G) : \mathsf{dist}_\omega(x,y) \leqslant R\right\}.$$

We can now state a special case of our main technical theorem; the connection to distributional limits and random walks is discussed subsequently.

Theorem 1.3. *For every $d, M, \tau \geqslant 1$ and every Ahlfors d-regular metric measure space $\mathcal{X}$ there is a constant C such that the following holds. If $G = (V, E)$ is a finite graph that is (τ, M)-quasi-packed in $\mathcal{X}$, then there is a conformal metric $\omega : V \to \mathbb{R}_+$ that satisfies*

$$\frac{1}{|V|}\sum_{x\in V} \omega(x)^d = 1,$$

and such that

$$\max_{x \in V(G)} |B_\omega(x, R)| \leqslant CR^d (\log R)^2 \quad \forall R \geqslant 1 .$$

The method of proof is based partially on a celebrated lemma of Benjamini and Schramm [BS01]. They show that if $\{G_n\}$ is a sequence of finite planar triangulations with uniformly bounded degrees and $\{G_n\}$ converges to a distributional limit (G, ρ), then almost surely any circle packing of G has at most one accumulation point in the plane. An analogous result holds for graphs sphere-packed in $\mathbb{R}^d$ when $d > 2$ [BC11].

We argue that, in a quantative sense, as long as the accumulation points remain separated, one can construct a multi-scale reweighting of the spheres in the packing, endowing the graph with a metric that reflects its d-dimensional structure with respect to the underlying counting measure. This is carried out in Section 3.

1.2 Conformal growth exponents

If (G, ρ) is random rooted graph, then a *conformal metric on* (G, ρ) is a random triple (G', ω, ρ') with $\omega : V(G) \to \mathbb{R}_+$ such that (G, ρ) and (G', ρ') have the same law. We say that the conformal weight is *normalized* if $\mathbb{E}\left[\omega(\rho)^2\right] = 1$. One thinks of such a metric $\omega : V(G) \to \mathbb{R}_+$ as deforming the geometry of the underlying graph subject to a bound on the total "area." As shown in [Lee17], normalized conformal metrics with nice geometric properties form a powerful tool in understanding the spectral geometry of (G, ρ).

In the present work, we consider *unimodular* random graphs (see Section 1.6); such graphs arise naturally as distributional limits of finite random rooted graphs $\{(G_n, \rho_n)\}$ where $\rho_n \in V(G_n)$ is chosen uniformly at random. We will consider only unimodular conformal metrics ω on (G, ρ); in other words, the setting where (G, ω, ρ) is unimodular as a marked network in the sense of [AL07].

Conformal growth exponents. Consider a unimodular random graph (G, ρ). In [Lee17], we defined the *upper and lower conformal growth exponents of* (G, ρ), respectively, by

$$\overline{\dim}_{\mathsf{cg}}(G, \rho) := \inf_{\omega} \limsup_{R \to \infty} \frac{\log \|\#B_\omega(\rho, R)\|_{L^\infty}}{\log R}, \tag{1.4}$$

$$\underline{\dim}_{\mathsf{cg}}(G, \rho) := \inf_{\omega} \liminf_{R \to \infty} \frac{\log \|\#B_\omega(\rho, R)\|_{L^\infty}}{\log R}, \tag{1.5}$$

where the infimum is over all normalized unimodular conformal metrics on (G, ρ), and we use $\|X\|_{L^\infty}$ to denote the essential supremum of a random variable X, and $\#S$ to denote the cardinality of a set S.

When $\overline{\dim}_{\mathsf{cg}}(G, \rho) = \underline{\dim}_{\mathsf{cg}}(G, \rho)$, define the *conformal growth exponent* by

$$\dim_{\mathsf{cg}}(G, \rho) := \overline{\dim}_{\mathsf{cg}}(G, \rho) = \underline{\dim}_{\mathsf{cg}}(G, \rho) .$$

Note that the quantities $\overline{\dim}_{\mathsf{cg}}, \underline{\dim}_{\mathsf{cg}}, \dim_{\mathsf{cg}}$ are functions of the law of (G, ρ); they are not defined on (fixed) rooted graphs.

The conformal growth exponent bears a philosophical resemblance to Pansu's notion of *conformal dimension* [Pan89]. The relationship between sphere packings in $\mathbb{R}^2$ and conformal mappings is classical and well-understood. For an emerging more general theory, we refer to Pansu's recent work [Pan16] which explores in detail the relationship between sphere packings and the theory of large-scale conformal maps.

L^q **conformal growth rate.** Let us define a generalization: If (G, ω, ρ) is a unimodular random conformal graph, we denote

$$\|\omega\|_{L^q} := \left(\mathbb{E}\, \omega(\rho)^q\right)^{1/q} .$$

Say that ω is *L^q-normalized* if $\|\omega\|_{L^q} = 1$.

Define the analogous L^q quantities: $\overline{\dim}^q_{\mathsf{cg}}, \underline{\dim}^q_{\mathsf{cg}}, \dim^q_{\mathsf{cg}}$ where now the infima in (1.4) and (1.5) are over all L^q-normalized conformal metrics on (G, ρ). Observe that, by monotonicity of L^q norms, we have

$$q \leqslant q' \implies \dim^q_{\mathsf{cg}}(G, \rho) \leqslant \dim^{q'}_{\mathsf{cg}}(G, \rho)\,.$$

The next theorem constitutes the main new technical theorem presented here. We use $\Rightarrow$ to denote convergence in the distributional sense; see Section 1.6.

Theorem 1.4. *For any $d \geqslant 2$, the following holds. If (G, ρ) is the distributional limit of finite graphs that are uniformly quasi-packed in an Ahlfors d-regular metric measure space, then there is an L^d-normalized unimodular conformal metric $\omega : V(G) \to \mathbb{R}_+$ such that almost surely, for all $R \geqslant 1$,*

$$\left|B_\omega(\rho, R)\right| \leqslant O(R^d (\log R)^2)\,. \tag{1.6}$$

In particular, $\overline{\dim}_{\mathsf{cg}}(G, \rho) \leqslant d$.

The last assertion follows from $\overline{\dim}_{\mathsf{cg}}(G, \rho) = \overline{\dim}^2_{\mathsf{cg}}(G, \rho) \leqslant \overline{\dim}^d_{\mathsf{cg}}(G, \rho)$. If $\mathcal{X}$ is Ahlfors d-regular with $d < 2$, the conclusion $\overline{\dim}_{\mathsf{cg}}(G, \rho) \leqslant 2$ still holds; see Section 3. We remark that some $(\log R)^{O(1)}$ factor is necessary even for the case of planar graphs; see [Lee17, §2].

A primary motivation for Theorem 1.4 is that such metrics can be used to obtain estimates on the heat kernel and spectral measure of G. For a locally finite, connected graph G, denote the discrete-time heat kernel

$$p^G_T(x, y) := \mathbb{P}[X_T = y \mid X_0 = x], \qquad x, y \in V(G)\,,$$

where $\{X_n\}$ is the standard random walk on G and $T \in \mathbb{N}$. We recall the *spectral dimension of G:*

$$\dim_{\mathsf{sp}}(G) := \lim_{n\to\infty} \frac{-2 \log p^G_{2n}(x, x)}{\log n}\,,$$

whenever the limit exists. If the limit does exist, then it is the same for all $x \in V(G)$.

Say that a real-valued random variable X has *negligible tails* if its tails decay faster than any inverse polynomial:

$$\lim_{n\to\infty} \frac{\log n}{|\log \mathbb{P}[|X| > n]|} = 0\,, \tag{1.7}$$

where we take $\log(0) = -\infty$ in the preceding definition (in the case that X is essentially bounded). The next theorem is from [Lee17]; it asserts that if $\overline{\dim}_{\mathsf{cg}}(G, \rho) \leqslant d$, then almost surely G admits d-dimensional lower bounds on the diagonal heat kernel:

$$p^G_{2n}(\rho, \rho) \geqslant n^{-d/2-o(1)} \quad \text{as} \quad n \to \infty\,.$$

Theorem 1.5. *Suppose that (G, ρ) is a unimodular random graph such that $\deg_G(\rho)$ has negligible tails. Then almost surely:*

$$\overline{\dim}_{\mathsf{sp}}(G) \leqslant \overline{\dim}_{\mathsf{cg}}(G, \rho)\,.$$

In particular, if there is a number d such that almost surely $\dim_{\mathsf{sp}}(G) = d$, then $d \leqslant \overline{\dim}_{\mathsf{cg}}(G, \rho)$.

In certain situations, one can give stronger estimates. Indeed, when the conformal growth rate has only a polylogarithmic correction as in (1.6), one obtains stronger results (see [Lee17, §4.2]).

Theorem 1.6. *Suppose (G,ρ) is the distributional limit of finite graphs that are uniformly quasi-packed in an Ahlfors d-regular metric measure space $\mathcal{X}$, and that $\deg_G(\rho)$ has exponential tails in the sense that*

$$\mathbb{P}[\deg_G(\rho) > k] \leqslant e^{-ck}$$

for some $c > 0$. Then there is a constant $C \geqslant 1$ such that for n sufficiently large,

$$\mathbb{P}\left[p_{2n}^G(\rho,\rho) \geqslant \frac{n^{-d_*/2}}{(\log n)^C}\right] \geqslant 1 - \frac{1}{\log n},$$

where $d_* = \max(d,2)$.

1.3 Gauged conformal growth and d-parabolicity

Consider a locally-finite connected graph $G = (V,E)$. Let Γ denote a collection of simple paths in G. The *ℓ_d-vertex extremal length of* Γ is defined by

$$\mathsf{VEL}_d(\Gamma) := \sup_{\omega} \inf_{\gamma \in \Gamma} \frac{\mathrm{len}_\omega(\gamma)}{\|\omega\|_{\ell_d(V)}},$$

where the infimum is over all conformal metrics on G, and $\|\omega\|_{\ell_d(V)} := \left(\sum_{v\in V} \omega(v)^d\right)^{1/d}$.

Fix a vertex $v_0 \in V$ and let $\Gamma_G(v_0)$ denote the set of infinite simple paths in G emenating from v_0. One says that G is *d-parabolic* if $\mathsf{VEL}_d(\Gamma_G(v_0)) = \infty$ (see [HS95, BS13]). One can check that this definition does not depend on the choice of $v_0 \in V$.

There are unimodular random graphs (G,ρ) where G is almost surely d-parabolic, but $\underline{\dim}_{\mathsf{cg}}^d(G,\rho) \geqslant \underline{\dim}_{\mathsf{cg}}(G,\rho) = \infty$, and other examples where $\dim_{\mathsf{cg}}(G,\rho) = d \geqslant 2$ but G is almost surely not d-parabolic; see Section 3.3.

Gauged growth. On the other hand, there is a common strengthening of the conditions. Say that (G,ρ) has *(C,R,d)-growth* if there is an L^d-normalized conformal metric $\omega : V(G) \to \mathbb{R}_+$ such that

$$\|\#B_\omega(\rho,R)\|_{L^\infty} \leqslant CR^d. \tag{1.8}$$

Say that (G,ρ) has *gauged d-dimensional conformal growth* if there is a constant $C \geqslant 1$ such that (G,ρ) has (C,R,d)-growth for all $R \geqslant 0$. A sequence $\{(G_n,\rho_n)\}$ has *uniform gauged d-dimensional conformal growth* if there is a constant $C \geqslant 1$ such that (G_n,ρ_n) has (C,R,d)-growth for all $R \geqslant 0$ and $n \geqslant 1$.

It is straightforward to see that if (G,ρ) has gauged d-dimensional growth, then $\overline{\dim}_{\mathsf{cg}}^d(G,\rho) \leqslant d$: For each $k \geqslant 1$, let ω_k denote an L^d-normalized conformal metric on (G,ρ) satisfying (1.8) and define

$$\hat{\omega} := \left(\frac{6}{\pi^2}\sum_{k\geqslant 1}\frac{\omega_k^d}{k^2}\right)^{1/d}.$$

(By unimodularity of the triple $(G,\hat{\omega},\rho)$, it holds that almost surely $\sup_{x\in V(G)} \hat{\omega}(x) < \infty$; see Section 1.6).

Establishing d-parabolicity is somewhat more involved; the $d = 2$ case of the following theorem is [Lee17, Thm. 2.1]. The general case is proved in Section 3.3.1.

Theorem 1.7. *For every $d \geqslant 1$, the following holds. If (G,ρ) is a unimodular random graph such that $\deg_G(\rho)$ is essentially bounded and (G,ρ) has gauged d-dimensional conformal growth, then G is almost surely d-parabolic.*

In order to establish Theorem 1.4, we prove the following stronger statement in Section 3.

Theorem 1.8. *For any $d \geqslant 1$, the following holds. If (G, ρ) is the distributional limit of finite graphs that are uniformly quasi-packed in an Ahlfors d-regular metric measure space, then (G, ρ) has gauged* $\max(d, 2)$*-dimensional conformal growth.*

Note that for the special case of planar graphs, the conjunction of Theorem 1.7 and Theorem 1.8 recovers the Benjamini-Schramm recurrence theorem [BS01] (that every distributional limit of finite planar graphs with uniformly bounded degrees is almost surely 2-parabolic).

1.4 The spectral measure of d-dimensional graphs

In order to obtain estimates like Theorem 1.5 and Theorem 1.6, it is clear that one needs to control the moments of the spectral measure at the root. Indeed, if (G, ρ) is a random rooted graph, then one can define the spectral measure $\mu := \mathbb{E}[\mu_G^\rho]$, there μ_G^v is the unique probability measure on $\mathbb{R}$ such that for all integers $T \geqslant 1$:

$$\deg_G(v) \int \theta^T \, d\mu_G^v(\theta) = \langle \mathbb{1}_v, P_G^T \mathbb{1}_v \rangle_{\ell^2(G)} .$$

Here, P_G is the random walk operator on G and $\ell^2(G)$ is the Hilbert space of functions $f : V(G) \to \mathbb{R}$ with $\langle f, g \rangle_{\ell^2(G)} := \sum_{x \in V(G)} \deg_G(x) f(x) g(x)$. (See, e.g., [Lee17, §4.4.1] and [BSV17, §1.4–1.5].) Note that μ is almost surely supported on $[-1, 1]$.

In this formulation, one has: For all integers $T \geqslant 1$,

$$\mathbb{E}\left[p_{2T}^G(\rho, \rho)\right] = \int \theta^{2T} d\mu(\theta) ,$$

hence an elementary calculation shows that for every $d \geqslant 1$ and $T \geqslant 1$:

$$\frac{1}{4} \mu\big([1 - \tfrac{1}{2T}, 1]\big) \leqslant \mathbb{E}\left[p_{2T}^G(\rho, \rho)\right] \leqslant T^{-d} + \mu\left(\left[1 - \tfrac{d \log T}{2T}, 1\right]\right) .$$

Almost sure (quenched) lower bounds on p_{2T}^G as in Theorem 1.5 are substantially more difficult to establish than lower bounds on $\mathbb{E}[p_{2T}^G(\rho, \rho)]$, but annealed estimates are already interesting, and one can draw a parallel to more classical settings.

The Weyl bound in $\mathbb{R}^d$. Consider a bounded domain $\Omega \subseteq \mathbb{R}^d$, and let $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$ be the corresponding Neumann eigenvalues. Let $N_\Omega(\lambda) := \#\{k : \lambda_k \leqslant \lambda\}$ denote the eigenvalue counting function. In 1912, addressing a conjecture of Lorentz, Weyl determined [Wey12] the first-order asymptotics of $N_\Omega(\lambda)$ as $\lambda \to \infty$:

$$N_\Omega(\lambda) \sim c_d \operatorname{vol}(\Omega) \lambda^{d/2} ,$$

where c_d is some constant depending only on the dimension.

In addressing a question of S. T. Yau on the spectrum of the Laplacian on orientable surfaces, Korevaar [Kor93] showed that if Ω is a subdomain of a complete d-dimensional Riemannian manifold (M, g_0) with nonnegative Ricci curvature, and $(M, \varphi g_0)$ is a finite-volume conformal metric, then

$$N_\Omega(\lambda) \geqslant C_d \operatorname{vol}(\Omega, \varphi g_0) \lambda^{d/2} , \tag{1.9}$$

where C_d is a constant depending only on the dimension d.

Analogous results can be obtained for distributional limits of finite graphs that are sphere-packed $\mathbb{R}^d$. Let ν denote the law of a random rooted graph (G, ρ) and define $\bar{d}_\nu : [0, 1] \to \mathbb{R}_+$ by

$$\bar{d}_\nu(\varepsilon) := \sup\left\{\mathbb{E}[\deg_G(\rho) \mid \mathcal{E}] : \mathbb{P}(\mathcal{E}) \geqslant \varepsilon\right\} ,$$

where the supremum is over all measurable sets $\mathcal{E}$ with $\mathbb{P}(\mathcal{E}) \geqslant \varepsilon$.

Theorem 1.9. *Consider $d \geqslant 1$ and an Ahlfors d-regular metric measure space $\mathcal{X}$. Suppose (G, ρ) is a distributional limit of finite graphs that are uniformly quasi-packed in $\mathcal{X}$. Then there is a number $c > 0$ such that the following holds. Let ν denote the law of (G, ρ), and let μ denote the corresponding spectral measure. For all $\varepsilon > 0$:*

$$\mu([1-\varepsilon, 1]) \geqslant c \frac{(\log(1/\varepsilon))^{-2}}{\bar{d}_\nu(\varepsilon)} \varepsilon^{d/2} . \tag{1.10}$$

The asymptotic dependence on ε is tight up to the $(\log(1/\varepsilon))^{-2}$ factor; see Remark 1.11.

The Laplacian spectrum of finite tangency graphs. Theorem 1.9 follows readily from an analogous result for finite graphs. Let $G = (V, E)$ denote a finite connected graph with $n = |V|$. Let $\{1 - \lambda_k(G) : k = 0, 1, \ldots, n-1\}$ be the eigenvalues of the random walk operator on G, where

$$0 = \lambda_0(G) \leqslant \lambda_1(G) \leqslant \cdots \leqslant \lambda_{n-1}(G) .$$

Define the corresponding counting function:

$$N_G(\lambda) := \#\{k > 0 : \lambda_k(G) \leqslant \lambda\} .$$

Define also

$$\Delta_G(k) := \max_{S \subseteq V : |S| \leqslant k} \sum_{x \in S} \deg_G(x) ,$$

where $\deg_G(x)$ denotes the degree of a vertex $x \in V$. Note, in particular, that $\Delta_G(1)$ is the maximum degree in G.

Denote $\bar{d}_G(\varepsilon) := \frac{\Delta_G(\varepsilon n)}{\varepsilon n}$. In [KLPT11], addressing a conjecture of Spielman and Teng [ST07], it is shown that there is a constant $c > 0$ such that if G is a planar graph, then for all $\lambda \in [0, 1]$,

$$N_G(\lambda) \geqslant \frac{c}{\Delta_G(1)} \lambda n . \tag{1.11}$$

In [Lee17], the author improves this bound to

$$N_G(\lambda) \geqslant \frac{c}{\bar{d}_G(\lambda)} \lambda n , \tag{1.12}$$

where $c > 0$ is some other constant.

While the utility of this improvement is not immediately apparent in the finite setting, one should observe that (1.11) yields no information for a distributional limit (G, ρ) in which there is no uniform bound on $\deg_G(\rho)$, whereas (1.12) yields (1.10) in the case $d = 2$ (and without the $\log(1/\varepsilon)^{-2}$ correction factor). Moreover, the correct quantitative dependence is essential to a spectral argument proving that the uniform infinite planar triangulation is almost surely recurrent [Lee17]; this fact was first established by Gurel-Gurevich and Nachmias [GN13] using effective resistance estimates. In Section 3.4, we use Theorem 1.3 to establish an analogous lower bound to (1.9) for graphs sphere-packed in $\mathbb{R}^d$ (and their generalizations).

Theorem 1.10 (Weyl bound for quasi-packed finite graphs)**.** *For every $d, \tau, M \geqslant 1$ and every Ahlfors d-regular metric measure space $\mathcal{X}$, there is a number $c > 0$ such that the following holds. If G is an n-vertex graph that is (τ, M)-quasi-packed in $\mathcal{X}$, then for all $\lambda \in [0, 1]$,*

$$N_G(\lambda) \geqslant \frac{c}{\bar{d}_G(\lambda)} \left(\log \frac{e}{\lambda}\right)^{-2} n \lambda^{d/2} .$$

Remark 1.11. Up to the factor of $(\log(e/\lambda))^2$, this bound is tight for a d-dimensional box $\{1, 2, \ldots, n^{1/d}\}^d$ considered as a subgraph of the integer lattice $\mathbb{Z}^d$. Whether the $(\log(1/\lambda))^2$ factor can be removed from the bound is an interesting open question.

1.5 Preliminaries

We use the notations $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$.

All graphs appearing in this paper are undirected and locally finite and without loops or multiple edges. If G is such a graph, we use $V(G)$ and $E(G)$ to denote the vertex and edge set of G, respectively. If $S \subseteq V(G)$, we use $G[S]$ for the induced subgraph on S. For $A, B \subseteq V(G)$, we write $E_G(A, B)$ for the set of edges with one endpoint in A and the other in B. We write dist_G for the unweighted path metric on $V(G)$, and $B_G(x, r) = \{y \in V(G) : \mathsf{dist}_G(x, y) \leqslant r\}$ to denote the closed r-ball around $x \in V(G)$. Also let $\deg_G(x)$ denote the degree of a vertex $x \in V(G)$, and $d_{\max}(G) = \sup_{x \in V(G)} \deg_G(x)$.

Write $G_1 \cong G_2$ to denote that G_1 and G_2 are isomorphic as graphs. If (G_1, ρ_1) and (G_2, ρ_2) are rooted graphs, we write $(G_1, \rho_1) \cong_\rho (G_2, \rho_2)$ to denote the existence of a rooted isomorphism.

1.6 Unimodular random graphs and distributional limits

We begin with a discussion of unimodular random graphs and distributional limits. One may consult the extensive reference of Aldous and Lyons [AL07]. The paper [BS01] offers a concise introduction to distributional limits of finite planar graphs. We briefly review some relevant points.

Let $\mathcal{G}$ denote the set of isomorphism classes of connected, locally finite graphs; let $\mathcal{G}_\bullet$ denote the set of *rooted* isomorphism classes of *rooted*, connected, locally finite graphs. Define a metric on $\mathcal{G}_\bullet$ as follows: $\mathbb{d}_{\mathrm{loc}}\left((G_1, \rho_1), (G_2, \rho_2)\right) = 1/(1 + \alpha)$, where

$$\alpha = \sup\left\{r > 0 : B_{G_1}(\rho_1, r) \cong_\rho B_{G_2}(\rho_2, r)\right\} .$$

$(\mathcal{G}_\bullet, \mathbb{d}_{\mathrm{loc}})$ is a separable, complete metric space. For probability measures $\{\mu_n\}, \mu$ on $\mathcal{G}_\bullet$, write $\{\mu_n\} \Rightarrow \mu$ when μ_n converges weakly to μ with respect to $\mathbb{d}_{\mathrm{loc}}$. If $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$, we say that (G, ρ) is the *distributional limit* of the sequence $\{(G_n, \rho_n)\}$.

The Mass-Transport Principle. Let $\mathcal{G}_{\bullet\bullet}$ denote the set of doubly-rooted isomorphism classes of doubly-rooted, connected, locally finite graphs. A probability measure μ on $\mathcal{G}_\bullet$ is *unimodular* if it obeys the following *Mass-Transport Principle:* For all Borel-measurable $F : \mathcal{G}_{\bullet\bullet} \to [0, \infty]$,

$$\int \sum_{x \in V(G)} F(G, \rho, x)\, d\mu((G, \rho)) = \int \sum_{x \in V(G)} F(G, x, \rho)\, d\mu((G, \rho)) . \tag{1.13}$$

If (G, ρ) is a random rooted graph with law μ, and μ is unimodular, we say that (G, ρ) is a *unimodular random graph*.

Distributional limits of finite graphs. As observed by Benjamini and Schramm [BS01], unimodular random graphs can be obtained from limits of finite graphs. Consider a sequence $\{G_n\} \subseteq \mathcal{G}$ of finite graphs, and let ρ_n denote a uniformly random element of $V(G_n)$. Then $\{(G_n, \rho_n)\}$ is a sequence of $\mathcal{G}_\bullet$-valued random variables, and one has the following.

Lemma 1.12. *If $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$, then (G, ρ) is unimodular.*

When $\{G_n\}$ is a sequence of finite graphs, we write $\{G_n\} \Rightarrow (G, \rho)$ for $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$ where $\rho_n \in V(G_n)$ is chosen uniformly at random.

Unimodular random conformal graphs. A *conformal graph* is a pair (G, ω), where G is a connected, locally finite graph and $\omega : V(G) \to \mathbb{R}_+$. Let $\mathcal{G}^*$ and $\mathcal{G}^*_\bullet$ denote the collections of isomorphism classes of conformal graphs and conformal rooted graphs, respectively. As in Section 1.6, one can define a metric on $\mathcal{G}_\bullet^*$ as follows: $\mathbb{d}_{\mathrm{loc}}^*\big((G_1,\omega_1,\rho_1),(G_2,\omega_2,\rho_2)\big) = 1/(\alpha+1)$, where

$$\alpha = \sup\left\{r > 0 : B_{G_1}(\rho_1, r) \cong_\rho B_{G_2}(\rho_2, r) \text{ and } \mathbb{d}\Big(\omega_1|_{B_{G_1}(\rho_1,r)}, \omega_2|_{B_{G_2}(\rho_2,r)}\Big) \leqslant \frac{1}{r}\right\},$$

where for two weights $\omega_1 : V(H_1) \to \mathbb{R}_+$ and $\omega_2 : V(H_2) \to \mathbb{R}_+$ on rooted-isomorphic graphs (H_1, ρ_1) and (H_2, ρ_2), we write

$$\mathbb{d}(\omega_1, \omega_2) := \inf_{\psi : V(H_1) \to V(H_2)} \big\|\omega_2 \circ \psi - \omega_1\big\|_{\ell^\infty}, \tag{1.14}$$

and the infimum is over all graph isomorphisms from H_1 to H_2 satisfying $\psi(\rho_1) = \rho_2$.

If $\{\mu_n\}$ and μ are probability measures on $\mathcal{G}_\bullet^*$, we abuse notation and write $\{\mu_n\} \Rightarrow \mu$ to denote weak convergence with respect to $\mathbb{d}_{\mathrm{loc}}^*$. One defines unimodularity of a random rooted conformal graph (G, ω, ρ) analogously to (1.13): It should now hold that for all Borel-measurable $F : \mathcal{G}_{\bullet\bullet}^* \to [0,\infty]$,

$$\int \sum_{x \in V(G)} F(G, \omega, \rho, x)\, d\mu((G, \omega, \rho)) = \int \sum_{x \in V(G)} F(G, \omega, x, \rho)\, d\mu((G, \omega, \rho)).$$

Indeed, such decorated graphs are a special case of the marked networks considered in [AL07], and again it holds that every distributional limit of finite unimodular random conformal graphs is a unimodular random conformal graph.

Suppose that (G, ρ) is a unimodular random graph. A *conformal weight on* (G, ρ) is a unimodular random conformal graph (G', ω, ρ') such that (G, ρ) and (G', ρ') have the same law. We will speak simply of a "conformal metric ω on (G, ρ)." Only such unimodular metrics are considered in this work.

1.6.1 Conformal growth rates under distributional limits

In order to establish our main result, we need to pass from a sequence of conformal metrics on finite graphs to a conformal metric on the distributional limit.

Theorem 1.13. *Consider $d, q \geqslant 1$. Suppose $\{(G_n, \rho_n)\}$ is a sequence of unimodular random graphs and $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$. If there is a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(R) \leqslant R^{o(1)}$ as $R \to \infty$, and a sequence of L^q-normalized unimodular random conformal graphs $\{(G_n, \omega_n, \rho_n)\}$ satisfying*

$$\|B_{\omega_n}(\rho_n, R)\|_{L^\infty} \leqslant R^d h(R), \tag{1.15}$$

then $\overline{\dim}_{\mathrm{cg}}^q(G, \rho) \leqslant d$. If the unimodular random graphs $\{(G_n, \rho_n)\}$ have uniform gauged d-dimensional growth, then (G, ρ) has gauged d-dimensional growth.

The preceding theorem follows immediately from the next lemma.

Lemma 1.14. *Consider a sequence $\{(G_n, \omega_n, \rho_n)\}$ of unimodular random conformal graphs satisfying the following conditions:*

1. *$\{(G_n, \rho_n)\}$ has a distributional limit.*
2. *$\limsup_{n\to\infty} \mathbb{E}[\omega_n(\rho_n)] < \infty$.*

Then $\{(G_n, \omega_n, \rho_n)\}$ has a subsequential weak limit in the metric $\mathbb{d}_{\mathrm{loc}}^*$.

Proof. By passing to a subsequence and scaling, we may assume that

$$\mathbb{E}[\omega_n(\rho_n)] \leqslant 1 \qquad \forall n \geqslant 1 . \tag{1.16}$$

Let μ_n denote the law of (G_n, ω_n, ρ_n). We will prove that the sequence $\{\mu_n\}$ is tight. Since $(\mathcal{G}^*_\bullet, \mathbb{d}^*_{\mathrm{loc}})$ is a complete, separable metric space, Prokhorov's theorem then implies that the sequence $\{\mu_n\}$ has a weak subsequential limit.

To establish tightness, it suffices to exhibit a sequence $\{K_j \subseteq \mathcal{G}^*_\bullet : j \geqslant 1\}$ such that each K_j is compact in the topology induced by $\mathbb{d}^*_{\mathrm{loc}}$ and

$$\lim_{j\to\infty} \lim_{n\to\infty} \mu_n(K_j) = 1 . \tag{1.17}$$

Let $\hat{\mu}_n$ denote the law of (G_n, ρ_n). Since (G_n, ρ_n) has a distributional limit and $(\mathcal{G}_\bullet, \mathbb{d}_{\mathrm{loc}})$ is complete, Prokhorov's theorem yields a sequence of compact sets $\{\hat{K}_j \subseteq \mathcal{G}_\bullet : j \geqslant 1\}$ such that

$$\lim_{j\to\infty} \lim_{n\to\infty} \hat{\mu}_n(\hat{K}_j) = 1 . \tag{1.18}$$

Denote the numbers:

$$b_{j,k} := \sup \left\{ |B_G(\rho, k)| : (G, \rho) \in \hat{K}_j \right\} .$$

Since each $\hat{K}_j$ is compact, we have $b_{j,k} < \infty$ for all $j, k \geqslant 1$.

Define the compact sets

$$K_j := \left\{ (G, \omega, \rho) : (G, \rho) \in \hat{K}_j \text{ and } \max_{x \in B_G(\rho,k)} \omega(x) \leqslant jk^2 b_{j,2k} \ \forall k \geqslant 1 \right\} .$$

We are left to verify that (1.17) holds.

To that end, we apply the Mass-Transport Principle to (G_n, ω_n, ρ_n) with the flow

$$F_{j,k}(G, \omega, x, y) := \omega(y) \mathbb{1}_{\{\mathsf{dist}_G(x,y) \leqslant k\}} \mathbb{1}_{\{(G,x) \in \hat{K}_j\}} ,$$

yielding

$$\begin{aligned} jk^2 b_{j,2k} \, \mathbb{P}\left[(G_n, \rho_n) \in \hat{K}_j \text{ and } \max_{x \in B_{G_n}(\rho_n,k)} \omega_n(x) > jk^2 b_{j,2k} \right] &\leqslant \mathbb{E}\left[\mathbb{1}_{\{(G_n,\rho_n)\in\hat{K}_j\}} \sum_{y \in B_{G_n}(\rho_n,k)} \omega_n(y) \right] \\ &= \mathbb{E}\left[\sum_{y \in V(G_n)} F_{j,k}(G_n, \omega_n, \rho_n, y) \right] \\ &= \mathbb{E}\left[\sum_{x \in V(G_n)} F_{j,k}(G_n, \omega_n, x, \rho_n) \right] \\ &= \mathbb{E}\left[\omega_n(\rho_n) \sum_{x \in B_{G_n}(\rho_n,k)} \mathbb{1}_{\{(G,x)\in\hat{K}_j\}} \right] \\ &\leqslant \mathbb{E}[\omega_n(\rho_n)] b_{j,2k} . \end{aligned}$$

Using (1.16), this gives

$$\mathbb{P}\left[(G_n, \omega_n, \rho_n) \in K_j \right] \geqslant \mathbb{P}\left[(G_n, \rho_n) \in \hat{K}_j \right] - \frac{1}{j} \sum_{k \geqslant 1} \frac{1}{k^2} .$$

In conjunction with (1.18), this yields (1.17). ◻

1.7 Ahlfors regularity and systems of dyadic cubes

Consider a complete, separable metric space (X, d). For $x \in X$ and two subsets $S, T \subseteq X$, we use the notations $d(S, T) := \inf_{x\in S, y\in T} d(x, y)$ and $d(x, S) = d(\{x\}, S)$. Define $\mathsf{diam}(S, d) := \sup_{x,y\in S} d(x, y)$ and for $R \geqslant 0$, define the closed balls

$$B_{(X,d)}(x, R) := \{y \in X : d(x, y) \leqslant R\} .$$

We omit the subscript (X, d) if the underlying metric space is clear from context. We say that (X, d) is *doubling* if there is a constant $\mathcal{D}$ such that every closed ball in X can be covered by $\mathcal{D}$ closed balls of half the radius, and we let $\mathcal{D}_{(X,d)}$ denote the infimal $\mathcal{D}$ for which this holds. (X, d) is a *length space* if, for every $x, y \in X$, the distance $d(x, y)$ is equal to the infimum of the length of continuous curves connecting x to y in X.

If μ is a measure on the Borel σ-algebra of X, we refer to (X, d, μ) as a *metric measure space.* Such a space is said to be *Ahlfors β-regular* if there are constants $c_1, c_2 > 0$ such that

$$c_1 R^\beta \leqslant \mu (B(x, R)) \leqslant c_2 R^\beta \qquad \forall x \in X, R \in [0, \mathsf{diam}(X)] .$$

It will occasionally be convenient to record the constants c_1, c_2, in which case we say that (X, d, μ) is *(c_1, c_2, β)-regular.* We recall the following elementary fact:

Fact 1.15. *If (X, d, μ) is Ahlfors β-regular for some $\beta > 0$, then (X, d) is doubling, and $\mathcal{D}_{(X,d)} \leqslant C$ for some constant $C = C(c_1, c_2, \beta)$ depending only on c_1, c_2, β.*

We will employ an appropriate system of hierarchical dyadic partitions of a doubling metric space (X, d) along the lines of [Chr90] and [Dav91]. Deterministic and stochastic constructions of this type are a basic tool in harmonic analysis and metric geometry (see, e.g., [LN05] and [HK12]).

For our purposes, it will be easiest to use a construction from [HK12] which we summarize here. Consider a metric space (X, d). A bi-infinite sequence $\boldsymbol{P} = \{\boldsymbol{P}_n : n \in \mathbb{Z}\}$ of partitions of X is said to be a *hierarchical system* if $\boldsymbol{P}_n$ is a refinement of $\boldsymbol{P}_{n+1}$ for all $n \in \mathbb{Z}$. We say that $\boldsymbol{P}$ is Δ*-adic* if

$$S \in \boldsymbol{P}_n \implies \mathsf{diam}_{(X,d)}(S) \leqslant \Delta^n \qquad \forall n \in \mathbb{Z} .$$

Theorem 1.16 ([HK12])**.** *Suppose (X, d) is a doubling metric space. Then there are numbers $Q, \ell, \Delta \geqslant 2$ that depend only on $\mathcal{D}(X, d)$ such that the following holds. There is a collection $\{\boldsymbol{P}^{(1)}, \ldots, \boldsymbol{P}^{(Q)}\}$ of Δ-adic hierarchical systems such that for every subset $S \subseteq X$ with $\mathsf{diam}_{(X,d)}(S) \leqslant \Delta^m$, there is a set*

$$C \in \bigcup_{i=1}^{Q} \boldsymbol{P}^{(i)}_{m+\ell}$$

such that $S \subseteq C$.

2 Quasi-packings and quasisymmetric invariance

We first demonstrate that the quasi-multiplicity condition (1.2) can be replaced by a simpler assumption whenever (X, dist) is an Ahlfors-regular length space and one uses only strict balls instead of quasi-balls.

2.1 Round balls, length spaces, and coarse packings

Let $\mathcal{B}$ denote the set of closed balls in (X, dist). Say that a graph G is *(τ, M)-coarsely packed in* (X, dist) if there is a map $\Phi : V(G) \to \mathcal{B}$ satisfying (1.1) as well as

$$\#\{v \in V(G) : x \in \Phi(v)\} \leqslant M \qquad \forall x \in X . \tag{2.1}$$

Recall that G *coarsely packs in* (X, dist) if it is (τ, M)-coarsely packed for some $\tau, M \geqslant 1$. Our goal in this section is to provide conditions on (X, dist) under which coarse packings yield quasi-packings.

Theorem 2.1. *Suppose that (X, dist, μ) is an Ahlfors d-regular metric measure space and additionally that (X, dist) is a length space. Then for every locally finite graph G:*

$$G \text{ coarsely packs in } (X, \mathsf{dist}) \implies G \text{ quasi-packs in } (X, \mathsf{dist})\,.$$

Quantitatively, if G is (τ, M)-coarsely packed in (X, dist), then it is (τ', M)-quasi-packed in (X, dist) with $\tau' \leqslant C\tau$, for some $C = C(X, \mathsf{dist})$.

We will prove the theorem after establishing a few preliminary results. Assume now that $\mathcal{X} = (X, \mathsf{dist}, \mu)$ is a complete, separable metric measure space. A Borel set $S \subseteq X$ is said to be *η-round* if the following holds: For every ball B in X whose center lies in $\bar{S}$ (the closure of S) and for which $S \not\subseteq B$, we have

$$\mu(S \cap B) \geqslant \eta \cdot \mu(B)\,. \tag{2.2}$$

Say that $\mathcal{X}$ is *η-round* if every ball in $\mathcal{X}$ is η-round, and that $\mathcal{X}$ is *uniformly round* if it is η-round for some $\eta > 0$. For instance, $\mathbb{R}^d$ with the Euclidean metric is 2^{-d}-round.

We recall that the measure μ is said to be *doubling* if there is a constant $C \geqslant 1$ such that

$$\mu(B(x, 2r)) \leqslant C\mu(B(x, r)) \tag{2.3}$$

for all $x \in X$ and $r \geqslant 0$.

Lemma 2.2. *If $\mathcal{X}$ is a length space and μ is doubling, then $\mathcal{X}$ is uniformly round. In particular, if (2.3) holds for some $C \geqslant 1$, then $\mathcal{X}$ is $1/(2C)$-round.*

Proof. Let $B_0 = B(x, r)$. Consider any $y \in B_0$ and $r' < r$. Since (X, dist) is a length space, there is a point $z \in B_0$ with $\mathsf{dist}(y, z) + \mathsf{dist}(z, x) = \mathsf{dist}(x, y)$ and satisfying

$$\begin{aligned} \mathsf{dist}(x, z) &\leqslant r - r'\,, \\ \mathsf{dist}(y, z) &\leqslant r'\,. \end{aligned}$$

In particular, it holds that $B(z, r') \subseteq B(y, r') \cap B(x, r)$, implying that

$$\mu\left(B_0 \cap B(y, r')\right) \geqslant \mu(B(z, r')) \geqslant C^{-1}\mu(B(z, 2r')) \geqslant C^{-1}\mu(B(y, r'))\,. \qquad \square$$

We will require the following elementary fact which states that a point in an Ahlfors d-regular space cannot be near too many pairwise-disjoint η-round bodies of large diameter.

Lemma 2.3. *Suppose $\mathcal{X}$ is (c_1, c_2, d)-regular and $S_1, S_2, \ldots, S_K \subseteq X$ are η-round sets that satisfy*

$$\#\{i \in \{1, \ldots, K\} : y \in S_i\} \leqslant s \qquad \forall y \in X\,, \tag{2.4}$$

and furthermore there is some $x \in X$ such that

$$\max_{i \in [K]} \mathsf{dist}(x, S_i) < \alpha \cdot \min_{i \in [K]} \mathsf{diam}(S_i)\,,$$

Then,

$$K \leqslant s\frac{c_2}{c_1\eta}(1 + 2\alpha)^d\,.$$

Proof. Let $\lambda = \max_{i\in[K]} \mathsf{dist}(x, S_i)$, and let $\{x_i\}$ be a collection of points such that $x_i \in \bar{S}_i$ and $\mathsf{dist}(x, x_i) \leqslant \lambda$. Consider the balls $B_i = B(x_i, \lambda/(2\alpha))$. By assumption, $\mathsf{diam}(S_i) > \lambda/\alpha$, hence $S_i \not\subseteq B_i$. Thus by the definition of η-round,

$$\mu(S_i \cap B_i) \geqslant \eta\mu(B_i) \geqslant \eta c_1(\lambda/(2\alpha))^d ,$$

where the latter inequality follows from the Ahlfors regularity of $\mathcal{X}$. But the sets $\{S_i\}$ satisfy $S_i \cap B_i \subseteq B(x, \lambda(1 + 1/(2\alpha))$ for every $i \in [K]$ and (2.4), implying that

$$K\eta c_1(\lambda/(2\alpha))^d \leqslant s \cdot \mu\left(B(x, \lambda(1 + 1/(2\alpha))\right) \leqslant sc_2\lambda^d(1 + 1/(2\alpha))^d ,$$

where again the final inequality uses the Ahlfors d-regularity. □

Proof of Theorem 2.1. Consider $\Phi : V(G) \to \mathcal{B}$ and suppose that (2.1) holds for some M'. Let $v_1, \ldots, v_M \in V(G)$ be such that $B(x, r) \cap \Phi(v_i) \neq \emptyset$ and $\mathsf{diam}(\Phi(v_i)) \geqslant r$ for each $i = 1, \ldots, M$. Under our assumptions, for some $c_1, c_2, \eta > 0$, Lemma 2.3 (applied with $s = M'$ and $\alpha = 1$) yields

$$M \leqslant M' \frac{c_2}{c_1\eta} 3^d$$ □

2.2 Quasisymmetric stability

Recall that if (X, d_X) and (Y, d_Y) are metric spaces, then a map $f : X \to Y$ is *η-quasisymmetric* if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for all $x, y, z \in X$:

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leqslant \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right). \tag{2.5}$$

The spaces (X, d_X) and (Y, d_Y) are said to be *quasisymmetrically equivalent* if, for some η, there is an η-quasisymmetric bijection from X to Y.

A metric space (X, d_X) is *uniformly perfect* if there is a number $\lambda \geqslant 1$ so that for every $x \in X$ and $r > 0$, the set $B_X(x, r) \setminus B_X(x, r/\lambda)$ is non-empty whenever $X \setminus B_X(x, r)$ is non-empty. We refer to [Hei01, §11] for background on these notions and their interplay. In particular, one has the following basic facts.

Lemma 2.4. *If (X, d_X) and (Y, d_Y) are quasisymmetrically equivalent, then (X, d_X) is uniformly perfect if and only if (Y, d_Y) is uniformly perfect.*

Lemma 2.5. *If $f : X \to Y$ is η-quasisymmetric and $A \subseteq B \subseteq X$, then*

$$\frac{1}{2\eta\left(\frac{\mathsf{diam}_X(B)}{\mathsf{diam}_X(A)}\right)} \leqslant \frac{\mathsf{diam}_Y(f(A))}{\mathsf{diam}_Y(f(B))} \leqslant \eta\left(\frac{2\,\mathsf{diam}_X(A)}{\mathsf{diam}_X(B)}\right).$$

Lemma 2.6 ([Tys01, Lem. 2.5])**.** *If $f : X \to Y$ is η-quasisymmetric and S is a τ-quasi-ball in X, then $f(S)$ is a $2\eta(\tau)$-quasi-ball in Y.*

The main result of this section is that, for uniformly perfect spaces, if X and Y are quasisymmetrically equivalent, then the classes of graphs that quasi-pack into X and Y coincide.

Theorem 2.7. *Suppose (X, d_X) and (Y, d_Y) are quasisymmetrically equivalent uniformly perfect spaces. Then there is a constant $K \geqslant 1$ such that a locally finite graph G is (τ, M)-quasi-packed in (X, d_X) if and only if it is (τ', M)-quasi-packed in (Y, d_Y), and moreover $K^{-1}\tau \leqslant \tau' \leqslant K\tau$.*

Proof. Let $f : X \to Y$ be an η-quasisymmetric bijection. Since f is a bijection, we will assume that $X = Y$. We use B_X and B_Y to denote balls in the metrics d_X and d_Y, respectively. Assume that (X, d_X) is uniformly perfect with constant $\lambda \geqslant 1$.

Suppose that G is (τ, M)-quasi-packed in (X, d_X), and let $\Phi : V(G) \to \mathcal{B}_\tau$ denote a mapping that verifies (1.1) and (1.2). Our goal is to establish that $f \circ \Phi$ witnesses a (τ', M)-quasi-packing in (Y, d_Y) for some $\tau' \geqslant 1$. For every $v \in V(G)$, let (x_v, r_v) be such that $B_X(x_v, r_v) \subseteq \Phi(v) \subseteq B_X(x_v, \tau r_v)$.

Quasi-tangency. Consider $\{u, v\} \in E(G)$ and suppose that $\mathsf{diam}_Y(\Phi(u)) \geqslant \mathsf{diam}_Y(\Phi(v))$. Observe that (1.1) implies there is a $z \in \Phi(u) \cap B_X(x_v, 2\tau^2 r_v)$. Thus Lemma 2.5 gives

$$\begin{aligned} d_Y(\Phi(u), \Phi(v)) &\leqslant \mathsf{diam}_Y(B_X(x_v, 2\tau^2 r_v)) \\ &\leqslant \mathsf{diam}_Y(\Phi(v)) \cdot 2\eta\left(\frac{4\tau^2 r_v}{\mathsf{diam}_X(\Phi(v))}\right) \\ &\leqslant \mathsf{diam}_Y(\Phi(v)) \cdot 2\eta\left(4\lambda\tau^2\right), \end{aligned}$$

where the second inequality employs Lemma 2.5, and in the last inequality we have used that X is uniformly perfect. Employing Lemma 2.6, we have thus verified that (1.1) holds for $f \circ \Phi$ with $\tau' = 2\eta(4\lambda\tau^2)$.

Quasi-multiplicity. Consider now some $x' \in Y$, $r' > 0$, and a subset $S \subseteq V(G)$ such that $B_Y(x', r') \cap \Phi(v) \neq \emptyset$ and $\mathsf{diam}_Y(\Phi(v)) \geqslant \tau' r'$ for all $v \in S$.

Let $D := \mathsf{diam}_X(B_Y(x', r'))$. Fix $v \in S$ and $z \in B_Y(x', r') \cap \Phi(v)$. Choose $z' \in \Phi(v)$ such that $d_Y(z, z') \geqslant \tau' r'/2$. Choose $z'' \in B_Y(x', r')$ so that $d_X(z, z'') \geqslant D/2$. Note that f^{-1} is η'-quasisymmetric with $\eta'(t) = 1/\eta^{-1}(1/t)$, therefore from (2.5):

$$\frac{\mathsf{diam}_X(B_Y(x', r'))}{\mathsf{diam}_X(\Phi(v))} \leqslant \frac{\mathsf{diam}_X(B_Y(x', r'))}{d_X(z, z')} \leqslant 2\frac{d_X(z, z'')}{d_X(z, z')} \leqslant \eta'\left(\frac{d_Y(z, z'')}{d_Y(z, z')}\right) \leqslant \eta'\left(\frac{4}{\tau'}\right). \tag{2.6}$$

Choose τ' large enough so that $\eta'(4/\tau') \leqslant 1/\tau$. Let $r := \mathsf{diam}_X(B_Y(x', r'))$ and fix any $x \in B_Y(x', r')$. By construction, $B_X(x, r) \cap \Phi(v) \neq \emptyset$ for every $v \in S$. By (2.6) and our choice of τ', we have

$$\mathsf{diam}_X(\Phi(v)) \geqslant \tau r \quad \forall v \in S.$$

Applying the quasi-multiplicity condition (1.2) to Φ, we see that $|S| \leqslant M$. We have thus verified that (1.2) holds also for $f \circ \Phi$ with τ' is chosen appropriately. □

3 Discrete conformal metrics on d-dimensional graphs

We first state the main technical result of this section. Recall the definition

$$d_* := \max(d, 2).$$

Theorem 3.1. *For every $d, \tau, M \geqslant 1$ and $c_1, c_2 > 0$, there is a number $C \geqslant 1$ such that the following holds. Suppose $G = (V, E)$ is a finite graph that is (τ, M)-quasi-packed in a (c_1, c_2, d)-regular space $\mathcal{X}$. Then for every $R \geqslant 0$, there is a conformal weight $\omega : V \to \mathbb{R}_+$ that satisfies*

$$\frac{1}{|V|}\sum_{x \in V} \omega(x)^{d_*} = 1, \tag{3.1}$$

and such that

$$\max_{x \in V} |B_\omega(x, R)| \leqslant C R^{d_*}. \tag{3.2}$$

Combining this with Theorem 1.13 yields Theorem 1.8.

3.1 Properties of quasi-packings

Suppose that G is (τ, M)-quasi-packed in a (c_1, c_2, d)-regular space (X, dist, μ) for some $d, \tau, M \geqslant 1$ and $c_1, c_2 > 0$. Let $\{S_v : v \in V(G)\}$ denote a family of τ-quasi-balls in X that satisfy (1.1) and (1.2). We now collect all the properties we will require of such a "packing" in proving the main theorem.

Throughout this section and the next, we will use the asymptotic notation $A \lesssim B$ to denote that $A \leqslant C \cdot B$ for some constant C that depends only the parameters d, c_1, c_2, τ, M. We use $A \asymp B$ to denote the conjunction of $A \lesssim B$ and $B \lesssim A$.

1. For every $v \in V(G)$,
$$\mathsf{diam}(S_v)^d \asymp \mu(S_v)\,. \tag{3.3}$$
This follows immediately from the definition of (c_1, c_2, d)-regular.

2. For every $x \in X$,
$$\#\{v \in V(G) : x \in S_v\} \lesssim 1\,.$$
This follows from (1.2) with $r = 0$.

3. For every $\{u, v\} \in E(G)$ and $x \in S_u, y \in S_v$:
$$\mathsf{dist}(x, y) \lesssim \mathsf{diam}(S_u) + \mathsf{diam}(S_v)\,. \tag{3.4}$$
This follows immediately from (1.1).

4. Consider a Borel set $Y \subseteq X$. It holds that
$$\sum_{v \in V(G): S_v \subseteq Y} \mu(S_v) \lesssim \mu(Y) \lesssim \mathsf{diam}(Y)^d\,. \tag{3.5}$$
The first inequality follows from (2) and the second from Ahlfors regularity.

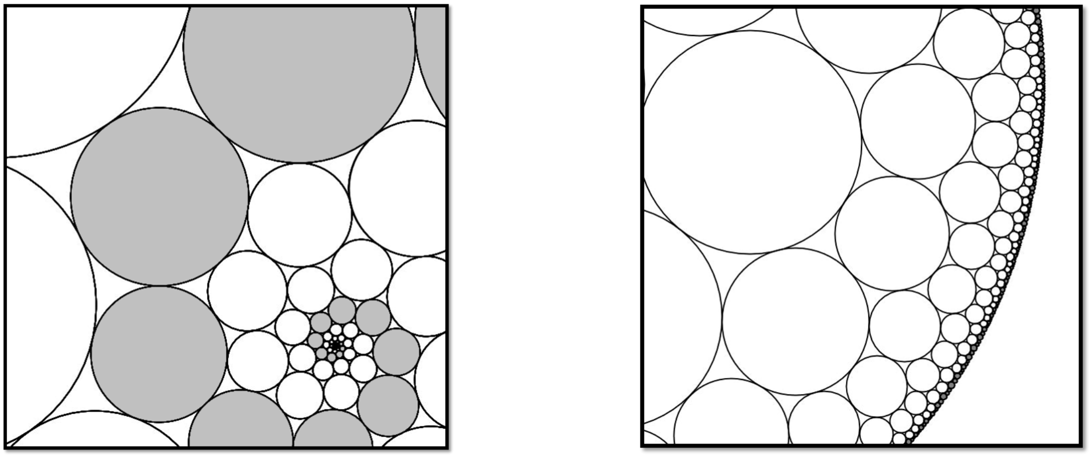
5. For any $\lambda \geqslant 1$, there is a number $C = C(\lambda, c_1, c_2, d, \tau)$ such that for all $x \in X$ and $r > 0$,
$$\#\{v \in V(G) : \mathsf{diam}(S_v) \geqslant r \text{ and } \mathsf{dist}(x, S_v) \leqslant \lambda r\} \lesssim C\,. \tag{3.6}$$
We derive this from (1.2) as follows. Cover $B(x, \lambda r)$ by balls $B_1, B_2, \ldots, B_{C'}$ of radius r/τ, where $C' = C'(c_1, c_2, d, \tau, \lambda)$. Now apply (1.2) to each B_i separately to obtain (3.6) with $C \leqslant C'M$.

3.2 Discrete uniformization

Our proof of Theorem 3.1 is inspired by the "isolation lemma" of Benjamini and Schramm [BS01] (see also [BC11, Gil14]). Suppose $G = (V, E)$ is sphere-packed in $\mathbb{R}^d$. When the spheres $\{S_v : v \in V\}$ in the packing have comparable radii, the background Euclidean metric provides a reasonable conformal weight; one sets $\omega(v)$ proportional to the radius of the sphere S_v.

Difficulties arise when the radii degenerate, for instance near an accumulation point (in the case of infinite G); see, for example, Figure 1(a). But if one imagines an *isolated* accumulation point as a cone, then it becomes rather tame: If we think of it as a metric on $\mathbb{S}^{d-1} \times [0, \infty)$, where the dth dimension is along the axis of the cone, then we merely need to do a "1-dimensional uniformization" along the axis (this can be seen in the use of the concavity of $x \mapsto x^{1/d}$ in Corollary 3.9 below). It would be problematic if the accumulation points themselves accumulated, e.g., as for a circle packing of a triangulation of the hyperbolic plane (e.g., Figure 1(b)). But the Benjamini-Schramm lemma asserts that this cannot happen for distributional limits of finite graphs packed in $\mathbb{R}^d$.

By default, we use the notation $\mathsf{diam}(\cdot)$ to denote the diameter in the metric dist. When we consider another metric, it will be explicitly specified.



(a) An isolated accumulation point (b) A continuum of accumulation points

Figure 1: Accumulation points

3.2.1 Construction of the conformal weight

Suppose now that $G = (V, E)$ is a finite graph that is (τ, M)-quasi-packed in (X, dist, μ). To each $v \in V$, associate a τ-quasi-ball $S_v \subseteq X$ so that Section 3.1(1)–(5) are satisfied.

Assume that $k \geqslant 3$ is given. We will establish the existence of a metric $\omega : V \to \mathbb{R}_+$ that satisfies $\frac{1}{|V|}\sum_{x \in V} \omega(x)^d \lesssim 1$ and such that any subset $U \subseteq V$ with $|U| = 2^k$ satisfies $\mathsf{diam}_\omega(U) \gtrsim 2^{k/d_*}$. This suffices to establish Theorem 3.1.

Identify v with an arbitrary point in S_v so that we may consider $V \subseteq X$. Define $\omega_0(v) := \mu(S_v)^{1/d}$. Then (3.3) gives:

$$\mathsf{diam}(S_v) \asymp \omega_0(v) . \tag{3.7}$$

Let $\boldsymbol{P} = \{\boldsymbol{P}_n : n \in \mathbb{Z}\}$ denote a Δ-adic hierarchical system in X (recall Section 1.7). Define

$$\hat{\boldsymbol{P}} := \left\{(C, n) : n \in \mathbb{Z}, C \in \boldsymbol{P}_n\right\} .$$

Consider a positive integer $s \lesssim 1$ to be chosen soon.

The level of a cube. For a pair $(C, n) \in \hat{\boldsymbol{P}}$, define

$$\mathrm{lev}_{\boldsymbol{P}}(C, n) := \max\left\{j \in \mathbb{N} : |(V \cap C) \setminus C')| \geqslant 2^j \text{ for all } C' \in \boldsymbol{P}_{n-s}\right\} .$$

The relevance of this definition is as follows. If $\mathrm{lev}_{\boldsymbol{P}}(C, n) = j$, then we are witnessing a "feature" of size $\approx 2^j$ that will not be fully seen by any cube at any lower scale. (For technical reasons, we actually shift by s scales, but $s \lesssim 1$.)

Thus we need to "uniformize" this feature at the current scale. Since we are trying to ensure d-dimensional volume growth, it should not be that this set of 2^j points is contained in a set of dist_ω-diameter significantly less than $2^{j/d}$ (for $d \geqslant 2$).

Let us first present a heuristic analysis. Suppose we consider a cube $C \in \boldsymbol{P}_n$ of diameter at most Δ^n and $\mathrm{lev}_{\boldsymbol{P}}(C, n) = j$. Moreover, suppose that for $v \in V \cap C$, it holds that $\omega_0(v) \lesssim \Delta^n$. (This is the case of "small bodies" in the arguments below; large bodies are handled by a separate argument.)

Then we should scale the metric ω_0 by $\approx \Delta^{-n} 2^{j/d}$ to ensure that we inflate this set to large enough diameter. (This is assuming that $\mathsf{diam}(V \cap C) \approx \Delta^n$; if the bulk $V \cap C$ has much smaller diameter, this feature will be detected at the correct scale in some other hierarchical system.) Thus we should endow the vertices $v \in V \cap C$ with weight $\omega(v) \geqslant \beta\omega_0(v)$, where $\beta \approx \Delta^{-n} 2^{j/d}$.

Consider now how much conformal weight we have spent. By a simple volume argument (3.5), the total ℓ_d-weight allocated is proportional to

$$\Delta^{-nd} 2^j \sum_{v \in V \cap C} \omega_0(v)^d \lesssim \Delta^{-nd} 2^j \Delta^{nd} \lesssim 2^j .$$

Thus if we hope to keep the total ℓ_d-weight bounded, it should be that we cannot detect too many level-j features. This is the content of the next lemma which follows [BS01, Lem 2.3].

Lemma 3.2. *For all integers $j \geqslant 0$,*

$$\#\left\{(C, n) \in \hat{\boldsymbol{P}} : \mathrm{lev}_{\boldsymbol{P}}(C, n) = j\right\} \leqslant \frac{2s|V|}{2^j} . \tag{3.8}$$

Proof. Fix $j \geqslant 0$. Denote

$$[\sigma] := \{n \in \mathbb{Z} : n \equiv \sigma \pmod{s}\} .$$

We will prove that for $\sigma \in \{0, 1, \ldots, s-1\}$,

$$\#\left\{(C, n) \in \hat{\boldsymbol{P}} : \mathrm{lev}_{\boldsymbol{P}}(C, n) = j \text{ and } n \in [\sigma]\right\} \leqslant \frac{2|V|}{2^j} . \tag{3.9}$$

Fix $\sigma \in \{0, 1, \ldots, s-1\}$. For a pair $(C, n) \in \hat{\boldsymbol{P}}$, define the set of children

$$\Lambda(C, n) := \{C' \subseteq C : C' \in \boldsymbol{P}_{n-s}\} .$$

Define a "flow" $F : (2^X \times [\sigma]) \times (2^X \times [\sigma]) \to \mathbb{R}$ "up" the hierarchical system $\boldsymbol{P}$ as follows: For every $n \in [\sigma]$,

$$F\Big((C', n-s), (C, n)\Big) = \begin{cases} \min\{2^{j-1}, |C' \cap V|\} & C \in \boldsymbol{P}_n, C' \in \Lambda(C, n) \\ 0 & \text{otherwise.} \end{cases}$$

Define also:

$$F_{\mathrm{in}}(C, n) := \sum_{(C', n') \in \hat{\boldsymbol{P}}} F\left((C', n'), (C, n)\right),$$

$$F_{\mathrm{out}}(C, n) := \sum_{(C', n') \in \hat{\boldsymbol{P}}} F\left((C, n), (C', n')\right),$$

$$F_{\mathrm{in}}^{(n)} := \sum_{C \in \boldsymbol{P}_n} F_{\mathrm{in}}(C, n) .$$

We make three observations:

1. First, notice that flow only goes "up" from a child set to a parent set, and thus from a lower level to a higher level:

$$F\left((C', n'), (C, n)\right) > 0 \implies n, n' \in [\sigma], n = n' + s, C' \in \Lambda(C, n) .$$

2. The flow out of (C, n) is always at most the flow into (C, n): $F_{\mathrm{out}}(C, n) \leqslant F_{\mathrm{in}}(C, n)$. This is because for $C \in \boldsymbol{P}_n$,

$$\sum_{C' \in \Lambda(C, n)} |C' \cap V| = |C \cap V| .$$

3. When $\mathrm{lev}_{\boldsymbol{P}}(C,n)=j$, the flow leaving (C,n) is less than the flow entering (C,n) by a least 2^{j-1} because by definition of $\mathrm{lev}_{\boldsymbol{P}}(C,n)$,

$$\sum_{C'\in\Lambda(C,n)} \min\{2^{j-1}, |C'\cap V|\} \geqslant 2^j .$$

In particular, combining this with observation (2) yields, for every $n\in\mathbb{Z}$,

$$F_{\mathrm{in}}^{(n+1)} \leqslant F_{\mathrm{in}}^{(n)} - 2^{j-1}\#\{C\in \boldsymbol{P}_n : \mathrm{lev}_{\boldsymbol{P}}(C,n)=j\} . \tag{3.10}$$

On the other hand, let $n_0\in[\sigma]$ be small enough so that every $C\in\boldsymbol{P}_{n_0}$ contains at most one point of V. Then $F_{\mathrm{in}}^{(n)}\leqslant |V|$ for all $n\leqslant n_0$. Combining this with (3.10) and the fact that $F\geqslant 0$ implies (3.9). ◻

Let us now assume additionally that $\boldsymbol{P}$ is Δ-adic for some $2\leqslant\Delta\lesssim 1$ to be fixed momentarily. Given $S\subseteq X$ and a parameter $n\in\mathbb{Z}$, we define the enlargements

$$N(S,R) := \{x\in X : \mathsf{dist}(x,S)\leqslant R\} .$$

Define also the truncated level function:

$$\mathrm{lev}^*_{\boldsymbol{P}}(C,n) := \min\{\mathrm{lev}_{\boldsymbol{P}}(C,n), k\} ,$$

where we recall that k is the parameter defined at the beginning of Section 3.2.1.

Remark 3.3. The motivation for this truncation lies in the definition (3.12) below, and the fact that we are only attempting to establish (3.2) for a single value of R or, equivalently, a single value of k. Considering "features" with level larger than k would incur a quantitative overhead that doesn't allow us to obtain a constant C in (3.2) that is independent of R.

Note that Lemma 3.2 gives

$$\#\left\{(C,n)\in\hat{\boldsymbol{P}} : \mathrm{lev}^*_{\boldsymbol{P}}(C,n)=j\right\} \leqslant \frac{4s|V|}{2^j} , \tag{3.11}$$

where the extra factor of 2 comes from the consequence

$$\#\left\{(C,n)\in\hat{\boldsymbol{P}} : \mathrm{lev}_{\boldsymbol{P}}(C,n)\geqslant j\right\} \leqslant \frac{4s|V|}{2^j} .$$

Recall that $d_*=\max(d,2)$. For every $(C,n)\in\hat{\boldsymbol{P}}$, we define a function $\theta_{\boldsymbol{P}}^{(C,n)} : V\to\mathbb{R}$ as follows:

$$\theta_{\boldsymbol{P}}^{(C,n)}(v) := \begin{cases} \dfrac{2^{\mathrm{lev}^*_{\boldsymbol{P}}(C,n)/d_*}}{\left(1+k-\mathrm{lev}^*_{\boldsymbol{P}}(C,n)\right)^{2/d_*}}\cdot\min\left\{\Delta^{-n}, \frac{1}{\omega_0(v)}\right\} & \text{if } S_v\cap N(C,2\tau\Delta^n)\neq\emptyset , \\ 0 & \text{otherwise.}\end{cases} \tag{3.12}$$

Define a conformal weight $\omega_{\boldsymbol{P}} : V\to\mathbb{R}_+$ by

$$\omega_{\boldsymbol{P}}(v) := \omega_0(v)\left(\sum_{(C,n)\in\hat{\boldsymbol{P}}}\left(\theta_{\boldsymbol{P}}^{(C,n)}(v)\right)^{d_*}\right)^{1/d_*}$$

The $1/\omega_0(v)$ factor in (3.12) is there to handle the case of a set S_v with $\mathsf{diam}(S_v) > \Delta^n$ intersecting the neighborhood of C. Denote

$$E_n(C) := \left\{ v \in V : \omega_0(v) > \Delta^n \text{ and } S_v \cap N(C, 2\tau\Delta^n) \neq \emptyset \right\}, \tag{3.13}$$

From (3.7), we have $v \in E_n(C) \implies \mathsf{diam}(S_v) \gtrsim \omega_0(v) \geqslant \Delta^n$, and therefore (3.6) implies that

$$|E_n(C)| \lesssim 1 \quad \text{for all} \quad (C, n) \in \hat{\boldsymbol{P}}\,. \tag{3.14}$$

Now write:

$$\sum_{v \in V} \omega_{\boldsymbol{P}}(v)^{d_*} = \sum_{j=0}^{k} \frac{2^j}{(1+k-j)^2} \sum_{n \in \mathbb{Z}} \sum_{\substack{C \in \boldsymbol{P}_n: \\ \mathrm{lev}^*_{\boldsymbol{P}}(C,n)=j}} \left(|E_n(C)| + \Delta^{-d_* n} \sum_{\substack{v \in V: \\ S_v \cap N(C, 2\tau\Delta^n) \neq \emptyset \\ \omega_0(v) \leqslant \Delta^n}} \omega_0(v)^{d_*} \right). \tag{3.15}$$

From (3.7), we have $\mathsf{diam}(S_v) \leqslant K_0 \omega_0(v)$ for some $1 \leqslant K_0 \lesssim 1$ and every $v \in V$. Thus in the case $d = d_*$, for a fixed $C \in \boldsymbol{P}_n$, we have

$$\begin{aligned}
\Delta^{-d_* n} \sum_{\substack{v \in V: \\ S_v \cap N(C, 2\tau\Delta^n) \neq \emptyset \\ \omega_0(v) \leqslant \Delta^n}} \omega_0(v)^{d_*} &= \Delta^{-dn} \sum_{\substack{v \in V: \\ S_v \cap N(C, 2\tau\Delta^n) \neq \emptyset \\ \omega_0(v) \leqslant \Delta^n}} \mu(S_v) \\
&\overset{(3.5)}{\lesssim} \Delta^{-dn} \mathsf{diam}(N(C, (2\tau + K_0)\Delta^n))^d \\
&\lesssim 1\,.
\end{aligned}$$

When $d < d_*$, use monotonicity of ℓ_p norms to write:

$$\left(\sum_{\substack{v \in V: \\ S_v \cap N(C, 2\tau\Delta^n) \neq \emptyset \\ \omega_0(v) \leqslant \Delta^n}} \left(\frac{\omega_0(v)}{\Delta^n} \right)^{d_*} \right)^{d/d_*} \leqslant \sum_{\substack{v \in V: \\ S_v \cap N(C, 2\tau\Delta^n) \neq \emptyset \\ \omega_0(v) \leqslant \Delta^n}} \left(\frac{\omega_0(v)}{\Delta^n} \right)^{d} = \Delta^{-dn} \sum_{\substack{v \in V: \\ S_v \cap N(C, 2\tau\Delta^n) \neq \emptyset \\ \omega_0(v) \leqslant \Delta^n}} \mu(S_v) \overset{(3.5)}{\lesssim} 1\,.$$

Using this in (3.15) together with (3.14), we conclude that

$$\begin{aligned}
\sum_{x \in V} \omega_{\boldsymbol{P}}(x)^{d_*} &\lesssim \sum_{j=0}^{k} \frac{2^j}{(1+k-j)^2} \# \left\{ (C, n) \in \hat{\boldsymbol{P}} : \mathrm{lev}^*_{\boldsymbol{P}}(C, n) = j \right\} \\
&\overset{(3.11)}{\leqslant} |V| \sum_{j=0}^{k} \frac{4s}{(1+k-j)^2} \\
&\lesssim |V|\,.
\end{aligned} \tag{3.16}$$

Since (X, dist) is doubling, Theorem 1.16 implies that for some positive integers $Q, \ell, \lesssim 1$ and $2 \leqslant \Delta \lesssim 1$, there is a sequence $\{\boldsymbol{P}^{(1)}, \ldots \boldsymbol{P}^{(Q)}\}$ of Δ-adic hierarchical systems in X such that:

$$S \subseteq X,\ \mathsf{diam}(S) \leqslant \Delta^m \implies S \subseteq C \text{ for some } (C, m + \ell) \in \bigcup_{i=1}^{Q} \hat{\boldsymbol{P}}^{(i)}\,. \tag{3.17}$$

Let us now set

$$s := \ell + 4$$

in the preceding construction. To construct our final weight, we define

$$\omega := \omega_{\boldsymbol{P}^{(1)}} + \cdots + \omega_{\boldsymbol{P}^{(Q)}} . \tag{3.18}$$

It follows that

$$\left(\frac{1}{|V|}\sum_{x\in V}\omega(x)^{d_*}\right)^{1/d_*} \lesssim \max\left\{\left(\frac{1}{|V|}\sum_{x\in V}\omega_{\boldsymbol{P}^{(i)}}(x)^{d_*}\right)^{1/d_*} : i = 1,\ldots,Q\right\} \overset{(3.16)}{\lesssim} 1\,,$$

where in the first inequality we used the fact that $Q \lesssim 1$.

3.2.2 The growth bound

The next lemma finishes the proof of Theorem 3.1.

Lemma 3.4. *For every subset of vertices $U \subseteq V$ with $|U| = 2^k$, there is an index $i \in \{1,\ldots,Q\}$ satisfying*

$$\mathsf{diam}_{\omega_{\boldsymbol{P}^{(i)}}}(U) \gtrsim 2^{k/d_*} . \tag{3.19}$$

Proof. Let us fix a subset $U \subseteq V$, and denote $D = \mathsf{diam}(U) > 0$. Let $n' := \lceil \log_\Delta D \rceil + \ell$. Then by (3.17), there is an index $i \in \{1,\ldots,Q\}$ such that $U \subseteq C$ for some $(C,n') \in \hat{\boldsymbol{P}}^{(i)}$. Let $\boldsymbol{P} = \boldsymbol{P}^{(i)}$.

We now define inductively a sequence of pairs $(C'_0, n'), (C'_1, n'-s), \ldots, (C'_{m'}, n' - m's) \in \hat{\boldsymbol{P}}$ as follows.

- Let $C'_0 := C$.
- If $|U \cap C'_i| \leqslant 1$, we set $m' := i$ and stop.

  Otherwise, we choose $C'_{i+1} \in \boldsymbol{P}_{n-s(i+1)}$ to be an element of the set $\{C' \in \boldsymbol{P}_{n-s(i+1)} : C' \subseteq C'_i\}$ that maximizes $|U \cap C'|$.

Let us then pass to the maximal subsequence $\{(C_0,n_0),(C_1,n_1),\ldots,(C_m,n_m)\}$ of the sequence $\{(C'_0,n),(C'_1,n-s),\ldots,(C'_{m'},n-m's)\}$ with $n_0 > n_1 > \cdots > n_m$ and the property that

$$n_i = \min\left\{n : \exists (C'_j, n' - js) \text{ with } n = n' - js \text{ and } C'_j \cap U = C_i \cap U\right\} .$$

In other words, we enforce the property that

$$C_{i+1} \cap U \neq C_i \cap U \quad \text{for each } i = 0,1,\ldots,m-1 . \tag{3.20}$$

Define $C_{m+1} = \emptyset$.

We have chosen the sequence $\{n_i\}$ in this way so that for every $i \in \{0,1,\ldots,m\}$,

$$\mathrm{lev}^*_{\boldsymbol{P}}(C_i, n_i) \geqslant \lfloor \log_2 |(U \cap C_i) \setminus C_{i+1}| \rfloor . \tag{3.21}$$

From our choice of $s = \ell + 4$ and the fact that $\boldsymbol{P}$ is Δ-adic with $\Delta \geqslant 2$, it holds that

$$\mathsf{diam}(C_1) \leqslant \Delta^{n'-s} \leqslant \Delta^{-3} D \leqslant \frac{D}{8} .$$

Since $\mathsf{diam}(U) = D$, there must exist some $u_0 \in U$ such that

$$\mathsf{dist}(u_0, C_1) > \frac{D}{4} > \Delta^{n_1} . \tag{3.22}$$

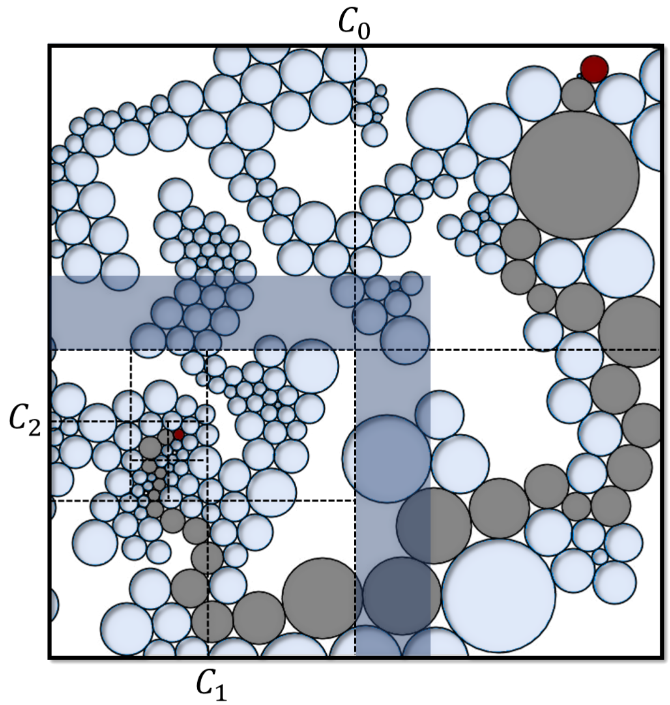


Figure 2: The path γ from $u_0 \in C_0$ to $u_m \in C_m$ passing through $N(C_1, \Delta^{n_1}) \setminus C_1$.

Fix also some $u_m \in C_m \cap U$. We will establish that $\mathsf{dist}_{\omega_P}(u_1, u_m)$ is large, certifying that $\mathsf{diam}_{\omega_P}(U)$ is large as well.

Let $N_i := |(U \cap C_i) \setminus C_{i+1}|$ for $i = 0, 1, \ldots, m$. Note that $N_i \geqslant 1$ from (3.20). Define

$$\ell_i := \text{lev}^*_{\boldsymbol{P}}(C_i, n_i) \quad \text{for } i \in \{0, 1, \ldots, m\},$$

and observe from (3.21) that

$$2^{\ell_i} \geqslant N_i/2. \tag{3.23}$$

And by construction,

$$\sum_{i=0}^{m} N_i = |U| = 2^k. \tag{3.24}$$

The length of a u_0-u_m path. Let $\gamma = \langle v_0, v_1, v_2, \ldots, v_t \rangle$ be an arbitrary simple path in G with $v_0 = u_0$ and $v_t = u_m$. Our goal is to prove that

$$\text{len}_{\omega_P}(\gamma) \gtrsim 2^{k/d_*}, \tag{3.25}$$

since if this holds for all such paths γ, it verifies (3.19).

The basic outline is as as follows. Informally, imagine that γ is parameterized by arclength in the metric dist. While γ need not spend much time in a cube C_i, it must cross from outside C_{i-1} to inside C_i, and therefore it must spend time $\asymp \Delta^{n_i}$ in the neighborhood $N(C_i, \Delta^{n_i})$, where its dist_{ω_P}-length experiences a reweighting by $\theta_{\boldsymbol{P}}^{(C_i,n_i)}$. See Figure 2. We will now split γ into subpaths $\gamma_0, \gamma_1, \ldots, \gamma_m$ accordingly and show that the reweighting is sufficient to yield (3.25).

For $i \in \{1, \ldots, m\}$, let s_i denote the largest index for which $v_{s_i} \in \gamma$ satisfies $v_{s_i} \notin N(C_i, \Delta^{n_i})$, and let t_i denote the smallest index for which $t_i > s_i$ and $v_{t_i} \in N(C_i, \Delta^{n_i}/2)$. Such indices must exist because γ begins at $u_0 \notin N(C_1, \Delta^{n_1})$ (recall (3.22)) and γ ends at $u_m \in C_m$. Let γ_i denote the subpath $\langle v_{s_i}, \dots, v_{t_i} \rangle$. Define γ_0 similarly unless $\gamma \subseteq N(C_0, \Delta^{n_0})$. In that case, we define $\gamma_0 := \gamma$. Observe that, by construction,

$$\mathrm{len}_{\mathsf{dist}}(\gamma_i) \gtrsim \Delta^{n_i} . \tag{3.26}$$

For $i \geqslant 1$, this follows from $v_{s_i} \notin N(C_i, \Delta^{n_i})$ but $v_{t_i} \in N(C_i, \Delta^{n_i}/2)$. If $i = 0$ and $\gamma_0 = \gamma$, it follows from

$$\mathrm{len}_{\mathsf{dist}}(\gamma) \geqslant \mathsf{dist}(u_0, u_m) \overset{(3.22)}{\geqslant} D/4 \gtrsim \Delta^{n_0} .$$

This yields a lower bound on the ω_0-length of each γ_i.

Lemma 3.5. *For each $i \in \{0, 1, \dots, m\}$,*

$$\mathrm{len}_{\omega_0}(\gamma_i) \gtrsim \Delta^{n_i} .$$

Proof. Parameterize $\gamma_i = \langle x_1, x_2, \dots, x_h \rangle$. From (3.4), we have

$$\mathsf{dist}(x_j, x_{j+1}) \lesssim \mathsf{diam}(S_{x_j}) + \mathsf{diam}(S_{x_{j+1}}) \lesssim \omega_0(x_j) + \omega_0(x_{j+1}) , \tag{3.27}$$

where the last inequality is (3.7).

We conclude that

$$\mathrm{len}_{\omega_0}(\gamma_i) \geqslant \frac{1}{2} \sum_{j=1}^{h} \omega_0(x_j) \overset{(3.27)}{\gtrsim} \sum_{j=1}^{h-1} \mathsf{dist}(x_j, x_{j+1}) = \mathrm{len}_{\mathsf{dist}}(\gamma_i) \gtrsim \Delta^{n_i} . \qquad \square$$

Toward proving (3.25), observe that

$$\mathrm{len}_{\omega_{\boldsymbol{P}}}(\gamma) \geqslant \frac{1}{2} \sum_{j=0}^{t} \omega_{\boldsymbol{P}}(v_j) \geqslant \frac{1}{2} \sum_{j=0}^{t} \omega_0(v_j) \left(\sum_{i=0}^{m} \left(\theta_{\boldsymbol{P}}^{(C_i, n_i)}(v_j) \right)^{d_*} \right)^{1/d_*} \tag{3.28}$$

Recall that $1 \leqslant K_0 \lesssim 1$ was chosen so that $\mathsf{diam}(S_v) \leqslant K_0 \omega_0(v)$ for all $v \in V$. Recalling (3.4), let $1 \leqslant K_1 \lesssim 1$ be such that

$$\max \left\{ \mathsf{dist}(x, y) : x \in S_u, y \in S_v \right\} \leqslant K_1 \left(\mathsf{diam}(S_u) + \mathsf{diam}(S_v) \right) \qquad \forall \{u, v\} \in E .$$

For each $v \in V$, denote

$$L(v) := \left\{ i \in \{0, 1, \dots, m\} : \omega_0(v) > \frac{\Delta^{n_i}}{8 K_0 K_1} \text{ and } S_v \cap N(C_i, 2\tau \Delta^{n_i}) \neq \emptyset \right\} .$$

This is the set of indices i such that S_v intersects the neighborhood of C_i but $\mathsf{diam}(S_v)$ is "large" with respect to $\mathsf{diam}(C_i)$.

Define the subset

$$\Lambda := \left\{ i \in \{0, 1, \dots, m\} : i \notin \bigcup_{v \in \gamma} L(v) \right\} ,$$

and the quantities

$$N_\Lambda := \sum_{i \in \Lambda} N_i$$
$$N_{\bar{\Lambda}} := 2^k - N_\Lambda .$$

Clearly the following two claims suffice to establish (3.25).

Lemma 3.6 (Large bodies)**.** *If $N_{\bar{\Lambda}} \geqslant 2^{k-1}$, then*

$$\mathrm{len}_{\omega_P}(\gamma) \gtrsim 2^{k/d_*}\,.$$

Lemma 3.7 (Small bodies)**.** *If $N_{\Lambda} \geqslant 2^{k-1}$, then*

$$\mathrm{len}_{\omega_P}(\gamma) \gtrsim 2^{k/d_*}\,.$$

In proving these two lemmas, we will need the following elementary estimate. It is a discretized version of the fact that the $x \mapsto (\log x)^{-2/d_*} x^{1/d_*}$ is concave on the interval $[c, \infty)$ for some $c > 1$.

Lemma 3.8. *For some integer $A \geqslant 2$, consider $S_A = \{(a_0, a_1, \ldots, a_k) \in \mathbb{Z}_+^{k+1} : A = a_0 2^k + a_1 2^{k-1} + \cdots + a_k\}$ Then the quantity*

$$\sum_{i=0}^{k} a_i \frac{2^{(k-i)/d_*}}{(7+i)^{2/d_*}} \tag{3.29}$$

is minimized over S_A when $a_1, \ldots, a_k \in \{0, 1\}$.

Proof. Consider any $(a_0, a_1, \ldots, a_k) \in S_A$ such that $a_i \geqslant 2$ for some $i > 0$. Then $(a'_0, a'_1, \ldots, a'_k) \in S_A$ where $a'_j = a_j$ if $j \notin \{i, i-1\}$, and $a'_i = a_i - 2, a'_{i-1} = a_{i-1} + 1$. We can calculate the change in the value of (3.29):

$$\begin{aligned} \frac{2^{(k-i)/d_*}}{(6+i)^{2/d_*}} - 2\frac{2^{(k-(i+1))/d_*}}{(7+i)^{2/d_*}} &= 2^{(k-i)/d_*}\left(\frac{1}{(6+i)^{2/d_*}} - \frac{2^{1-1/d_*}}{(7+i)^{2/d_*}}\right) \\ &= \frac{2^{(k-i)/d_*}}{(7+i)^{2/d_*}}\left(\left(1 + \frac{1}{6+i}\right)^{2/d_*} - 2^{1-1/d_*}\right) < 0\,, \end{aligned}$$

where we have used $d_* \geqslant 2$. ◻

Corollary 3.9. *Suppose that for some $a_0, a_1, a_2, \ldots, a_k \in \mathbb{Z}_+$, it holds that $a_0 2^k + a_1 2^{k-1} + \cdots + a_k \geqslant 2^{k-2}$. Then,*

$$\sum_{i=0}^{k} a_i \frac{2^{(k-i)/d_*}}{(1+i)^{2/d_*}} \geqslant \frac{2^{(k-2)/d_*}}{9}\,.$$

Proof. Applying Lemma 3.8 gives

$$\sum_{i=0}^{k} a_i \frac{2^{(k-i)/d_*}}{(1+i)^{2/d_*}} \geqslant \sum_{i=0}^{k} a_i \frac{2^{(k-i)/d_*}}{(7+i)^{2/d_*}} \geqslant \frac{2^{(k-2)/d_*}}{9^{2/d_*}} \geqslant \frac{2^{(k-2)/d_*}}{9}\,.$$ ◻

Contribution from large bodies. Now we can prove Lemma 3.6.

Proof of Lemma 3.6. From the definition (3.12), we have

$$i \in L(v) \implies \left(\omega_0(v)\theta_P^{(C_i,n_i)}(v)\right)^{d_*} \gtrsim \frac{2^{\ell_i}}{(1+k-\ell_i)^2} \tag{3.30}$$

Using (3.28) in conjunction with (3.30) yields

$$\mathrm{len}_{\omega_P}(\gamma) \gtrsim \sum_{v\in\gamma}\left(\sum_{i\in L(v)} \frac{2^{\ell_i}}{(1+k-\ell_i)^2}\right)^{1/d_*} \geqslant \sum_{v\in\gamma}\sum_{i\in L(v)} \frac{2^{\ell_i/d_*}}{(1+k-\ell_i)^{2/d_*}}\,. \tag{3.31}$$

Now from (3.23), we have

$$\sum_{v\in\gamma}\sum_{i\in L(v)} 2^{\ell_i} \geqslant N_{\bar{\Lambda}}/2 \geqslant 2^{k-2}\,.$$

Thus Corollary 3.9 in conjunction with (3.31) yields the desired bound. ◻

Contribution from small bodies. Once we restrict ourselves to subpaths γ_i composed of bodies that are "small" with respect to the scale of the cube C_i, we can argue that the corresponding subpaths are well-behaved.

Lemma 3.10. *For every $i \in \Lambda$, if $\gamma_i = \langle x_1, \dots, x_h \rangle$, then*

$$\mathsf{dist}(x_j, x_{j+1}) \leqslant \frac{\Delta^{n_i}}{4} \quad \textit{for} \quad j = 1, 2, \dots, h-1 .$$

In particular, it holds that $\gamma_i \subseteq N(C_i, 2\Delta^{n_i}) \setminus N(C_i, \Delta^{n_i}/4)$.

Proof. By construction, we have $x_2, \dots, x_h \in N(C_i, \Delta^{n_i})$ and $x_1, \dots, x_{h-1} \notin N(C_i, \Delta^{n_i}/2)$. Thus the second assertion of the lemma follows from the first.

To verify the former, note that since $x_2, \dots, x_h \in N(C_i, \Delta^{n_i})$, we have $S_{x_j} \cap N(C_i, \Delta^{n_i}) \neq \emptyset$ for $j = 2, \dots, h$. Therefore since $i \in \Lambda$, it holds that $\omega_0(x_j) \leqslant \frac{\Delta^{n_i}}{8K_0K_1}$ for $j = 2, \dots, h$. In particular, $\mathsf{diam}(S_{x_2}) \leqslant K_0 \omega_0(x_2) \leqslant \frac{\Delta^{n_i}}{8}$. Since $\{x_1, x_2\} \in E$, the quasi-tangency condition (1.1) gives

$$\mathsf{dist}(S_{x_1}, S_{x_2}) \leqslant \tau \cdot \mathsf{diam}(S_{x_2}) \leqslant \tau \frac{\Delta^{n_i}}{8} ,$$

and therefore

$$S_{x_2} \cap N(C_i, \Delta^{n_i}) \neq \emptyset \implies S_{x_1} \cap N(C_i, 2\tau\Delta^{n_i}) \neq \emptyset .$$

Since $i \in \Lambda$, we have $\omega_0(x_1) \leqslant \frac{\Delta^{n_i}}{8K_0K_1}$ as well.

Using this in conjuction with (3.4), it holds that for $j = 1, 2, \dots, h-1$, since $\{x_j, x_{j+1}\} \in E(G)$,

$$\mathsf{dist}(x_j, x_{j+1}) \leqslant K_1 \left(\mathsf{diam}(S_{x_j}) + \mathsf{diam}(S_{x_{j+1}})\right) \leqslant K_0K_1(\omega_0(x_j) + \omega_0(x_{j+1})) \leqslant 2K_0K_1 \frac{\Delta^{n_i}}{8K_0K_1} \leqslant \frac{\Delta^{n_i}}{4} . \quad \square$$

Recall that $\gamma = \langle v_0, v_1, \dots, v_t \rangle$.

Lemma 3.11. *For each $j \in \{0, 1, \dots, t\}$, v_j occurs in at most one subpath $\{\gamma_i : i \in \Lambda\}$.*

Proof. Note that since $n_{i+1} \leqslant n_i - s$ for all $i = 0, 1, \dots, m-1$, and $\Delta \geqslant 2, s \geqslant 4$, the sets $N(C_i, 2\Delta^{n_i}) \setminus N(C_i, \Delta^{n_i}/4)$ are pairwise disjoint for all $i = 0, 1, \dots, m$. Hence the result follows from Lemma 3.10. $\square$

We can now finish the proof.

Proof of Lemma 3.7. First, note that Lemma 3.11 implies that for every $j \in \{0, 1, \dots, t\}$,

$$\left(\sum_{\substack{i \in \Lambda: \\ v_j \in \gamma_i}} \left(\theta_{\boldsymbol{P}}^{(C_i, n_i)}(v_j) \right)^{d_*} \right)^{1/d_*} = \sum_{\substack{i \in \Lambda: \\ v_j \in \gamma_i}} \theta_{\boldsymbol{P}}^{(C_i, n_i)}(v_j) .$$

Using this in (3.28) yields

$$\mathsf{len}_{\omega_{\boldsymbol{P}}}(\gamma) \geqslant \frac{1}{2} \sum_{j=0}^{t} \omega_0(v_j) \sum_{\substack{i \in \Lambda: \\ v_j \in \gamma_i}} \theta_{\boldsymbol{P}}^{(C_i, n_i)}(v_j) = \frac{1}{2} \sum_{i \in \Lambda} \left(\sum_{v \in \gamma_i} \theta_{\boldsymbol{P}}^{(C_i, n_i)}(v) \omega_0(v) \right) . \tag{3.32}$$

From Lemma 3.5, we know that

$$\sum_{v \in \gamma_i} \omega_0(v) \gtrsim \Delta^{n_i} . \tag{3.33}$$

For $i \in \Lambda$, Lemma 3.10 yields $\gamma_i \subseteq N(C_i, 2\Delta^{n_i})$, hence $S_v \cap N(C_i, 2\Delta^{n_i}) \neq \emptyset$ for each $v \in \gamma_i$. From the definition of Λ, this yields $\omega_0(v) \leqslant \frac{\Delta^{n_i}}{8K_0}$, thus from the definition (3.12),

$$v \in \gamma_i \implies \theta_{\boldsymbol{P}}^{(C_i,n_i)}(v) \gtrsim \Delta^{-n_i} \frac{2^{\ell_i/d_*}}{(1+k-\ell_i)^{2/d_*}} .$$

Combining this with (3.32) and (3.33) gives

$$\mathrm{len}_{\omega_{\boldsymbol{P}}}(\gamma) \gtrsim \sum_{i \in \Lambda} \frac{2^{\ell_i/d_*}}{(1+k-\ell_i)^{2/d}} . \tag{3.34}$$

By (3.23) and our assumption that $N_\Lambda = \sum_{i \in \Lambda} N_i \geqslant 2^{k-1}$, we have $\sum_{i \in \Lambda} 2^{\ell_i} \geqslant 2^{k-2}$. Thus Corollary 3.9 in conjunction with (3.34) yields

$$\mathrm{len}_{\omega_{\boldsymbol{P}}}(\gamma) \gtrsim 2^{k/d_*} ,$$

completing the proof. □

□

3.3 d-parabolicity

We first discuss two examples showing that for distributional limits of finite graphs with uniformly bounded degrees, d-parabolicity and the property that $\overline{\dim}_{\mathsf{cg}}^{d}(G,\rho) \leqslant d$ are incomparable.

First, we remark on the following general construction. Let $\{(H_n, \rho_n) : n \geqslant 1\}$ be a sequence of non-isomorphic, finite rooted graphs, and let p be a probability on $\mathbb{N}$. Let $(\boldsymbol{H}, \boldsymbol{h})$ be the random rooted graph that arises by choosing (H_n, ρ_n) with probability $p(n)$. Suppose furthermore that

$$\mathbb{E}\left[|V(\boldsymbol{H})|\right] = \sum_{n \geqslant 1} p(n)|V(H_n)| < \infty . \tag{3.35}$$

Consider a path P_N of length $N \geqslant 1$, and attach to each vertex of P_N an independent copy of $(\boldsymbol{H}, \boldsymbol{h})$ (we identify $\boldsymbol{h}$ with the corresponding vertex in P_N). This yields a random graph G_N, and we choose a root $r_N \in V(G_N)$ uniformly at random. We claim that $\{(G_N, r_N)\}$ has a distributional limit (G, ρ). To see this, note that

$$q(n) := \lim_{N \to \infty} \mathbb{P}[r_N \text{ is in a copy of } H_n] = \frac{p(n)|V(H_n)|}{\mathbb{E}[V(\boldsymbol{H})]} .$$

Now (3.35) implies that q is a probability on $\mathbb{N}$.

It is then straightforward to describe the limit: (G, ρ) is a bi-infinite path P with some fixed vertex $v_0 \in V(P)$. At v_0, we attach a copy H of (H_n, ρ_n) with probability $q(n)$, and choose $\rho \in V(H)$ uniformly at random. At every vertex in $V(P) \setminus \{v_0\}$, we attach an independent copy of $(\boldsymbol{H}, \boldsymbol{h})$.

Using the weight $W(v) := \frac{\mathbb{1}_{V(P)}(v)}{1+\mathsf{dist}_G(v_0,v)}$ verifies the following claim.

Claim 3.12. G is almost surely 2-parabolic.

Example 3.13 (Infinite conformal growth exponent but 2-parabolic)**.** Now let $\{H_n : n \geqslant 1\}$ denote an infinite family of connected, transitive, d-regular graphs with $|V(H_n)| \in [n, 2n]$ and

$$\mathsf{diam}(H_n) < C \log(n+1) , \tag{3.36}$$

for some $C > 0$. (The diameter here refers to the graph metric.) For instance, one can take a family of expanding Cayley graphs.

Lemma 3.14. *If $\rho_n \in V(H_n)$ is uniformly random, then for any $\omega : V(H_n) \to \mathbb{R}_+$:*

$$\max_{x \in V(H_n)} \left| B_\omega \left(x, 2C \log(n+1) \sqrt{\mathbb{E}[\omega(\rho_n)^2]} \right) \right| \geqslant \frac{n}{4}. \tag{3.37}$$

Proof. Consider the following family of convex sets indexed by $D > 0$:

$$C_D := \left\{ \omega : \mathbb{E}[\omega(\rho_n)^2] \leqslant 1 \text{ and } \frac{1}{|V(H_n)|^2} \sum_{x,y \in V(H_n)} \mathsf{dist}_\omega(x,y) \geqslant D \right\}.$$

By convexity and transitivity of H_n, $\omega_0 \in C_D \iff C_D \neq \emptyset$, where $\omega_0 \equiv 1$ is the uniform weight. Note that dist_{ω_0} is simply the graph metric dist_{H_n}, hence (3.36) implies that $C_{C\log(n+1)} = \emptyset$.

Thus for any $\omega : V(H_n) \to \mathbb{R}_+$, there is an $x_0 \in V(H_n)$ such that

$$\frac{1}{|V(H_n)|} \sum_{x \in V(H_n)} \mathsf{dist}_\omega(x, x_0) < C \sqrt{\mathbb{E}[\omega(\rho_n)^2]} \cdot \log(n+1).$$

In particular, for $R := C\sqrt{\mathbb{E}[\omega(\rho_n)^2]} \cdot \log(n+1)$, it holds that

$$|B_\omega(x_0, 2R)| > \tfrac{1}{2} |V(H_n)|,$$

completing the proof. □

Define $p(n) := \frac{c'}{n^2(\log(n+1))^2}$, where the constant c' is chosen so that p is a probability on $\mathbb{N}$. Then (3.35) is satisfied, hence there is a distributional limit (G, ρ) as above. By Claim 3.12, G is almost surely 2-parabolic.

Let ω denote a (unimodular) L^2-normalized conformal weight on (G, ρ), and define the numbers

$$W_n := \sqrt{\mathbb{E}\left[\omega(\rho)^2 \mid \rho \text{ is in a copy of } H_n\right]}.$$

Since ω is L^2-normalized, we have

$$\sum_{n \geqslant 1} q(n) W_n^2 \leqslant 1.$$

Because $q(n) \asymp \frac{1}{n(\log n)^2}$, there must exist an infinite set $I \subseteq \mathbb{N}$ such that $n \in I \implies W_n \leqslant \log n$.

Note that the Mass-Transport Principle yields, for $n \geqslant 2$,

$$\mathbb{E}\left[\frac{1}{|V(H)|} \sum_{x \in V(H)} \omega(x)^2 \;\middle|\; \rho \text{ is in a copy } H \text{ of } H_n \right] = W_n^2,$$

hence Markov's inequality gives

$$\mathbb{P}\left[\frac{1}{|V(H)|} \sum_{x \in V(H)} \omega(x)^2 > (\log n)^2 W_n^2 \;\middle|\; \rho \text{ is in a copy } H \text{ of } H_n \right] \leqslant \frac{1}{\log n}.$$

Applying the Mass-Transport Principle again, a straightforward application of Borel-Cantelli shows that almost surely there are infinitely many $n \in I$ such that G contains a copy H of H_n with

$$\frac{1}{|V(H)|} \sum_{x \in V(H)} \omega(x)^2 < (\log n)^2 W_n^2 \leqslant (\log n)^4.$$

And in this case, (3.37) yields

$$\max_{v\in V(H)} \left|B_\omega(v, 2C\log(n+1)^3\right| \geqslant \frac{n}{4}\,,$$

clearly ruling out any finite growth exponent. This demonstrates that $\underline{\dim}_{\mathsf{cg}}(G,\rho) = \infty$.

Example 3.15 (2-dimensional conformal growth, but not 2-parabolic)**.** We will exhibit a unimodular random graph $(\widehat{T},\rho)$ with $\deg_{\widehat{T}}(\rho) \leqslant 6$ almost surely, and such that $\widehat{T}$ is almost surely transient (and hence *not* 2-parabolic), yet $\overline{\dim}_{\mathsf{cg}}(\widehat{T},\rho) \leqslant 2$.

Denote by T_n the complete 4-ary tree of height $n \geqslant 1$. Let us obtain a graph $\widetilde{T}_n$ by replacing every edge at distance h from the leaves by $f(h)$ parallel paths of length $g(h)$, with

$$f(h) := 2^h\,,$$
$$g(h) := \left\lceil 2^{h-\sqrt{h}} \right\rceil .$$

Observe that for any $x \in V(\widetilde{T}_n)$ and $i \geqslant 0$, it holds that

$$|B_{\widetilde{T}_n}(x, 2^{i-\sqrt{i}})| \leqslant O(1)\sum_{j=1}^{i} 4^{i-j} f(j) g(j) \leqslant O(4^i)\,, \tag{3.38}$$

and moreover there is a flow from a leaf of $\widetilde{T}_n$ to the root with energy at most

$$O(1)\sum_{j=1}^{h} \frac{g(j)}{f(j)} \leqslant O(1)\,. \tag{3.39}$$

Thus if we let (T,ρ) denote the distributional limit of $\{(\widetilde{T}_n,\rho_n)\}$ with $\rho_n \in V(\widetilde{T}_n)$ chosen uniformly at random, then (3.39) implies that T is almost surely transient, and (3.38) implies that $\overline{\dim}_{\mathsf{cg}}(T,\rho) \leqslant 2$ (using the normalized conformal weight $\omega \equiv 1$).

The only remaining issue is that the vertex degrees in (T,ρ) are not bounded. Since every distributional limit of finite planar graphs with uniformly bounded degrees *is* 2-parabolic, replacing the parallel paths with bounded-degree subgraphs will require the final step in our construction to be non-planar.

To obtain uniformly bounded degrees, we replace every vertex $x \in V(T_n)$ at distance $h = 0,1,2,\ldots$ from the leaves with a cloud C_x containing $f(h) = 2^h$ vertices. Moreover, if $y \in V(T_n)$ is a child of x, we connect every vertex in C_y to exactly two vertices of C_x via internally-disjoint paths of length $g(h)$ to obtain a graph $\widehat{T}_n$.

Clearly one can do this in a manner so that if x is an internal node of T_n, then the degree of every vertex in C_x in $\widehat{T}_n$ is precisely 6 (one path from each of its four children and two paths to its parent), unless x is the root of T_n, in which case the vertices in C_x have degree 4. Now let $(\widehat{T},\rho)$ denote the distributional limit of $\{(\widehat{T}_n,\rho_n)\}$ where $\rho_n \in V(\widehat{T}_n)$ is chosen uniformly at random.

It is straightforward that both the growth and energy estimates (3.38) and (3.39) hold for $\widehat{T}_n$ as well, where now the flow is from a leaf to the cloud C_r of the root $r \in V(\widehat{T}_n)$. Therefore $(\widehat{T},\rho)$ is a unimodular random graph with essentially bounded degrees that is almost surely transient (and hence *not* 2-parabolic) but which satisfies $\overline{\dim}_{\mathsf{cg}}(\widehat{T},\rho) \leqslant 2$.

Using the duality between d-parabolicity and the $\ell^{d'}$ energy of a flow to ∞ (where $d' = \frac{d}{d-1}$ is the dual exponent to d), one can similarly construct examples, for every $d \geqslant 2$, of unimodular random graphs (G,ρ) such that is almost surely not d-parabolic but satisfies $\overline{\dim}^{d}_{\mathsf{cg}}(G,\rho) \leqslant d$.

3.3.1 Gauged conformal growth and vertex extremal length

We now prove that gauged d-dimensional conformal growth implies d-parabolicity when the degree of the root is almost surely uniformly bounded.

Proof of Theorem 1.7. Fix $d \geqslant 1$ and a unimodular random graph (G, ρ) with gauged d-dimensional conformal growth and such that $\deg_G(\rho)$ is essentially bounded. For each $R \geqslant 0$, let ω_R be an L^d-normalized conformal metric on (G, ρ) that satisfies

$$\|B_{\omega_R}(\rho, R)\|_{L^\infty} \leqslant CR^d \tag{3.40}$$

for some constant $C \geqslant 1$.

From [Lee17, Lem. 2.6], we may assume that for each $R \geqslant 0$, the following additional properties hold almost surely:

1. For all $x \in V(G)$, $\omega_R(x) \geqslant 1/2$.
2. For all $\{x, y\} \in E(G)$, we have $\omega_R(x) \leqslant C'\omega_R(y)$, where $C' > 1$ is a constant depending only on $\|\deg_G(\rho)\|_{L^\infty}$.

Moreover, these additional properties are sufficient to guarantee that we can compare dist_{ω_R} balls to dist_G balls in the following sense (see [Lee17, Lem. 2.5]): Almost surely, for every $x \in V(G)$ and $R, r \geqslant 0$,

$$B_G\left(x, \frac{\log \frac{r}{2\omega_R(x)}}{\log C'}\right) \subseteq B_{\omega_R}(x, r) \subseteq B_G(x, 2r)\,. \tag{3.41}$$

Fix $\varepsilon \in (0, 1), n \geqslant 1$. Let $\{r_j\}$ be the sequence of numbers with $r_1 = 1$ and, that satisfies, for $j > 1$,

$$\frac{\log \frac{\varepsilon r_j}{16C'}}{\log C'} = 2r_{j-1}\,.$$

Denote

$$\Lambda_G := \left\{x \in V(G) : \omega_{r_j}(x) \leqslant \frac{1}{\varepsilon} \text{ for } j \leqslant n\right\}\,.$$

For $x \in V(G)$, let

$$A_j(x) := B_{\omega_{r_j}}(x, r_j) \setminus B_{\omega_{r_j}}\left(x, \frac{r_j}{8C'}\right)\,.$$

By our choice of the sequence $\{r_j\}$ and (3.41), for every $x \in \Lambda_G$, we have

$$B_{\omega_{r_{j-1}}}(x, r_{j-1}) \subseteq B_G(x, 2r^{j-1}) \subseteq B_{\omega_{r_j}}(x, r_j/(8C'))\,, \tag{3.42}$$

hence if $x \in \Lambda_G$, then the sets $A_1(x), A_2(x), \ldots, A_n(x)$ are pairwise disjoint.

Consider now the following conformal weight which depends on the choice of some $z \in V(G)$:

$$\omega_{(z)}(x) := \left(\sum_{j=1}^{n} r_j^{-d}\omega_{r_j}(x)^d \mathbb{1}_{A_j(z)}(x)\right)^{1/d}\,.$$

By construction, if $z \in \Lambda_G$, then

$$\sum_{x \in V(G)} \omega_{(z)}(x)^d \leqslant \sum_{j=0}^{n} r_j^{-d}\mathcal{V}_{\omega_{r_j}}(z, r_j)\,, \tag{3.43}$$

where

$$\mathcal{V}_\omega(x,r) := \sum_{y\in B_\omega(x,r)} \omega(y)^d\,,$$

we used the fact established earlier that $z \in \Lambda_G$ implies that the sets $A_j(z)$ are pairwise disjoint for $j = 1, 2, \ldots, n$.

Now observe that

$$\mathsf{dist}_{\omega_{(z)}}(z,x) \geqslant \sum_{j=1}^{n} \frac{\mathsf{dist}_{\omega_{r_j}\mathbb{1}_{A_j(z)}}(z,x)}{r_j}\,. \tag{3.44}$$

Suppose that $x \in V(G) \setminus B_G(z, 2r_n)$ and consider any path γ from z to x in G. Let γ_j denote the portion of γ which lies inside $A_j(z)$. Every vertex $u \in B_{\omega_j}(z, r_j/(8C'))$ satisfies $\omega_{r_j}(u) \leqslant r_j/(4C')$ by definition of $\mathsf{dist}_{\omega_{r_j}}$, thus if $\{u,v\} \in E(G)$, then by Property (2) above, $\omega(v) \leqslant r_j/4$.

In particular,

$$\mathrm{len}_{\omega_{r_j}\mathbb{1}_{A_j(z)}}(\gamma) = \mathrm{len}_{\omega_{r_j}}(\gamma_j) \geqslant \frac{r_j}{2C'}\,.$$

Using (3.44), we conclude that

$$z \in \Lambda_G \implies \mathsf{dist}_{\omega_{(z)}}(z, V(G) \setminus B_G(z, 2r_n)) \geqslant \sum_{j=1}^{n} \frac{r_j}{2C' r_j} \geqslant \frac{n}{2C'}\,. \tag{3.45}$$

Let us now return to (3.43). For a conformal metric $\omega : V(G) \to \mathbb{R}_+$ and some $R > 0$, define the transport

$$F(G,\omega,x,y) = \omega(x)^d\, \mathbb{1}_{\{\mathsf{dist}_\omega(x,y)\leqslant R\}}\,.$$

Then by the Mass-Transport Principle,

$$\begin{aligned}\mathbb{E}\left[\mathcal{V}_\omega(\rho,R)\right] = \mathbb{E}\left[\sum_{x\in V(G)} F(G,\omega,x,\rho)\right] &= \mathbb{E}\left[\sum_{x\in V(G)} F(G,\omega,\rho,x)\right] \\ &= \mathbb{E}\left[\omega(\rho)^d |B_\omega(\rho,R)|\right] \leqslant \|B_\omega(\rho,R)\|_{L^\infty}\, \mathbb{E}\left[\omega(\rho)^d\right]\,.\end{aligned}$$

We conclude from (3.40) that for each $j \leqslant n$,

$$\mathbb{E}\left[\mathcal{V}_{\omega_{r_j}}(\rho, r_j) \mid \rho \in \Lambda_G\right] \leqslant \frac{C r_j^d}{\mathbb{P}[\rho \in \Lambda_G]}\,,$$

hence

$$\mathbb{E}\left[\sum_{x\in V(G)} \omega_{(\rho)}(x)^d \mid \rho \in \Lambda_G\right] \leqslant \frac{Cn}{1-\varepsilon^d n}\,,$$

where we have used Markov's inequality and a union bound to assert that $\mathbb{P}[\rho \in \Lambda_G] \geqslant 1 - \varepsilon^d n$.

Take $\varepsilon = 1/n$ and $n \geqslant 2$ in the preceding construction and define the event

$$\mathcal{E}(n) := \left\{\omega_{r_j}(\rho) \leqslant n \text{ for } j \leqslant n \text{ and } \|\omega_{(\rho)}\|^d_{\ell^d(V(G))} \leqslant 2Cn^{1.5}\right\}\,.$$

By Markov's inequality and a union bound, we have

$$\mathbb{P}\left(\mathcal{E}(n)\right) \geqslant 1 - \frac{2}{\sqrt{n}}\,.$$

Moreover from (3.45),

$$\mathcal{E}(n) \implies \frac{\mathsf{dist}_{\omega_{(\rho)}}(\rho, V(G) \setminus B_G(\rho, 2r_n))}{\|\omega_{(\rho)}\|_{\ell_d(V(G))}} \geqslant \frac{n}{4C'C^{1/d}n^{1.5/d}} \geqslant \frac{n^{1/4}}{4C'\sqrt{C}}.$$

In other words, for every $n \geqslant 1$, it holds that

$$\mathbb{P}\left[\mathsf{VEL}_d(\Gamma_G(\rho)) \geqslant \frac{n^{1/4}}{4C'\sqrt{C}}\right] \geqslant 1 - \frac{2}{\sqrt{n}}.$$

Sending $n \to \infty$, it follows that

$$\mathbb{P}\left[\mathsf{VEL}_d(\Gamma_G(\rho)) = \infty\right] = 1,$$

i.e., almost surely G is d-parabolic. □

3.4 Spectral bounds for the graph Laplacian

We now prove the following generalization of Theorem 1.10.

Theorem 3.16. *For every $d, \tau, M \geqslant 1$, $c_1, c_2 > 0$, there is a constant $C \geqslant 1$ such that the following holds. Suppose $G = (V, E)$ is an n-vertex graph that is (τ, M)-quasi-packed in a (c_1, c_2, d)-regular space (X, dist, μ). Then for $k = 1, 2, \ldots, n-1$,*

$$\lambda_k(G) \leqslant C \frac{\Delta_G(k)}{k} \left(\log \frac{n}{k}\right)^2 \left(\frac{k}{n}\right)^{2/d}.$$

Consider a finite connected graph $G = (V, E)$. Define the *Rayleigh quotient* $\mathcal{R}_G(f)$ of non-zero $f : V \to \mathbb{R}$ by

$$\mathcal{R}_G(f) := \frac{\sum_{\{x,y\} \in E} |f(x) - f(y)|^2}{\sum_{x \in V} \deg_G(x) f(x)^2}.$$

It is an elementary fact (see, e.g., [Lee17, Cor. 3.1]) that to establish Theorem 3.16, it suffices to find k disjointly supported functions $\varphi_1, \varphi_2, \ldots, \varphi_k : V \to \mathbb{R}$ such that for each $i = 1, 2, \ldots, k$,

$$\mathcal{R}_G(\varphi_i) \leqslant C \frac{\Delta_G(k)}{k} \left(\log \frac{n}{k}\right)^2 \left(\frac{k}{n}\right)^{2/d}.$$

Toward this end, we now state [Lee17, Thm. 3.12]. For a finite graph $G = (V, E)$, denote $\bar{d}_G(\varepsilon) := \frac{\Delta_G(\varepsilon|V|)}{\varepsilon|V|}$.

Theorem 3.17. *There is a constant $C \geqslant 1$ such that the following holds. Consider a finite graph $G = (V, E)$ with $n = |V|$. Suppose that $\omega : V \to \mathbb{R}_+$ is a conformal metric on G satisfying*

1. $\frac{1}{|V|} \sum_{x \in V} \omega(x)^2 \leqslant 1$,
2. *For some numbers $R > 0, K \geqslant 2$:*

$$\max_{x \in V} |B_\omega(x, R)| \leqslant K \leqslant n/2. \tag{3.46}$$

Then there exist disjoint supported functions $\varphi_1, \varphi_2, \ldots, \varphi_k : V \to \mathbb{R}_+$ with $k \geqslant n/16K$, and such that

$$\max\left\{\mathcal{R}_G(\varphi_1), \ldots, \mathcal{R}_G(\varphi_k)\right\} \leqslant C \frac{(\log K)^2 \left(\bar{d}_G(1/K) + \bar{d}_G\left(1/R^2\right)\right)}{R^2}.$$

Remark 3.18. The statement of [Lee17, Thm. 3.12] contains an additional parameter α, and here we have used the fact that one can take $\alpha \leqslant O(\log K)$. This is a basic and well-known estimate; it follows, for instance, from [Lee17, Lem. 4.5] which it itself a reference to [LN05, Lem. 3.11].

Now Theorem 3.16 is a consequence of the following proposition combined with Theorem 3.1.

Proposition 3.19. *Suppose that $G = (V, E)$ is an n-vertex graph with (c, R, d)-growth for some numbers $c \geqslant 1, d \geqslant 2$ and all $R \geqslant 0$. Then for $k = 1, 2, \ldots, n-1$,*

$$\lambda_k(G) \leqslant O(1) \frac{\Delta_G(k)}{k} \left(\log \frac{n}{k}\right)^2 \left(\frac{ck}{n}\right)^{2/d} .$$

Proof. For each $R \geqslant 0$, let $\omega_R : V \to \mathbb{R}_+$ be a conformal metric on G satisfying

$$\frac{1}{|V|} \sum_{x \in V} \omega_R(x)^d = 1 ,$$

and

$$\max_{x \in V} |B_\omega(x, R)| \leqslant cR^d .$$

Note that from Hölder's inequality,

$$\frac{1}{|V|} \sum_{x \in V} \omega_R(x)^2 \leqslant \left(\frac{1}{|V|} \sum_{x \in V} \omega_R(x)^d\right)^{2/d} = 1 .$$

So we can apply Theorem 3.17 with ω_R and $K = cR^d$ to obtain, for $k \leqslant n/(16cR^d)$,

$$\lambda_k(G) \leqslant O(1) \frac{(d \log R)^2 \bar{d}_G(\frac{1}{cR^d})}{R^2} .$$

Setting $R := (n/16ck)^{1/d}$ yields

$$\lambda_k(G) \leqslant O(1) \left(\frac{ck}{n}\right)^{2/d} \left(\log \frac{n}{k}\right)^2 \frac{\Delta_G(k)}{k} .$$

completing the proof. ∎

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