Spectral dimension, Euclidean embeddings, and the metric growth exponent

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Abstract

For reversible random networks, we exhibit a relationship between the almost sure spectral dimension and the Euclidean growth exponent, which is the smallest asymptotic rate of volume growth over all embeddings of the network into a Hilbert space. Using metric embedding theory, it is then shown that the Euclidean growth exponent coincides with the metric growth exponent. This simplifies and generalizes a powerful tool for bounding the spectral dimension in random networks.

Contents

1 Introduction

Let G be a locally-finite, connected graph and denote by $\{X_n\}$ the simple random walk on G. For vertices $u, v \in V(G)$ and $n \geqslant 1$, denote the discrete-time heat kernel by

$$\mathsf{p}_n(x,y) := \mathbb{P}[X_n = x \mid X_0 = y]\,.$$

The *spectral dimension of G is equal to d_s* if it holds that for some vertex $v_0 \in V(G)$,

$$d_s = \lim_{n\to\infty} \frac{-2\log \mathsf{p}_n(v_0,v_0)}{\log n}.$$

One can check that the choice of $v_0 \in V(G)$ does not affect the limit (assuming it exists).

Consider now a discrete metric space $(X, \mathfrak{d})$, and define the *asymptotic growth exponent*

$$\bar{d}_f(X,\mathfrak{d}) := \limsup_{R\to\infty} \frac{\log |B_{\mathfrak{d}}(x_0,R)|}{\log R}, \tag{1.1}$$

where

$$B_{\mathfrak{d}}(x,R) := \{y \in X : \mathfrak{d}(x,y) \leqslant R\},$$

and again the choice of $x_0 \in X$ is irrelevant.

The present work explores a general setting of random rooted graphs (G, ρ) where the almost sure spectral dimension of G coincides with the smallest asymptotic growth exponent over a natural class of metrics on $V(G)$. This provides a powerful set of tools for understanding the spectral dimension of these graphs via the construction of discrete metric spaces on their vertex set.

Our main analytic tools are K. Ball's notion of Markov type [Bal92] and the extension to maximal Markov type appearing in [NPSS06], as well as Bourgain's metric embedding theorem [Bou85]. Both tools have been previously applied fruitfully in contexts outside the non-linear geometry of Banach spaces; see, e.g., [LMN02] and [LLR95], respectively.

Reversible random networks. Consider a random rooted network (G, ρ, κ, ξ) where G is a locally-finite, connected graph, $\rho \in V(G)$, and $\{\kappa_{uv} \geqslant 0 : \{u, v\} \in E(G)\}$ are edge conductances (where $\kappa_{uv} = \kappa_{vu}$ for $\{u, v\} \in E(G)$). We allow $E(G)$ to contain self-loops $\{v, v\}$ for $v \in V(G)$. Here, $\xi : V(G) \cup E(G) \to \Xi$ is an auxiliary marking, where Ξ is some Polish mark space.

We will sometimes use the notation $(G, \rho, \xi_1, \xi_2, \ldots, \xi_k)$ to reference a random rooted network with marks $\xi_i : V(G) \cup E(G) \to \Xi_i$, which we intend as shorthand for $(G, \rho, (\xi_1, \xi_2, \ldots, \xi_k))$, where the mark space is the Cartesian product $\Xi_1 \times \cdots \times \Xi_k$. We usually allow the conductances to remain implicit, writing simply (G, ρ, ξ).

The random walk $\{X_n\}$ (conditioned on (G, ρ, ξ)) is defined by the transition kernel

$$\mathsf{p}_1(u, v) := \mathbb{P}[X_1 = v \mid X_0 = u] := \frac{\kappa_{uv}}{\kappa_u}, \tag{1.2}$$

where we denote $\kappa_u := \sum_{v:\{u,v\}\in E(G)} \kappa_{uv}$. Let us also define, for $S \subseteq V(G)$, the volume $\mathrm{vol}(S) := \sum_{u\in S} \kappa_u$.

Definition 1.1. Say that (G, ρ, ξ) is a *reversible random network* if (G, X_0, X_1, ξ) and (G, X_1, X_0, ξ) have the same law.

Reversible random networks are closely related to unimodular random networks by a change of measure. This connection is laid out in [BC12], and one can find a detailed explanation in [Lee21b, §1.1]. One can also consult [AL07] for an extensive reference on unimodular random networks.

Euclidean embeddings. Suppose (G, ρ) is a reversible random network and fix a separable Hilbert space $\mathcal{H}$. A *proper Euclidean embedding* of (G, ρ) is a random mapping $\Psi : V(G) \to \mathcal{H}$ such that (G, ρ, Ψ) is a reversible random network with $\mathbb{E}\,\|\Psi(X_0) - \Psi(X_1)\|_{\mathcal{H}}^2 < \infty$. Such an embedding induces a distance $\mathfrak{d}_\Psi(u, v) := \|\Psi(u) - \Psi(v)\|_{\mathcal{H}}$ on $V(G)$.

Definition 1.2 (Euclidean growth exponent)**.** The *Euclidean growth exponent* $d^{\star}_{\mathrm{euc}}$ *of* (G, ρ) is the infimal value d such that almost surely the asymptotic growth exponent of $(V(G), \mathfrak{d}_\Psi)$ is at most d, where the infimum is taken over all proper Euclidean embeddings $\Psi : V(G) \to \mathcal{H}$. Equivalently, it is the infimal value d such that almost surely

$$\limsup_{R\to\infty} \frac{\log \mathrm{vol}\left(\Psi^{-1}\left(B_{\mathcal{H}}(0, R)\right)\right)}{\log R} \leqslant d,$$

where $B_{\mathcal{H}}(0, R) := \{x \in \mathcal{H} : \|x\|_{\mathcal{H}} \leqslant R\}$.

Let us also define the *(upper) spectral dimension* $\bar{d}_s$ *of* (G,ρ) as the infimal value d such that almost surely

$$\limsup_{n\to\infty} \frac{-2\log \mathsf{p}_{2n}(\rho,\rho)}{\log n} \leqslant d\,.$$

Our first result bounds the spectral dimension by the Euclidean growth exponent.

Theorem 1.3. *If* (G,ρ) *is a reversible random network, then* $\bar{d}_s \leqslant d^\star_{\mathrm{euc}}$.

This result is proved in Section 2 by extending basic results in Markov type theory to the setting of reversible random networks.

Spectral concentration. There are known examples (see [AHNR18] and Section 3.3) where $\bar{d}_s$ is finite, while $d^\star_{\mathrm{euc}} = \infty$, so this inequality cannot be reversed in general. We now present an upper bound that holds whenever the return probabilities $\mathsf{p}_{2n}(\rho,\rho)$ are sufficiently concentrated.

Definition 1.4 (Annealed spectral dimension)**.** The *(lower) annealed spectral dimension* $\underline{d}_s^{\mathcal{A}}$ *of* (G,ρ) is given by

$$\underline{d}_s^{\mathcal{A}} := \liminf_{n\to\infty} \frac{-2\log \mathbb{E}[\mathsf{p}_{2n}(\rho,\rho)]}{\log n}.$$

Note that, by concavity of the logarithm, we have $\underline{d}_s^{\mathcal{A}} \leqslant \bar{d}_s$. If $\underline{d}_s^{\mathcal{A}} = \bar{d}_s$, then the inequality in Theorem 1.3 can be reversed.

Theorem 1.5. *Suppose* (G,ρ) *is a reversible random network satisfying* $\mathbb{E}[1/\kappa_\rho] < \infty$. *Then,*

$$d^\star_{\mathrm{euc}} \leqslant \frac{\bar{d}_s}{\left(1 + (\underline{d}_s^{\mathcal{A}} - \bar{d}_s)/2\right)_+}\,.$$

This is proved in Section 3, where an appropriate Euclidean metric is constructed from the discrete-time heat flow on G. Note that the theorem is vacuous unless $\bar{d}_s < \underline{d}_s^{\mathcal{A}} + 2$. In Section 3.3, we present examples where the reverse inequality $d^\star_{\mathrm{euc}} \leqslant \bar{d}_s$ fails to hold (in particular, these examples satisfy $\bar{d}_s > \underline{d}_s^{\mathcal{A}} + 2$).

The next conclusion follows from the conjunction of Theorem 1.3 and Theorem 1.5.

Corollary 1.6. *If* (G,ρ) *is a reversible random network with* $\mathbb{E}[1/\kappa_\rho] < \infty$ *and* $\bar{d}_s = \underline{d}_s^{\mathcal{A}}$, *then the spectral dimension of* G *is almost surely equal to* $d^\star_{\mathrm{euc}}$.

Note that $\mathbb{E}[1/\kappa_\rho] < \infty$ is automatically satisfied, for instance, when (G,ρ) is a reversible random graph (i.e., almost surely a network with unit conductances on the edges).

The Euclidean and metric growth exponents coincide. If (G,ρ) is a reversible random network, then an L^2 *change of metric on* (G,ρ) is a random metric $\mathfrak{d} : V(G)\times V(G) \to [0,\infty)$ such that $(G,\rho,\mathfrak{d})$ is a reversible random network with $\mathbb{E}\,\mathfrak{d}(X_0,X_1)^2 < \infty$.

Definition 1.7 (Metric growth exponent)**.** The *metric growth exponent* $d^\star_f$ *of* (G,ρ) is the infimal value d such that almost surely $\bar{d}_f(V(G),\mathfrak{d}) \leqslant d$ (recall (1.1)), where the infimum is taken over all L^2 changes of metric on (G,ρ).

Theorem 1.8 (Euclidean vs. metric growth exponent)**.** *If (G,ρ) is a reversible random network satisfying $\mathbb{E}[1/\kappa_\rho] < \infty$, then*

$$d^\star_{\mathrm{euc}} = d^\star_f \,.$$

This result is proved in Section 4 using Bourgain's embedding method [Bou85] to construct embeddings of G into a Hilbert space. The following corollary of Theorem 1.3 and Theorem 1.8 has so far had the most utility in studying concrete models.

Corollary 1.9. *If (G,ρ) is a reversible random network satisfying $\mathbb{E}[1/\kappa_\rho] < \infty$, then $\bar{d}_s \leqslant d^\star_f$. If, moreover, $\bar{d}_s = \underline{d}_s^{\mathcal{A}}$, then almost surely the spectral dimension of G is $d^\star_f$.*

Indeed, a weaker version of the first half of Corollary 1.9 appears in [Lee21a], where it is stated for unimodular random graphs.

If (G,ρ) is a unimodular random graph with $\mathbb{E}[\deg_G(\rho)] < \infty$, then there is a random rooted graph $(\tilde{G},\tilde{\rho})$ whose law is absolutely continuous with respect to that of (G,ρ) and such that $(\tilde{G},\tilde{\rho})$ is a reversible random graph (see [BC12] and [Lee21b, §1.1]). Thus Corollary 1.9 is is somewhat stronger the results of [Lee21a] which requires the law of $\deg_G(\rho)$ to have tails that decrease faster than any inverse polynomial. Moreover, our definition of $d^\star_f$ only requires almost sure asymptotic control on the volume of balls around the root, while [Lee21a] requires almost sure control at all scales.

While the added generality is a benefit, certainly the main advantage of our probabilistic proof of Corollary 1.9 is that it is substantially simpler and more elegant than the spectral graph-theoretic arguments in [Lee21a].

Applications of Corollary 1.9. Suppose $\{G_n\}$ is a sequence of finite random networks and $\rho_n \in V(G_n)$ has the law of the stationary measure on G_n. It is not difficult to check that each (G_n,ρ_n) is a reversible random network (indeed, (G_n, X_0, X_1) and (G_n, X_1, X_0) have the same law conditioned on G_n).

For $R \geqslant 0$, denote $B^{G_n}(\rho_n, R) := \{v \in V(G_n) : \mathrm{dist}_{G_n}(\rho_n, v) \leqslant R\}$, where dist denotes the path distance in G_n. One says that a random rooted network (G,ρ) is the *distributional limit* of $\{(G_n,\rho_n\})$ if the law of $(B^{G_n}(\rho_n,R),\rho_n)$ (considered as a rooted network, up to rooted network isomorphism) converges to the law of $B^G(\rho,R)$ for every $R \geqslant 0$. Let us write $\{(G_n,\rho_n)\} \Rightarrow (G,\rho)$ to denote such convergence.

If (G,ρ) is the distributional limit of $\{(G_n,\rho_n)\}$, then (G,ρ) is a reversible random network. One can consult [AL07] and [BS01] in the unimodular setting, and [BC12] for the connection between unimodular and reversible random graphs. Thus distributional limits of finite networks provide a rich family of reversible random networks.

Using Corollary 1.9, the following theorems offer an interesting interplay between probability and geometry.

Theorem 1.10 ([Lee21a])**.** *Suppose $\{(G_n,\rho_n)\}$ is a sequence of reversible random networks, where each G_n is almost surely a finite planar graph. If $\{(G_n,\rho_n)\} \Rightarrow (G,\rho)$, then the metric growth exponent of (G,ρ) satisifes $d^\star_f \leqslant 2$.*

This theorem shows that many well-studied models of random planar maps have spectral dimension at most 2. For instance, it applies to the uniform infinite planar triangulation (UIPT) [AS03]. This upper bound was established in [Lee21a] and [GM21] later showed that the spectral dimension is almost surely equal to 2. For the uniform infinite planar quadrangulation (UIPQ) (see, e.g., [Kri08]), it implies that the spectral dimension is almost surely at most 2 (if the a.s. limit exists), but presently a matching lower bound remains elusive.

Theorem 1.10 can be generalized in two ways: To broader notions of 2-dimensional graphs, or to higher dimensions.

Theorem 1.11 (Exclued minors, [Lee21a]). *Let H be a fixed finite graph. Suppose $\{(G_n, \rho_n)\}$ is a sequence of reversible random networks, where each G_n is almost surely a finite graph excluding H as a graph minor. If $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$, then the metric growth exponent of (G, ρ) satisifes $d_f^\star \leqslant 2$.*

Say that G *sphere packs in* $\mathbb{R}^d$ if G is the intersection graph of interior-disjoint closed balls in $\mathbb{R}^d$.

Theorem 1.12 ([Lee18]). *Suppose $\{(G_n, \rho_n)\}$ is a sequence of reversible random networks, where each G_n almost surely sphere-packs in $\mathbb{R}^d$. If $\{(G_n, \rho_n)\} \Rightarrow (G, \rho)$, then the metric growth exponent of (G, ρ) satisifes $d_f^\star \leqslant d$.*

This extends substantially to the much more general setting of "quasi-packings" of graphs in any Ahlfors d-regular metric measure space [Lee18].

2 Spectral dimension and Euclidean growth

In order to prove Theorem 1.3, we will extend K. Ball's notion of Markov type 2 [Bal92] from finite state spaces to reversible random networks.

Theorem 2.1. *Suppose that (G, ρ) is a reversible random network and $\Psi : V(G) \to \mathcal{H}$ is a proper Euclidean embedding of (G, ρ). Then it holds that*

$$\mathbb{E} \max_{0 \leqslant t \leqslant n} \|\Psi(X_n) - \Psi(X_0)\|_{\mathcal{H}}^2 \leqslant 128n\, \mathbb{E}\, \|\Psi(X_0) - \Psi(X_1)\|_{\mathcal{H}}^2.$$

This theorem is proved in Section 2.1. For now, we record a corollary and then use it to establish Theorem 1.3.

Corollary 2.2. *Suppose (G, ρ) is a reversible random network and $\Psi : V(G) \to \mathcal{H}$ is a proper Euclidean embedding of (G, ρ). Then almost surely it holds that*

$$\Psi(X_0), \Psi(X_1), \ldots, \Psi(X_n) \in B_{\mathcal{H}}\left(\Psi(X_0), \sqrt{n} \log n\right)$$

for all but finitely many n.

Proof. Consider the reversible random network (G, ρ, Ψ). Define $K := \mathbb{E}\, \|\Psi(X_0) - \Psi(X_1)\|_{\mathcal{H}}^2$, and recall that $K < \infty$ since Ψ is a proper Euclidean embedding. By Theorem 2.1, we have $\mathbb{E} \max_{0 \leqslant t \leqslant n} \|\Psi(X_n) - \Psi(X_0)\|_{\mathcal{H}}^2 \leqslant 128Kn$ for $n \geqslant 1$.

For $n \geqslant 1$, denote the events

$$\mathcal{E}_n := \left\{\Psi(X_0), \Psi(X_1), \ldots, \Psi(X_n) \in B_{\mathcal{H}}\left(\Psi(X_0), \sqrt{n/2} \log(n/2)\right)\right\} .$$

For n sufficiently large, it holds that

$$\neg\mathcal{E}_n \implies \max_{0 \leqslant t \leqslant n} \|\Psi(X_n) - \Psi(X_0)\|_{\mathcal{H}}^2 > \tfrac{1}{4} n (\log n)^2,$$

and then Markov's inequality gives

$$\mathbb{P}(\neg\mathcal{E}_n) \leqslant \frac{512K}{(\log n)^2}\,.$$

Therefore, by the Borel-Cantelli Lemma, it holds that almost surely $\mathcal{E}_{2^k}$ occurs for all but finitely many k, yielding the desired claim. □

Lemma 2.3. *Suppose $(G,\rho,\mathcal{S})$ is a reversible random network, where $\mathcal{S} \subseteq V(G)$ is a random finite set of vertices. Then almost surely, for any $n \geqslant 0$,*

$$\frac{\mathsf{p}_{2n}(\rho,\rho)}{\kappa_\rho} \geqslant \frac{\left(\mathbb{P}[X_n \in \mathcal{S} \mid (G,\rho,\mathcal{S})]\right)^2}{\mathrm{vol}(\mathcal{S})}\,.$$

Proof. Using reversibility of the random walk (conditioned on $(G,\rho,\mathcal{S})$), we have $\kappa_\rho \mathsf{p}_n(\rho,x) = \kappa_x \mathsf{p}_n(x,\rho)$, and this gives

$$\begin{aligned}
\mathsf{p}_{2n}(\rho,\rho) \geqslant \sum_{x\in\mathcal{S}} \mathsf{p}_n(\rho,x)\mathsf{p}_n(x,\rho) &= \kappa_\rho \sum_{x\in\mathcal{S}} \frac{\mathsf{p}_n(\rho,x)^2}{\kappa_x} \\
&\geqslant \kappa_\rho \frac{\left(\sum_{x\in\mathcal{S}} \mathsf{p}_n(\rho,x)\right)^2}{\mathrm{vol}(\mathcal{S})} \\
&= \kappa_\rho \frac{\left(\mathbb{P}[X_n \in \mathcal{S} \mid (G,\rho,\mathcal{S})]\right) 2}{\mathrm{vol}(\mathcal{S})}\,,
\end{aligned}$$

where the second inequality is an application of Cauchy-Schwarz. □

The next result implies Theorem 1.3.

Corollary 2.4. *Suppose (G,ρ) is a reversible random network and $\Psi : V(G) \to \mathcal{H}$ is a proper Euclidean embedding of (G,ρ). Then almost surely,*

$$\limsup_{n\to\infty} \frac{-2\log \mathsf{p}_{2n}(\rho,\rho)}{\log n} \leqslant \limsup_{R\to\infty} \frac{\log \mathrm{vol}\left(\Psi^{-1}(B_{\mathcal{H}}(0,R))\right)}{\log R}. \tag{2.1}$$

Proof. Define $\mathcal{S}_n := \Psi^{-1}\left(B_{\mathcal{H}}(\Psi(X_0), \sqrt{n}\log n)\right)$. By Corollary 2.2, it holds that almost surely, for n sufficiently large, $X_n \in \mathcal{S}_n$. Applying Lemma 2.3 to $(G,\rho,\mathcal{S}_{2n})$ gives $\mathsf{p}_{2n}(\rho,\rho) \geqslant \kappa_\rho/\mathrm{vol}(\mathcal{S}_{2n})$ for n sufficiently large, and therefore, almost surely,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{-2\log \mathsf{p}_{2n}(\rho,\rho)}{\log n} &\leqslant \limsup_{R\to\infty} \frac{2\log \mathrm{vol}\left(\Psi^{-1}\left(B_{\mathcal{H}}(\Psi(X_0), \sqrt{2n}\log(2n))\right)\right)}{\log n} \\
&\leqslant \limsup_{R\to\infty} \frac{\log \mathrm{vol}\left(\Psi^{-1}\left(B_{\mathcal{H}}(\Psi(X_0), R)\right)\right)}{\log R} \\
&= \limsup_{R\to\infty} \frac{\log \mathrm{vol}\left(\Psi^{-1}\left(B_{\mathcal{H}}(0, R)\right)\right)}{\log R}\,.
\end{aligned}$$ □

2.1 Maximal Markov type and reversible random graphs

Our goal now is to prove Theorem 2.1. Consider a reversible random network (G, ρ, ξ), and recall that $\{X_t\}$ is the random walk on G with $X_0 = \rho$. Fix a time $n \geqslant 0$, and denote the reversed walk $\tilde{X}_t := X_{n-t}$ for $0 \leqslant t \leqslant n$. The next lemma follows from reversibility, since $(G, X_0, X_1, \ldots, X_n, \xi)$ and $(G, \tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_n, \xi)$ have the same law.

Lemma 2.5. *For $s \in \{1, \ldots, n-1\}$, the law of X_{s+1} conditioned on $\{(G, X_0, \xi), X_s = w, X_0 = u, x_n = v\}$ is equal to the law of $\tilde{X}_{n-(s+1)}$ conditioned on $\{(G, \tilde{X}_0, \xi), \tilde{X}_{n-s} = w, \tilde{X}_0 = u, \tilde{X}_n = v\}$.*

We now adapt the forward-backward martingale decomposition from [NPSS06] to our setting.

Lemma 2.6. *Let $\mathfrak{X}$ be a normed space and let (G, ρ, Ψ) be a reversible random network with $\Psi : V(G) \to \mathfrak{X}$. Then for every $n \geqslant 0$, there are two $\mathfrak{X}$-valued processes $\{A_t : t = 0, 1, \ldots, n\}$ and $\{B_t : t = 0, 1, \ldots, n\}$ with the following properties:*

1. *Almost surely, conditioned on (G, X_0, X_n, Ψ), both $\{A_t\}$ and $\{B_t\}$ are martingales.*
2. *It holds that almost surely, for all $t \in \{0, 1, \ldots, n\}$,*
$$\Psi(X_{2t}) - \Psi(X_0) = A_t - B_t\,.$$
3. *For every $1 \leqslant t \leqslant n$ and $q \geqslant 1$,*
$$\max\left\{\mathbb{E}\,\|A_t - A_{t-1}\|_{\mathfrak{X}}^q, \mathbb{E}\,\|B_t - B_{t-1}\|_{\mathfrak{X}}^q\right\} \leqslant 2^q\,\mathbb{E}\,\|\Psi(X_0) - \Psi(X_1)\|_{\mathfrak{X}}^q\,.$$

Proof. Fix $n \geqslant 0$ and let us condition on (G, X_0, X_{2n}, Ψ). We write $\mathbb{E}_{(G,u,v,\Psi)}$ for the expectation conditioned on $\{(G, u, \Psi), X_0 = u, X_{2n} = v\}$. Define the reversed walk by $\tilde{X}_t := X_{2n-t}$ for $t = 0, 1, \ldots, 2n$.

Define $M_0 := \Psi(X_0)$ and $N_0 := \Psi(\tilde{X}_0)$, and for $0 \leqslant s \leqslant 2n-1$,

$$M_{s+1} - M_s := \Psi(X_{s+1}) - \Psi(X_s) - \mathbb{E}_{(G,X_0,X_{2n},\Psi)}[\Psi(X_{s+1}) - \Psi(X_s) \mid X_s] \tag{2.2}$$
$$N_{s+1} - N_s := \Psi(\tilde{X}_{s+1}) - \Psi(\tilde{X}_s) - \mathbb{E}_{(G,\tilde{X}_0,\tilde{X}_{2n},\Psi)}[\Psi(\tilde{X}_{s+1}) - \Psi(\tilde{X}_s) \mid \tilde{X}_s]\,. \tag{2.3}$$

Observe that, conditioned on (G, X_0, X_{2n}, Ψ), it holds that $\{M_s\}$ is a martingale with respect to the filtration induced on $\{X_0, X_1, \ldots, X_{2n}\}$, and $\{N_s\}$ is a martingale with respect to the filtration induced on $\{\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_{2n}\}$.

Observing that $X_s = \tilde{X}_{2n-s}$, by Lemma 2.5, it holds that

$$\mathbb{E}_{(G,X_0,X_{2n},\Psi)}[\Psi(X_{s+1}) - \Psi(X_s) \mid X_s] = \mathbb{E}_{(G,\tilde{X}_0,\tilde{X}_{2n},\Psi)}\left[\Psi(\tilde{X}_{2n-(s+1)}) - \Psi(\tilde{X}_{2n-s}) \mid \tilde{X}_{2n-s}\right],$$

and therefore

$$\begin{aligned}\Psi(X_{s+1}) - \Psi(X_{s-1}) &= \Psi(X_{s+1}) - \Psi(X_s) - \mathbb{E}_{(G,X_0,X_{2n},\Psi)}[\Psi(X_{s+1}) - \Psi(X_s) \mid X_s] \\ &\quad - \left(\Psi(\tilde{X}_{2n-s-1}) - \Psi(\tilde{X}_{2n-s}) - \mathbb{E}_{(G,\tilde{X}_0,\tilde{X}_{2n},\Psi)}[\Psi(\tilde{X}_{2n-s-1}) - \Psi(\tilde{X}_{2n-s}) \mid \tilde{X}_{2n-s}]\right) \\ &= (M_{s+1} - M_s) - (N_{2n-s+1} - N_{2n-s}).\end{aligned} \tag{2.4}$$

Define the processes $\{A_t : t = 0, 1, \ldots, n\}$ and $\{B_t : t = 0, 1, \ldots, n\}$ by

$$A_t := \sum_{s=1}^{t} M_{2s} - M_{2s-1}$$

$$B_t := \sum_{s=1}^{t} N_{2n-s+1} - N_{2n-s} ,$$

and observe that almost surely, conditioned on (G, X_0, X_{2n}, Ψ), $\{A_t\}$ and $\{B_t\}$ are also martingales with respect to the forward and backward filtrations, respectively. Summing (2.4) over odd values $s = 1, 2, \ldots, 2t-1$ gives claim (2) of the lemma.

For claim (3), note that $A_t - A_{t-1} = M_{2t} - M_{2t-1}$, and then employ (2.2). An identical argument holds for $B_t - B_{t-1}$ using (2.3). $\square$

Let us now use this to prove Theorem 2.1.

Proof of Theorem 2.1. Note that if $\{M_t\}$ is a martingale in some normed space $\mathfrak{X}$, then $\{\|M_t\|_{\mathfrak{X}}\}$ is a submartingale, and Doob's L^2 maximal inequality yields

$$\mathbb{E} \max_{0 \leqslant t \leqslant n} \|M_t\|_{\mathfrak{X}}^2 \leqslant 4 \, \mathbb{E} \, \|M_n\|_{\mathfrak{X}}^2.$$

Therefore applying Lemma 2.6(2) gives

$$\mathbb{E} \max_{0 \leqslant t \leqslant n} \|\Psi(X_{2t}) - \Psi(X_0)\|_{\ell^2}^2 \leqslant 2 \, \mathbb{E} \max_{0 \leqslant t \leqslant n} (\|A_t\|_{\mathcal{H}}^2 + \|B_t\|_{\mathcal{H}}^2) \leqslant 8 \left(\mathbb{E} \, \|A_n\|_{\mathcal{H}}^2 + \mathbb{E} \, \|B_n\|_{\mathcal{H}}^2 \right).$$

Now using orthogonality of martingale difference sequences, we have

$$\mathbb{E} \, \|A_n\|_{\mathcal{H}}^2 \leqslant \sum_{t=1}^{n} \mathbb{E} \, \|A_t - A_{t-1}\|_{\mathcal{H}}^2 \leqslant 4n \, \mathbb{E} \, \|\Psi(X_0) - \Psi(X_1)\|_{\mathcal{H}}^2,$$

where the last inequality uses Lemma 2.6(3). Applying the same reasoning to $\{B_t\}$ then gives us

$$\mathbb{E} \max_{0 \leqslant t \leqslant n} \|\Psi(X_{2t}) - \Psi(X_0)\|_{\ell^2}^2 \leqslant 64n \, \mathbb{E} \, \|\Psi(X_0) - \Psi(X_1)\|_{\mathcal{H}}^2 .$$

Finally, to handle odd times, one notes that

$$\|\Psi(X_{2n+1}) - \Psi(X_0)\|_{\mathcal{H}} \leqslant \|\Psi(X_{2n+1}) - \Psi(X_{2n})\|_{\mathcal{H}} + \|\Psi(X_{2n}) - \Psi(X_0)\|_{\mathcal{H}},$$

and therefore

$$\begin{aligned} \mathbb{E} \, \|\Psi(X_{2n+1}) - \Psi(X_0)\|_{\mathcal{H}}^2 &\leqslant 128n \left(\mathbb{E} \, \|\Psi(X_0) - \Psi(X_1)\|_{\mathcal{H}}^2 + \mathbb{E} \, \|\Psi(X_{2n+1}) - \Psi(X_{2n})\|_{\mathcal{H}}^2 \right) \\ &= 256n \, \mathbb{E} \, \|\Psi(X_0) - \Psi(X_1)\|_{\mathcal{H}}^2 , \end{aligned}$$

where the last equality uses that (G, X_{2n}, Ψ) and (G, X_{2n+1}, Ψ) have the same law since (G, ρ, Ψ) is reversible. $\square$

3 Change of metric via heat flow

Our objective is now to prove Theorem 1.5. First, we need to review some basics of spectral theory on infinite graphs.

3.1 Spectral theory on locally-finite networks

Fix a network G with conductances κ, and write $\ell^2(G)$ for the Hilbert space of functions $f : V(G) \to \mathbb{R}$ equipped with the inner product

$$\langle f, g \rangle_{\ell^2(G)} := \sum_{u \in V(G)} \kappa_u f(u) g(u).$$

Define the averaging operator $P : \ell^2(G) \to \ell^2(G)$ by

$$P\psi(u) := \sum_{v:\{u,v\} \in E(G)} \frac{\kappa_{uv}}{\kappa_u} \psi(v)\,.$$

Observe that P is self-adjoint with respect to the $\ell^2(G)$ inner product:

$$\langle \varphi, P\psi \rangle_{\ell^2(G)} = \sum_{x \in V(G)} \kappa_x \varphi(x) \sum_{y:\{x,y\} \in E(G)} \frac{\kappa_{xy}}{\kappa_x} \psi(y) = 2 \sum_{\{x,y\} \in E(G)} \kappa_{xy} \varphi(x) \psi(y).$$

Since P is an averaging operator, it is also bounded, and therefore the spectral theorem yields a resolution of the identity I_P so that $P = \int_{-1}^{1} \lambda \, dI_P(\lambda)$.

Given a vertex $v \in V(G)$, one defines the associated *spectral measure* μ_G^v *at* v by

$$\mu_G^v((-\infty, \lambda)) := \frac{\langle \mathbb{1}_v, I_P((-\infty, \lambda)) \mathbb{1}_v \rangle_{\ell^2(G)}}{\kappa_v}\,.$$

This is the unique probability measure μ_G^v on $[-1, 1]$ such that for all integers $n \geqslant 1$,

$$\kappa_v \int_{[-1,1]} \lambda^n \, d\mu_G^v(\lambda) = \langle \mathbb{1}_v, P^n \mathbb{1}_v \rangle_{\ell^2(G)}\,. \tag{3.1}$$

Note that for any $u, v \in V(G)$ and $n \geqslant 0$, we have

$$\langle P^n \mathbb{1}_u, P^n \mathbb{1}_v \rangle_{\ell^2(G)} = \left\langle \mathbb{1}_u, P^{2n} \mathbb{1}_v \right\rangle_{\ell^2(G)} = \kappa_u \mathsf{p}_{2n}^G(u, v). \tag{3.2}$$

Note also that $P\mathbb{1}_u = \sum_{v:\{u,v\} \in E(G)} \frac{\kappa_{uv}}{\kappa_v} \mathbb{1}_v$, and therefore

$$\begin{aligned} \|P^n \mathbb{1}_u\|_{\ell^2(G)}^2 - \sum_{v:\{u,v\} \in E(G)} \kappa_{uv} \frac{\langle P^n \mathbb{1}_u, P^n \mathbb{1}_v \rangle_{\ell^2(G)}}{\kappa_v} &= \left\langle \mathbb{1}_u, (I - P) P^{2n} \mathbb{1}_u \right\rangle_{\ell^2(G)} \\ &= \kappa_u \int_{[-1,1]} (1 - \lambda) \lambda^{2n} \, d\mu_G^u(\lambda). \end{aligned} \tag{3.3}$$

Define the map $\Phi_n : V(G) \to \ell^2(G)$ by $\Phi_n(v) := \frac{P^n \mathbb{1}_v}{\kappa_v}$. Then for any $u \in V(G)$,

$$\begin{aligned}
&\sum_{v:\{u,v\}\in E(G)} \kappa_{uv} \left\|\Phi_n(u) - \Phi_n(v)\right\|^2_{\ell^2(G)} \\
&\quad = \frac{\|P^n \mathbb{1}_u\|^2_{\ell^2(G)}}{\kappa_u} + \sum_{v:\{u,v\}\in E(G)} \frac{\kappa_{uv}}{\kappa_v^2} \left\|P^n \mathbb{1}_v\right\|^2_{\ell^2(G)} - 2 \sum_{v:\{u,v\}\in E(G)} \frac{\kappa_{uv}}{\kappa_u \kappa_v} \langle P^n \mathbb{1}_u, P^n \mathbb{1}_v \rangle_{\ell^2(G)} \\
&\quad \overset{(3.3)}{=} 2 \int_{[-1,1]} (1-\lambda)\lambda^{2n} d\mu_G^u(\lambda) + \left(\sum_{v:\{u,v\}\in E(G)} \frac{\kappa_{uv}}{\kappa_v^2} \left\|P^n \mathbb{1}_v\right\|^2_{\ell^2(G)} - \frac{\|P^n \mathbb{1}_u\|^2_{\ell^2(G)}}{\kappa_u} \right).
\end{aligned} \tag{3.4}$$

Volume of spectral balls. For $u \in V(G)$, define the set

$$\mathcal{S}_n(u) := \left\{ v \in V(G) : \left\|\Phi_n(u) - \Phi_n(v)\right\|^2_{\ell^2(G)} \leqslant \frac{1}{2} \left\|\Phi_n(u)\right\|^2_{\ell^2(G)} \right\}.$$

Lemma 3.1. *If $v \in \mathcal{S}_n(u)$, then $\mathsf{p}_{2n}(v,u) \geqslant \frac{1}{4}\mathsf{p}_n(u,u)$.*

Proof. Note that $\langle x, y\rangle = \frac{1}{2}\left(\|x\|^2 + \|y\|^2 - \|x-y\|^2\right) \geqslant \frac{1}{4}\|x\|^2$ whenever $\|x-y\|^2 \leqslant \frac{1}{2}\|x\|^2$, hence $v \in \mathcal{S}_n(u)$ implies that

$$\langle \Phi_n(u), \Phi_n(v) \rangle_{\ell^2(G)} \geqslant \frac{1}{4} \left\|\Phi_n(u)\right\|^2_{\ell^2(G)} .$$

By definition, this gives

$$\frac{\langle P^n \mathbb{1}_u, P^n \mathbb{1}_v \rangle_{\ell^2(G)}}{\kappa_v} \geqslant \frac{1}{4} \frac{\|P^n \mathbb{1}_u\|^2_{\ell^2(G)}}{\kappa_u},$$

and by (3.2), this is precisely the inequality

$$\mathsf{p}_{2n}(v,u) \geqslant \frac{1}{4}\mathsf{p}_{2n}(u,u) .$$

□

Lemma 3.2. *For any $u \in V(G)$, it holds that*

$$\mathrm{vol}\left(\mathcal{S}_n(u)\right) \leqslant 16\kappa_u \frac{\mathsf{p}_{4n}(u,u)}{\mathsf{p}_{2n}(u,u)^2} \leqslant \frac{16\kappa_u}{\mathsf{p}_{2n}(u,u)}.$$

Proof. Using $\kappa_u \mathsf{p}_n(u,v) = \kappa_v \mathsf{p}_n(v,u)$, we have

$$\mathsf{p}_{4n}(u,u) \geqslant \sum_{v\in\mathcal{S}_n(u)} \mathsf{p}_{2n}(u,v)\mathsf{p}_{2n}(v,u) = \sum_{v\in\mathcal{S}_n(u)} \frac{\kappa_v}{\kappa_u} \mathsf{p}_{2n}(v,u)^2 \geqslant \frac{\mathrm{vol}(\mathcal{S}_n(u))}{16} \frac{\mathsf{p}_{2n}(u,u)^2}{\kappa_u},$$

where the last inequality follows from Lemma 3.1. This yields the first claimed inequality, and the second follows from monotonicity of the even return times: $\mathsf{p}_{2n}(u,u) \geqslant \mathsf{p}_{4n}(u,u)$. □

3.2 Construction of the metric

Note that in proving Theorem 1.5, we may assume that $\frac{\kappa_{uu}}{\kappa_u} \geqslant \frac{1}{2}$ for all $u \in V(G)$. This does not affect the spectral dimension of G, and it only affects the volume of sets by a constant factor. This assumption ensures that P is a nonnegative operator, and therefore the spectral measures μ_G^v are supported on $[0,1]$.

Consider now a reversible random network (G,ρ) with $\mathbb{E}[1/\kappa_\rho] < \infty$ and define the measure $\mu := \mathbb{E}\left[\mu_G^\rho/\kappa_\rho\right]$.

Lemma 3.3. *It holds that, for any $d \geqslant 1$ and $n \geqslant 1$,*

$$\frac{1}{2}\,\mathbb{E}\left[\|\Phi_n(X_0)-\Phi_n(X_1)\|^2_{\ell^2(G)}\right] \leqslant n^{-d} + \frac{d\log n}{n}\,\mathbb{E}\left[\mathsf{p}_{2n}(\rho,\rho)\right].$$

Proof. Dividing (3.4) by κ_u, setting $u = X_0$, and taking expectations gives

$$\frac{1}{2}\,\mathbb{E}\left[\|\Phi_n(X_0)-\Phi_n(X_1)\|^2_{\ell^2(G)}\right] = \int_{[0,1]}(1-\lambda)\lambda^{2n}\,d\mu(\lambda) + \frac{1}{2}\,\mathbb{E}\left[\frac{\|P^n\mathbb{1}_{X_1}\|^2_{\ell^2(G)}}{\kappa^2_{X_1}} - \frac{\left\|P^n\mathbb{1}_{X_0}\right\|^2_{\ell^2(G)}}{\kappa^2_{X_0}}\right].$$

Note that, from (3.2), it holds that $\frac{\|P^n\mathbb{1}_\rho\|^2_{\ell^2(G)}}{\kappa_\rho^2} = \frac{\mathsf{p}_{2n}(\rho,\rho)}{\kappa_\rho}$. Therefore since $\mathbb{E}[1/\kappa_\rho] < \infty$, the latter expectation is bounded and is, in fact, equal to 0 since (G,X_0) and (G,X_1) have the same law.

To finish, split the integral into two pieces depending on whether $\lambda \leqslant 1 - \frac{d\log n}{n}$:

$$\begin{aligned}\int_{[0,1]}(1-\lambda)\lambda^{2n}\,d\mu(\lambda) &\leqslant \left(1-\frac{d\log n}{n}\right)^{2n} + \frac{d\log n}{n}\int_{[0,1]}\lambda^{2n}\,d\mu(\lambda)\\ &\overset{(3.1)}{\leqslant} n^{-d} + \frac{d\log n}{n}\,\mathbb{E}[\mathsf{p}_{2n}(\rho,\rho)]\,.\end{aligned}$$

□

For an integer $k \geqslant 1$, define

$$\begin{aligned}\mathfrak{d}_k(x,y) &:= \left\|\Phi_{2^k}(x)-\Phi_{2^k}(y)\right\|_{\ell^2(G)}\\ W_k &:= \mathbb{E}[\mathfrak{d}_k(X_0,X_1)^2],\end{aligned}$$

and then

$$\mathfrak{d}(x,y) := \sqrt{\sum_{k\geqslant 1}\frac{1}{k^2W_k}\mathfrak{d}_k(x,y)^2}\,.$$

By construction we have $\mathbb{E}[\mathfrak{d}(X_0,X_1)^2] \leqslant \sum_{k\geqslant 1}k^{-2} < \infty$, hence $\mathfrak{d}$ is an L^2 change of metric. The next result implies Theorem 1.5.

Theorem 3.4. *If $\bar{d}_s - \underline{d}_s^{\mathcal{A}} < 2$, then*

$$\limsup_{R\to\infty}\frac{\log\mathrm{vol}(B_{\mathfrak{d}}(\rho,R))}{\log R} \leqslant \frac{\bar{d}_s}{1+(\underline{d}_s^{\mathcal{A}}-\bar{d}_s)/2}\,.$$

Proof. By Lemma 3.3, for any $d \geqslant 1$, we have

$$W_k \leqslant 2^{-kd} + dk2^{-k}\,\mathbb{E}[\mathsf{p}_{2^{k+1}}(\rho,\rho)]\,.$$

Choosing $d := 2\underline{d}_s^{\mathcal{A}}$, by definition of $\underline{d}_s^{\mathcal{A}}$, this implies that

$$W_k \leqslant 2^{-k}2^{-k(\underline{d}_s^{\mathcal{A}}-o(1))/2}, \quad k \to \infty. \tag{3.5}$$

By construction, it holds that

$$\mathfrak{d}(\rho,x)^2 \geqslant \frac{1}{k^2 W_k}\left\|\Phi_{2^k}(\rho)-\Phi_{2^k}(x)\right\|_{\ell^2(G)}^2.$$

Therefore for any $\varepsilon > 0$,

$$x \in B_{\mathfrak{d}}\left(\rho, \varepsilon\frac{2^{k/2}}{k}\right) \implies \left\|\Phi_{2^k}(\rho)-\Phi_{2^k}(x)\right\|_{\ell^2(G)}^2 \leqslant \varepsilon^2 2^k W_k\,.$$

Choosing

$$\varepsilon_k := 2^{-(k+1)/2}\frac{\left\|\Phi_{2^k}(\rho)\right\|_{\ell^2(G)}}{\sqrt{W_k}}$$

gives $x \in B_{\mathfrak{d}}\left(\rho, \varepsilon_k\frac{2^{k/2}}{k}\right) \implies \left\|\Phi_{2^k}(\rho)-\Phi_{2^k}(x)\right\|_{\ell^2(G)}^2 \leqslant \frac{1}{2}\left\|\Phi_{2^k}(\rho)\right\|_{\ell^2(G)}^2$ and then Lemma 3.2 implies that

$$\mathrm{vol}\left(B_{\mathfrak{d}}\left(\rho, \varepsilon_k\frac{2^{k/2}}{k}\right)\right) \leqslant \mathrm{vol}(\mathcal{S}_{2^k}(\rho)) \leqslant \frac{16\kappa_\rho}{\mathsf{p}_{2^{k+1}}(\rho,\rho)}.$$

By definition of $\bar{d}_s$, it holds that almost surely

$$\frac{\mathsf{p}_{2^k}(\rho,\rho)}{\kappa_\rho} \geqslant 2^{-k\bar{d}_s/2-o(1)}, \quad k \to \infty,$$

which implies that almost surely

$$\mathrm{vol}\left(B_{\mathfrak{d}}\left(\rho, \varepsilon_k\frac{2^{k/2}}{k}\right)\right) \leqslant 2^{k\bar{d}_s/2+o(1)}, \quad k \to \infty. \tag{3.6}$$

Moreover, using $\left\|\Phi_{2^k}(\rho)\right\|_{\ell^2(G)}^2 = \mathsf{p}_{2^{k+1}}(\rho,\rho)/\kappa_\rho$ (cf. (3.2)) and (3.5), we have that almost surely

$$\varepsilon_k \geqslant 2^{k(\underline{d}_s^{\mathcal{A}}-\bar{d}_s-o(1))/4}, \quad k \to \infty. \tag{3.7}$$

When $\bar{d}_s - \underline{d}_s^{\mathcal{A}} < 2$, then $\varepsilon_k\frac{2^{k/2}}{k} \to \infty$ and combining (3.6) and (3.7) gives

$$\limsup_{R\to\infty}\frac{\log\mathrm{vol}(B_{\mathfrak{d}}(\rho,R))}{\log R} \leqslant \frac{\bar{d}_s}{1+(\underline{d}_s^{\mathcal{A}}-\bar{d}_s)/2}. \qquad \square$$

3.3 Lower bound examples

We review constructions based on a technique from [AHNR18], where the authors add random tails to the vertices of a 3-regular tree. These show that, without the assumption $\underline{d}_s^{\mathcal{A}} = \bar{d}_s$, the reverse inequality $d_{\mathrm{euc}}^{\star} \leqslant \bar{d}_s$ can fail (recall Theorem 1.3).

Let $\{p_\ell : \ell \geqslant 1\}$ denote a probability distribution on positive integers with bounded first moment: $\sum_{\ell \geqslant 1} p_\ell \cdot \ell < \infty$. Define the probability distribution $\{\tilde{p}_\ell : \ell \geqslant 1\}$ by

$$\tilde{p}_\ell = \frac{p_\ell(\ell+1)}{\sum_{k \geqslant 1} p_k(k+1)}.$$

Consider a connected, locally-finite, vertex-transitive graph H and fix a vertex $v_0 \in V(H)$. Let $\{L_v : v \in V(H)\}$ denote an independent family of random variables where L_v has law p for $v \neq v_0$ and L_{v_0} has law $\tilde{p}$.

Definition 3.5 (Adding tails)**.** Let $\tilde{H}$ be the random graph that results from attaching a path P_v of length L_v to every $v \in V(H)$, and let $\rho \in V(P_{v_0})$ be a random vertex with law

$$\mathbb{P}[\rho = v] = \frac{\deg_{\tilde{H}}(v)}{\sum_{u \in V(P_{v_0})} \deg_{\tilde{H}}(u)}.$$

The next lemma is a straightforward exercise; see [Lee21a, §4.4] where H is a complete 3-ary tree.

Lemma 3.6. *If (H, v_0) is the distributional limit of a sequence of finite reversible random graphs, then $(\tilde{H}, \rho)$ is a reversible random graph.*

We will also consider $H = \mathbb{Z}^d$ for some $d \geqslant 3$. Note that since H is amenable, one can take a Følner sequence $\{S_n \subseteq V(H)\}$. Let $\rho_n \in S_n$ have the law of the stationary measure on the induced graph $H[S_n]$, and then $\{(H[S_n], \rho_n)\} \Rightarrow (H, 0)$.

Let us define

$$p_\ell := c\ell^{-2}(\log(\ell+1))^{-2}, \quad \ell \geqslant 1, \tag{3.8}$$

where the constant $c > 0$ is chosen so that p is a probability.

Lemma 3.7. *There is a constant $C > 0$ such that when H is the infinite 3-regular tree or $H = \mathbb{Z}^d$ for $d \geqslant 1$, the spectral dimension of $(\tilde{H}, \rho)$ satisfies $\bar{d}_s \leqslant C$.*

Proof sketch. From the definition of p, we see that if $|B^H(v_0, R)| \gg n$, then there is likely some $v^* \in B^H(v_0, R)$ with $L_{v^*} \geqslant n^{1/2-o(1)}$. And in such a tail, the random walk has constant probability to get trapped for at least L_{v^*} steps. In the 3-regular tree, a ball of radius $O(\log n)$ likely contains a vertex v^* whose tail has length at least $\sqrt{n}$. The random walk goes to v^*, remains in the tail for time $\approx n$ and then returns to v_0, all with probability at least n^{-K} for some (large) constant $K > 0$ (because $\mathrm{dist}_H(v_0, v^*) \leqslant O(\log n)$).

For $H = \mathbb{Z}^d$, standard estimates of the Green function show that when $R \gg n^{1/d}$, there is likely some $v^* \in B^H(0, R)$ with $L_{v^*} \geqslant \sqrt{n}$, and the probability for the random walk on H to start at 0, visit v^*, and then return to 0 is roughly $R^{-2(d-2)} \geqslant n^{-2-o(1)}$, yielding a uniform upper bound on $\bar{d}_s$. □

On the other hand, we have the following.

Lemma 3.8. *When H is the infinite 3-regular tree, then $(\tilde{H}, \rho)$ satisfies $d^{\star}_{\mathrm{euc}} = \infty$. When $H = \mathbb{Z}^d$ for $d \geqslant 1$, then $(\tilde{H}, \rho)$ satisfies $d^{\star}_{\mathrm{euc}} = d$.*

Since $d^{\star}_{\mathrm{euc}} < \infty$ entails amenability, the result for the 3-regular tree follows. For $H = \mathbb{Z}^d$, one can use the following.

Lemma 3.9. *If $H = \mathbb{Z}^d$ for $d \geqslant 1$, then the Euclidean growth exponent of $(\tilde{H}, \rho)$ satisfies $d^{\star}_{\mathrm{euc}} = d$.*

Proof. Note that for an integer $R \geqslant 1$, the law of $|B^{\tilde{H}}(\rho, R)|$ is equal to the law of

$$L_0 + \sum_{j=1}^{N} L_j ,$$

where $\{L_j\}$ are independent random variables where L_0 has law $\tilde{p}$ and L_j has law p for $j \geqslant 1$, and $N = |B^H(0, R)|$. Since p has bounded first moments, by the strong law of large numbers, almost surely

$$\lim_{R \to \infty} \frac{\log |B^{\tilde{H}}(\rho, R)|}{\log R} = d \,.$$

It follows that $d^{\star}_{\mathrm{euc}} \leqslant d$ can be obtained from the mapping $\Psi : V(\tilde{H}) \to \mathbb{R}^d$ given by $\Psi(v) = \hat{v}$.

To establish that $d^{\star}_{\mathrm{euc}} \geqslant d$, it suffices to prove the same for the rooted subgraph $(H, 0)$. Consider a proper Euclidean embedding $\Psi : \mathbb{Z}^d \to \mathcal{H}$, where $\mathcal{H}$ is a separable Hilbert space. Let (Ω, μ) be probability space underlying $(H, 0, \Psi)$ and denote by $\bar{\mathcal{H}}$ the Hilbert space of measurable functions $f : \Omega \to \mathcal{H}$ with norm $\|f\|_{\bar{\mathcal{H}}} := \sqrt{\int \|f(x)\|^2_{\mathcal{H}} \, d\mu(x)}$. Since (Ω, μ, Ψ) is reversible and $\mathbb{E} \, \|\Psi(X_0) - (X_1)\|^2_{\mathcal{H}} < \infty$, it holds that the map $F : \mathbb{Z}^d \to \bar{\mathcal{H}}$ given by $(F(x))(\omega) = (\Psi(\omega))(x)$ satisfies

$$\mathbb{E}\left[\|F(X_0) - F(X_1)\|^2_{\bar{\mathcal{H}}} \mid X_0 = x \right] = \mathbb{E} \, \|\Psi(X_0) - \Psi(X_1)\|^2_{\mathcal{H}} , \tag{3.9}$$

and

$$\limsup_{R \to \infty} \frac{\log \left|\Psi^{-1}\left(B_{\mathcal{H}}(0, R)\right)\right|}{\log R} = \limsup_{R \to \infty} \frac{\log \left|F^{-1}\left(B_{\bar{\mathcal{H}}}(0, R)\right)\right|}{\log R} \,. \tag{3.10}$$

Define $C := \max_{\{u,0\} \in E(H)} \|F(u) - F(0)\|_{\bar{\mathcal{H}}}$ and note that $C < \infty$ follows from (3.9). Moreover, the triangle inequality in $\bar{\mathcal{H}}$ implies the path metric on $\mathbb{Z}^d$ satisfies

$$\mathrm{dist}_{\mathbb{Z}^d}(u, v) \geqslant C^{-1} \|F(u) - F(v)\|_{\bar{\mathcal{H}}} ,$$

and therefore (3.10) is at least d (the growth rate for the path metric on $\mathbb{Z}^d$). ◻

4 Euclidean embeddings and the metric growth exponent

Let us now prove Theorem 1.8.

4.1 The Mass-Transport Principle

Let $\mathcal{G}_\bullet$ denote the collection of isomorphism classes of rooted, connected, locally-finite networks, and let $\mathcal{G}_{\bullet\bullet}$ denote the collection of isomorphism classes of doubly-rooted, connected, locally-finite networks. We will consider functionals $F : \mathcal{G}_{\bullet\bullet} \to [0, \infty)$. Equivalently, these are functionals $F(G_0, x_0, y_0, \xi_0)$ that are invariant under automorphisms of ψ of G_0: $F(G_0, x_0, y_0, \xi_0) = F(\psi(G_0), \psi(x_0), \psi(y_0), \xi_0 \circ \psi^{-1})$.

The *mass-transport principle (MTP)* for a random rooted network (G, ρ, ξ) asserts that for any nonnegative Borel $F : \mathcal{G}_{\bullet\bullet} \to [0, \infty)$, it holds that

$$\mathbb{E}\left[\sum_{x\in V(G)} F(G, \rho, x, \xi)\right] = \mathbb{E}\left[\sum_{x\in V(G)} F(G, x, \rho, \xi)\right].$$

Unimodular random networks are precisely those that satisfy the MTP (see [AL07]).

Using the fact that biasing the law of a reversible random network (G, ρ, ξ) with $\mathbb{E}[1/\kappa_\rho] < \infty$ by $1/\kappa_\rho$ (see [BC12, Prop. 2.5]) yields a unimodular random network, one arrives at the following biased MTP.

Lemma 4.1. *If (G, ρ, ξ) is a reversible random network with $\mathbb{E}[1/\kappa_\rho] < \infty$, then for any nonnegative Borel functional $F : \mathcal{G}_{\bullet\bullet} \to [0, \infty)$, it holds that*

$$\mathbb{E}\left[\frac{1}{\kappa_\rho}\sum_{x\in V(G)} F(G, \rho, x, \xi)\right] = \mathbb{E}\left[\frac{1}{\kappa_\rho}\sum_{x\in V(G)} F(G, x, \rho, \xi)\right]. \tag{4.1}$$

We use the mass-transport principle to relate the cardinality of balls to their volume.

Lemma 4.2. *Suppose $(G, \rho, \mathfrak{d})$ is a reversible random network with $\mathbb{E}[1/\kappa_\rho] < \infty$. Then,*

$$\limsup_{R\to\infty} \frac{\log |B_{\mathfrak{d}}(\rho, R)|}{\log R} \leqslant \limsup_{R\to\infty} \frac{\log \operatorname{vol}(B_{\mathfrak{d}}(\rho, R))}{\log R}.$$

Proof. Define the mass transportation

$$F(G, x, y, \mathfrak{d}) := \kappa_x \frac{\mathbb{1}_{\{\mathfrak{d}(x,y)\leqslant R\}}}{\operatorname{vol}(B_{\mathfrak{d}}(y, R))}.$$

Then the mass transport principle (Lemma 4.1) gives

$$\mathbb{E}\left[\frac{|B_{\mathfrak{d}}(\rho, R)|}{\operatorname{vol}(B_{\mathfrak{d}}(\rho, 2R))}\right] \leqslant \mathbb{E}\left[\frac{1}{\kappa_\rho}\sum_{x\in V(G)} F(G, \rho, x, \mathfrak{d})\right] = \mathbb{E}\left[\frac{1}{\kappa_\rho}\sum_{x\in V(G)} F(G, x, \rho, \mathfrak{d})\right] = \mathbb{E}[1/\kappa_\rho].$$

Using Markov's inequality and the Borel-Cantelli lemma shows that almost surely $|B_{\mathfrak{d}}(\rho, 2^k)| \leqslant k^2 \operatorname{vol}(B_{\mathfrak{d}}(\rho, 2^{k+1}))$ holds for all but finitely many k, completing the proof. □

4.2 Choice of a separable Hilbert space

For a rooted graph (G,ρ), let us write $[G,\rho] \in \mathscr{G}_\bullet$ for its correpsonding rooted isomorphism class. As shown in [AL07, §2], there is a continuous map F from $\mathscr{G}_\bullet$ to the space of networks on vertex set $\mathbb{N}$ with root 0 such that (G,ρ) and $\left(F([G,\rho]), 0\right)$ are isomorphic. We may assume therefore that if (G,ρ) is a random reversible network, then $V(G)=\mathbb{N}$ and $\rho = 0$.

Let $\boldsymbol{\mathcal{U}} := \{\boldsymbol{U}_v : v \in \mathbb{N}\}$ be a family of i.i.d. uniform $[0,1]$ random variables, and let (Ω,μ) denote the underlying probability space. Define the Hilbert space $\mathcal{H}$ of measurable mappings $f : \Omega \to \ell_2$ with norm $\|f\|_{\mathcal{H}}^2 := \int \|f(\omega)\|_{\ell_2}^2 d\mu(\omega)$. Equivalently, we may envision $\mathcal{H}$ as the Hilbert space of ℓ_2-valued random variables on (Ω,μ) with $\|\boldsymbol{X}\|_{\mathcal{H}}^2 := \mathbb{E}\,\|\boldsymbol{X}\|_{\ell_2}^2$.

4.3 Construction of the embedding

The next result implies that there is a proper Euclidean embedding $\Psi : V(G) \to \mathcal{H}$ such that almost surely

$$\limsup_{R\to\infty} \frac{\log \mathrm{vol}\left(\Psi^{-1}(B_{\ell_2}(0,R))\right)}{\log R} \leqslant \limsup_{R\to\infty} \frac{\log |B_{\mathfrak{d}}(\rho,R)|}{\log R},$$

and combined with Lemma 4.2, this completes the proof of Theorem 1.8.

Theorem 4.3 (Euclidean embedding theorem)**.** *Suppose $(G,\rho,\mathfrak{d})$ is a reversible random network with $\mathbb{E}\,\mathfrak{d}(X_0,X_1)^2 < \infty$. Then there is a proper Euclidean embedding $\Psi : V(G) \to \mathcal{H}$ such that almost surely, for all $R \geqslant 0$,*

$$\|\Psi(\rho) - \Psi(x)\|_{\mathcal{H}} \geqslant \frac{\mathfrak{d}(\rho,x)}{(\log |B_{\mathfrak{d}}(\rho,2R)|)^{3/2}}, \qquad \forall x \in B_{\mathfrak{d}}(\rho,R).$$

Proof. As discussed in Section 4.2, let us assume that $V(G)=\mathbb{N}$ and $\rho=0$. For each integer $t \geqslant 1$, define the random set

$$\boldsymbol{V}_t := \{v \in \mathbb{N} : \boldsymbol{U}_v \leqslant 2^{-t}\},$$

and define the random map $\boldsymbol{\psi}_t : \mathbb{N} \to \mathbb{R}$ by

$$\boldsymbol{\psi}_t(x) := \frac{\mathfrak{d}(x,\boldsymbol{V}_t)}{t}.$$

Finally, define $\Psi : V(G) \to \mathcal{H}$ by

$$\Psi(v) := \left(\boldsymbol{\psi}_1(v), \boldsymbol{\psi}_2(v), \ldots\right).$$

Note that for any $u,v \in V(G)$,

$$\|\Psi(u)-\Psi(v)\|_{\mathcal{H}}^2 = \sum_{t\geqslant 1} t^{-2}\,\mathbb{E}\,|\boldsymbol{\psi}_t(u) - \boldsymbol{\psi}_t(v)|^2 \leqslant \sum_{t\leqslant 1} t^{-2}\mathfrak{d}(u,v)^2 \leqslant \frac{\pi^2}{6}\mathfrak{d}(u,v)^2.$$

Therefore $\mathbb{E}\,\|\Psi(X_0)-\Psi(X_1)\|_{\mathcal{H}}^2 \leqslant \frac{\pi^2}{6}\,\mathbb{E}\,\mathfrak{d}(X_0,X_1)^2 < \infty$, and $\Psi : V(G) \to \mathcal{H}$ is a proper Euclidean embedding.

Consider now $x \in B_{\mathfrak{d}}(\rho,R)$. For convenience, let us define the open ball

$$B_{\mathfrak{d}}^\circ(x,R) := \{y \in V(G) : \mathfrak{d}(x,y) < R\}.$$

For $t \geqslant 1$, let r_t be the smallest radius such that $\max\{|B_{\mathfrak{d}}(\rho, r_t)|, |B_{\mathfrak{d}}(x, r_t)|\} \geqslant 2^t$. Let t^* be the smallest value of t such that $r_t \geqslant \mathfrak{d}(\rho, x)/4$, and reassign $r_{t^*} := \mathfrak{d}(\rho, x)/4$. Denote $r_0 := 0$. Then by construction,

$$\frac{\mathfrak{d}(\rho, x)}{4} = (r_1 - r_0) + (r_2 - r_1) + (r_3 - r_2) + \cdots + (r_{t^*} - r_{t^*-1}) \,. \tag{4.2}$$

Consider now some $t \in \{1, 2, \ldots, t^*\}$. Note that, by definition of r_t, it holds that $|B_{\mathfrak{d}}(\rho, r_{t-1})| \geqslant 2^{t-1}$ or $|B_{\mathfrak{d}}(x, r_{t-1})| \geqslant 2^{t-1}$. Without loss of generality, assume this is achieved by ρ. It is also true that $|B^{\circ}_{\mathfrak{d}}(x, r_t)| < 2^t$, and therefore since $B_{\mathfrak{d}}(x, r_t)$ and $B_{\mathfrak{d}}(\rho, r_{t-1})$ are disjoint, there is some (universal) constant $q > 0$ such that

$$\mathbb{P}\left[\boldsymbol{V}_t \cap B_{\mathfrak{d}}(\rho, r_{t-1}) \neq \emptyset \wedge \boldsymbol{V}_t \cap B^{\circ}_{\mathfrak{d}}(x, r_t) = \emptyset \mid (G, \rho, \mathfrak{d})\right] \geqslant q \,.$$

In particular, we have

$$\mathbb{E}\left[|\boldsymbol{\psi}_t(\rho) - \boldsymbol{\psi}_t(x)|^2 \mid (G, \rho, \mathfrak{d})\right] \geqslant \frac{q}{t^2}(r_t - r_{t-1})^2,$$

and therefore almost surely,

$$\|\Psi(\rho) - \Psi(x)\|^2_{\mathcal{H}} \geqslant \frac{q}{(t^*)^2} \sum_{t=1}^{t^*} (r_t - r_{t-1})^2 \geqslant \frac{q}{(t^*)^3} \left(\sum_{t=1}^{t^*} (r_t - r_{t-1})\right)^2 \overset{(4.2)}{=} \frac{q}{16(t^*)^3} \mathfrak{d}(\rho, x)^2 \,,$$

where the second inequality is an application of Cauchy-Schwarz. Since $t^* \leqslant \log_2 |B_{\mathfrak{d}}(\rho, 2R)|$, the desired result follows. (After rescaling Ψ by a universal constant.) □

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