On expanders from the action of $GL(2, \mathbb{Z})$

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Abstract

Consider the undirected graph $G_n = (V_n, E_n)$ where $V_n = (\mathbb{Z}/n\mathbb{Z})^2$ and E_n contains an edge from (x, y) to $(x+1, y)$, $(x, y+1)$, $(x+y, y)$, and $(x, y+x)$ for every $(x, y) \in V_n$. Gabber and Galil, following Margulis, gave an elementary proof that $\{G_n\}$ forms an expander family. In this expository note, we present a somewhat simpler proof of this fact, and demonstrate its utility by isolating a key property of the linear transformations $(x, y) \mapsto (x+y, x), (x, y+x)$ that yields expansion.

As an example, take any invertible, integral matrix $S \in GL_2(\mathbb{Z})$ and let $G_n^S = (V_n, E_n^S)$ where E_n^S contains, for every $(x, y) \in V_n$, an edge from (x, y) to $(x+1, y)$, $(x, y+1)$, $S(x, y)$, and $S^\top(x, y)$, and $S^\top$ denotes the transpose of S. Then $\{G_n^S\}$ forms an expander family if and only if the infinite graph

$$G^S = \left(\mathbb{Z}^2 \setminus \{0\}, \left\{\{z, Sz\}, \{z, S^\top z\} : z \in \mathbb{Z}^2 \setminus \{0\}\right\}\right)$$

has positive Cheeger constant.

This latter property turns out to be elementary to analyze: For any $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$, the graph G^S has positive Cheeger constant if and only if $(a+d)(b-c) \neq 0$. The case $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ recovers the Margulis-Gabber-Galil graphs. We also present some other generalizations.

1 Introduction

Expander graphs have played a fundamental role in many areas of mathematics and computer science; we refer to the monograph [HLW06]. Margulis [Mar73] discovered the first explicit construction of expanders. Based on his work, Gabber and Galil [GG81] later presented an elementary construction and analysis. The Gabber-Galil graphs still provide the simplest, most succinct description of expanders to date.

Consider the undirected graph $G_n = (V_n, E_n)$ with vertex set $V_n = (\mathbb{Z}/n\mathbb{Z})^2$ and edge set E_n which contains, for every $(x, y) \in V_n$, an edge to each of $(x \pm 1, y), (x, y \pm 1), (x \pm y, y), (x, y \pm x)$. Then $\{G_n : n \geqslant 2\}$ forms a family of expander graphs with vertex degree at most 8. Jimbo and Maruoka [JM87], using discrete Fourier analysis, presented another proof that the Gabber-Galil graphs are expanders. Both these analyses contain at least one non-trivial and arguably opaque technical analytic step. For instance, the survey [HLW06] gives an elementary proof along the lines of [JM87] but still refers to the argument as "subtle and mysterious."

We present a somewhat simpler proof, or at least one whose pieces are each well-motivated. The "technical step" is replaced by an application of the discrete Cheeger inequality and a very

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simple combinatorial lemma inspired by a paper of Linial and London [LL06] (cf. Lemma 2.2). Moreover, the basic approach allows us to analyze a variety of similar families.

Given any two invertible, integral matrices $S, T \in GL_2(\mathbb{Z})$, one can consider the family of graphs $G_n^{S,T} = (V_n, E_n^{S,T})$, where $E_n^{S,T}$ contains edges from every $(x,y) \in V_n$ to each of

$$(x \pm 1, y), (x, y \pm 1), S(x,y), S^{-1}(x,y), T(x,y), T^{-1}(x,y).$$

The Gabber-Galil graphs correspond to the choice $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

Consider also the countably infinite graph $G^{S,T}$ with vertex set $\mathbb{Z}^2 \setminus \{0\}$ and edges

$$E^{S,T} \stackrel{\text{def}}{=} \left\{\{z, Sz\}, \{z, Tz\} : z \in \mathbb{Z}^2 \setminus \{0\}\right\} .$$

In Section 3, we prove the following relationship.

Theorem 1.1. *For any $S, T \in GL_2(\mathbb{Z})$, if $G^{S^\top, T^\top}$ has positive Cheeger constant, then $\{G_n^{S,T}\}$ is a family of expander graphs.*

An infinite graph $G = (V, E)$ with uniformly bounded degrees has positive Cheeger constant if there is a number $\varepsilon > 0$ such that every finite subset $U \subseteq V$ has at least $\varepsilon|U|$ edges with exactly one endpoint in U. While Theorem 1.1 may not seem particularly powerful, it turns out that in many interesting cases, proving a non-trivial lower bound on the Cheeger constant of $G^{S,T}$ is elementary. For the Gabber-Galil graphs, the argument is especially simple; see Lemma 2.2.

One can generalize the Gabber-Galil graphs in a few different ways. As a prototypical example, consider the family $\{G_n^{S,S^\top}\}$ for any $S \in GL_2(\mathbb{Z})$. In Section 4, we give the following characterization.

Theorem 1.2. *For any $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$, it holds that $\{G_n^{S,S^\top}\}$ is an expander family if and only if $(a+d)(b-c) \neq 0$.*

For instance, the preceding theorem implies that if S has order 4 then $\{G_n^{S,S^\top}\}$ is not a family of expander graphs, but if S has order 6 and $S \neq S^\top$ then the graphs are expanders.

Earlier, Cai [Cai03] considered a different generalization. Let $R = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ be the reflection across the line $y = x$. The Gabber-Galil graphs can also be seen as $G_n^{S,T}$ where $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $T = RSR$. In Section 4.1, we give the following characterization.

Theorem 1.3. *For any $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$, it holds that $\{G_n^{S,RSR}\}$ is an expander family if and only if $(a+d)(b+c) \neq 0$.*

Cai [Cai03] considers the situation $\det(S) = 1$ and $|a+d| \geqslant 2, |b+c| \geqslant 2$. However, his work does not prove that $\{G_n^{S,RSR}\}$ are expanders. In fact, the graphs he associates to a matrix S are somewhat complicated and need to refer to the action of S on the torus. Moreover, they do not have uniformly bounded degree; the degree of his graphs grow linearly in $\|S\|_1$ (the sum of the magnitudes of the entries of S). The maximum degree of our graphs is clearly bounded by 8. Interestingly, Cai states that $\{G_n^{S,S^\top}\}$ is a more natural generalization, but the main technical tool of the Gabber-Galil style analysis (see Theorem 4.10) does not work for these graphs.

2 The Margulis-Gabber-Galil graphs

Consider an undirected graph $G = (V, E)$ with an at most countable vertex set. For $A, B \subseteq V$, we use $E(A, B)$ to denote the set of edges with one endpoint in A and one in B. We write $E(A) = E(A, \bar{A})$ where $\bar{A}$ denotes the complement of A in V. We define the expansion of a subset $U \subseteq V$ by

$$h_G(U) \stackrel{\text{def}}{=} \frac{|E(U)|}{|U|}\,.$$

For G finite, we set $h(G) \stackrel{\text{def}}{=} \min_{|U| \leqslant \frac{1}{2}|V|} h_G(U)$. If G is infinite, we put $h(G) \stackrel{\text{def}}{=} \min_{U \subseteq V : |U| < \infty} h_G(U)$. In both the finite and infinite case, we refer to $h(G)$ as the *Cheeger constant of* G.

We also have the Rayleigh quotient of a function $f : V \to \mathbb{C}$ given by

$$\mathcal{R}_G(f) \stackrel{\text{def}}{=} \frac{\sum_{\{u,v\} \in E} |f(u) - f(v)|^2}{\sum_{u \in V} |f(u)|^2}\,,$$

and for finite G, we put $\lambda_2(G) \stackrel{\text{def}}{=} \min\{\mathcal{R}_G(f) : \sum_{u \in V} f(u) = 0\}$. This is the smallest non-zero eigenvalue of the combinatorial Laplacian (see, e.g., the book [Chu97]). An infinite family of finite graphs $\{G_n\}$ with uniformly bounded degrees is called an *expander family* if $\lambda_2(G_n) \geqslant c > 0$ for some $c > 0$. We will assume familiarity with the following discrete Cheeger inequality.

Lemma 2.1. *For any countable graph $G = (V, E)$ with maximum degree Δ and any function $f : V \to \mathbb{C}$ with $\sum_{v \in V} |f(v)|^2 < \infty$, there exists a finite subset $U \subseteq \{v \in V : f(v) \neq 0\}$ such that*

$$h_G(U) \leqslant \sqrt{2\Delta \mathcal{R}_G(f)}\,.$$

Proof. Let $U_t = \{v \in V : |f(v)|^2 \geqslant t\}$. Observe that for each $t > 0$, one has $U_t \subseteq \{v \in V : f(v) \neq 0\}$ and U_t is finite since $\sum_{v \in V} |f(v)|^2$ is finite. Now we have:

$$\begin{aligned}
\int_0^\infty |E(U_t, \bar{U}_t)|\,dt &= \sum_{\{u,v\} \in E} \left| |f(u)|^2 - |f(v)|^2 \right| \\
&= \sum_{\{u,v\} \in E} (|f(u)| + |f(v)|)(|f(u)| - |f(v)|) \\
&\leqslant \sqrt{\sum_{\{u,v\} \in E} (|f(u)| + |f(v)|)^2} \sqrt{\sum_{\{u,v\} \in E} |f(u) - f(v)|^2} \\
&\leqslant \sqrt{2\Delta \sum_{u \in V} |f(u)|^2} \sqrt{\sum_{\{u,v\} \in E} |f(u) - f(v)|^2}.
\end{aligned}$$

On the other hand, $\int_0^\infty |U_t|\,dt = \sum_{u \in V} |f(u)|^2$, thus

$$\int_0^\infty |E(U_t, \bar{U}_t)|\,dt \leqslant \sqrt{2\Delta \mathcal{R}_G(f)} \int_0^\infty |U_t|\,dt\,,$$

implying there exists a $t > 0$ such that $h_G(U_t) \leqslant \sqrt{2\Delta \mathcal{R}_G(f)}$. ◻

An initial expanding object. We will start with an initial "expanding object," and then try to construct a family of graphs out of it. First, consider the infinite graph $\mathcal{G} = (\mathbb{Z}^2, E)$ whose edges are given by two maps $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S(x, y) = (x, x + y)$ and $T(x, y) = (x + y, y)$. Each vertex $z \in \mathbb{Z}^2$ is connected to $S(z), S^{-1}(z), T(z), T^{-1}(z)$. So every vertex has degree at most four. Clearly $(0, 0)$ is not adjacent to anything. Using an argument from [LL06], we will show that this graph is an expander in the following sense.

Lemma 2.2. *For any finite subset $A \subseteq \mathbb{Z}^2 \setminus \{0\}$, we have $|E(A, \bar{A})| \geqslant |A|$.*

Proof. Define $Q_1 = \{(x, y) \in \mathbb{Z}^2 : x > 0, y \geqslant 0\}$. This is the first quadrant, without the y-axis and the origin. Define Q_2, Q_3, Q_4 similarly by rotating Q_1 by 90, 180, and 270 degrees, respectively, and note that we have a partition $\mathbb{Z}^2 \setminus \{0\} = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$.

Let $A_i = A \cap Q_i$. We will show that $|E(A_1, \bar{A} \cap Q_1)| \geqslant |A_1|$. Since our graph is invariant under rotations of the plane by 90°, this will imply our goal:

$$|E(A, \bar{A})| \geqslant \sum_{i=1}^{4} |E(A_i, \bar{A} \cap Q_i)| \geqslant \sum_{i=1}^{4} |A_i| = |A| .$$

It is immediate that $S(A_1), T(A_1) \subseteq Q_1$. Furthermore, we have $S(A_1) \cap T(A_1) = \emptyset$ because S maps points in Q_1 above (or onto) the line $y = x$ and T maps points of Q_1 below the line $y = x$. Furthermore, S and T are bijections, thus $|S(A_1) + T(A_1)| = |S(A_1)| + |T(A_1)| = 2|A_1|$. In particular, this yields $|E(A_1, \bar{A} \cap Q_1)| \geqslant |A_1|$, as desired. □

Of course, $\mathcal{G}$ is not a finite graph, so for a number $n \geqslant 2$, we define the graph $G_n = (V_n, E_n)$ with vertex set $V_n = (\mathbb{Z}/n\mathbb{Z})^2$. There are four types of edges in E_n: A vertex (x, y) is connected to the vertices

$$\{(x, y \pm 1), (x \pm 1, y), (x, x \pm y), (x \pm y, y)\} ,$$

where arithmetic is taken modulo n. This yields a graph of degree at most 8. We now state the main result of this section.

Theorem 2.3. *There is a constant $c > 0$ such that for every $n \geqslant 2$,*

$$\lambda_2(G_n) \geqslant c .$$

In other words, $\{G_n\}$ forms an expander family.

Passing to the continuous torus. Our results for $\mathbb{Z}^2$ do not seem immediately useful for analyzing these finite graphs. We will first pass from the discrete graphs $\{G_n\}$ to the continuous torus. This is a reassuring step, as it means our analysis is not going to rely on number theoretic considerations of the modulus n.

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus equipped with the Lebesgue measure and consider the complex Hilbert space

$$L^2(\mathbb{T}^2) = \left\{ f : \mathbb{T}^2 \to \mathbb{C} : \int_{\mathbb{T}^2} |f|^2 < \infty \right\} .$$

equipped with the inner product $\langle f, g \rangle_{L^2} = \int_{\mathbb{T}^2} f \bar{g}$.

We also define a related value

$$\lambda_2(\mathbb{T}^2_{S,T}) \stackrel{\text{def}}{=} \min_{f\in L^2(\mathbb{T}^2)} \left\{ \frac{\|f - f\circ S\|^2_{L^2} + \|f - f\circ T\|^2_{L^2}}{\|f\|^2_{L^2}} : \int_{\mathbb{T}^2} f = 0 \right\} . \tag{2.1}$$

Lemma 2.4. *There is some $\varepsilon > 0$ such that for any $n \geqslant 2$, we have $\lambda_2(G_n) \geqslant \varepsilon\lambda_2(\mathbb{T}^2_{S,T})$.*

Proof. Suppose we are given some map $f : V_n \to \mathbb{C}$ such that $\sum_{u\in V_n} f(u) = 0$. We define its continuous extension $\tilde{f} : \mathbb{T}^2 \to \mathbb{C}$ as follows. There is a natural embedding of V_n into $[0,1]^2$ which we represent as follows: Given a point $w = (x/n, y/n) \in [0,1]^2$, with $x, y \in \{0, 1, \ldots, n\}$, we write $[w]$ for the corresponding element of V_n.

Every point $z \in [0,1)^2$ sits inside a grid square with four corners u_1, u_2, u_3, u_4 such that $[u_1], [u_2], [u_3], [u_4] \in V_n$. We call such a square (thought of as a subset of $\mathbb{T}^2$) a *canonical square.* Define $\tilde{f}(z)$ as the average

$$\tilde{f}(z) = \frac{\sum_{i=1}^4 (\frac{1}{n} - \|u_i - z\|_\infty) f([u_i])}{\sum_{i=1}^4 (\frac{1}{n} - \|u_i - z\|_\infty)} . \tag{2.2}$$

Observe that this is well-defined; e.g., if z lies on the segment between u_1 and u_2 then the coefficients of $f([u_3])$ and $f([u_4])$ are zero. By symmetry, it follows immediately that $\int_{\mathbb{T}^2} \tilde{f} = 0$.

It is also easy to verify that $\|\tilde{f}\|^2_{L^2} \geqslant \frac{c}{n^2}\sum_{v\in V} f(v)^2$ for some $c > 0$. For any square with corners $\{u_1, u_2, u_3, u_4\}$, let $i \in \{1, 2, 3, 4\}$ be such that $f([u_i])^2$ is maximal and let B denote an ℓ_∞ ball of radius $\frac{1}{8n}$ around u_i. Then $\int_B |\tilde{f}|^2 \geqslant \frac{c}{n^2}\sum_{i=1}^4 f([u_i])^2$ for some universal constant $c > 0$. Summing over all the squares yields the claim.

So to finish the proof, we are left to argue that

$$\|\tilde{f} - \tilde{f}\circ S\|^2_{L^2} + \|\tilde{f} - \tilde{f}\circ S\|^2_{L^2} \leqslant \frac{c}{n^2} \sum_{\{u,v\}\in E_n} (f(u) - f(v))^2 \tag{2.3}$$

for some $c > 0$. Consider any point $z \in \mathbb{T}^2$ contained in a square $\square_1$ and suppose $S(z)$ is in $\square_2$. Note that $\square_1 = \square_2$ is a possibility. Let C be the set of (at most) eight vertices of V_n that comprise the corners of $\square_1$ and $\square_2$. Then any pair of vertices in C can reach each other using a path of length at most five in G_n. This is the only place where we need to use the fact that edges of the form $(x, y) \leftrightarrow (x, y \pm 1)$ and $(x, y) \leftrightarrow (x \pm 1, y)$ are present in G_n. On the other hand, we clearly have

$$|\tilde{f}(z) - \tilde{f}(S(z))|^2 \leqslant \max_{u,v\in C} |f(u) - f(v)|^2 ,$$

since $\tilde{f}(z)$ is a convex combination of the f-values at the corners of $\square_1$ and $\tilde{f}(S(z))$ is a convex combination of the f-values at the corners of $\square_2$.

Now consider a canonical square $\square \subseteq \mathbb{T}^2$, which has measure $1/n^2$. Let $E(\square)$ to denote the set of edges in G_n that occur on some path of length at most 5 emanating from the corners of $\square$. Then the preceding argument yields

$$\int_\square |\tilde{f}(z) - \tilde{f}(S(z))|^2 dz \leqslant \frac{1}{n^2} \max_{\{u,v\}\in E(\square)} |f(u) - f(v)|^2 \leqslant O\left(\frac{1}{n^2}\right) \sum_{\{u,v\}\in E(\square)} |f(u) - f(v)|^2 ,$$

using the fact that $|E(\square)| = O(1)$ because G_n has degree at most 8. Summing the preceding inequality over all canonical squares yields

$$\int_{\mathbb{T}^2} |\tilde{f}(z) - \tilde{f}(S(z))|^2 dz \leqslant O\left(\frac{1}{n^2}\right) \sum_{\{u,v\}\in E} |f(u) - f(v)|^2 ,$$

since every edge occurs in some set $E(\square)$ at most $O(1)$ times. An identical argument holds for T, yielding (2.3). $\square$

Using the Fourier transform to unwrap the torus. Our final goal is to show that $\lambda_2(\mathbb{T}^2_{S,T}) > 0$. Our approach is based on the fact that S and T, being shift operators, will act rather nicely on the Fourier basis.

We recall that if $m, n \in \mathbb{N}$ and we define $\chi_{m,n} \in L^2(\mathbb{T}^2)$ by $\chi_{m,n}(x, y) = \exp(2\pi i(mx + ny))$, then $\{\chi_{m,n} : m, n \in \mathbb{Z}\}$ forms an orthonormal Hilbert basis for $L^2(\mathbb{T}^2)$. In particular, every $f \in L^2(\mathbb{T}^2)$ can be written as

$$f = \sum_{m,n\in\mathbb{Z}} \hat{f}(m, n)\chi_{m,n} , \tag{2.4}$$

where $\hat{f}(m, n) = \langle f, \chi_{m,n}\rangle_{L^2}$ and convergence in (2.4) is in the $L^2(\mathbb{T}^2)$ norm (see, for instance, [Kat04, §I.5]). Putting $\ell^2(\mathbb{Z}^2) = \{f : \mathbb{Z}^2 \to \mathbb{C} : \sum_{z\in\mathbb{Z}^2} |f(z)|^2 < \infty\}$, the Fourier transform is the linear isometry $f \mapsto \hat{f}$ from $L^2(\mathbb{T}^2)$ to $\ell^2(\mathbb{Z}^2)$.

For any $m, n \in \mathbb{Z}$, we have

$$\chi_{m,n} \circ S = \chi_{m,n+m} \quad \text{and} \quad \chi_{m,n} \circ T = \chi_{m+n,n} .$$

Thus for any $f \in L^2(\mathbb{T}^2)$, we have

$$\begin{aligned} \widehat{f \circ S} &= \sum_{m,n} \hat{f}(m, n)\chi_{m,n+m} = \sum_{m,n} \hat{f}(m, n - m)\chi_{m,n} = \hat{f} \circ T^{-1} \\ \widehat{f \circ T} &= \sum_{m,n} \hat{f}(m, n)\chi_{m+n,n} = \sum_{m,n} \hat{f}(m - n, n)\chi_{m,n} = \hat{f} \circ S^{-1} . \end{aligned}$$

The final thing to note is that $\hat{f}(0, 0) = \langle f, \chi_{0,0}\rangle = \int_{\mathbb{T}^2} f$. So now if we simply apply the Fourier transform (a linear isometry) to the expression in (2.1), we arrive at

$$\begin{aligned} \lambda_2(\mathbb{T}^2_{S,T}) &= \min_{f\in L^2(\mathbb{T}^2)} \left\{ \frac{\|\hat{f} - \widehat{f \circ S}\|^2_{\ell^2} + \|\hat{f} - \widehat{f \circ T}\|^2_{\ell^2}}{\|\hat{f}\|^2_{\ell^2}} : \int_{\mathbb{T}^2} f = 0 \right\} \\ &= \min_{\hat{f}\in\ell^2(\mathbb{Z}^2)} \left\{ \frac{\sum_{z\in\mathbb{Z}^2} |\hat{f}(z) - \hat{f}(T^{-1}(z))|^2 + |\hat{f}(z) - \hat{f}(S^{-1}(z))|^2}{\|\hat{f}\|^2_{L^2}} : \hat{f}(0, 0) = 0 \right\} . \end{aligned}$$

In other words,

$$\lambda_2(\mathbb{T}^2_{S,T}) = \min_{\hat{f}\in\ell^2(\mathbb{Z}^2)} \{\mathcal{R}_{\mathcal{G}}(\hat{f}) : \hat{f}(0, 0) = 0\} ,$$

where $\mathcal{G}$ is our initial graph defined on $\mathbb{Z}^2$. Applying the discrete Cheeger inequality (Lemma 2.1) with $\Delta = 4$, yields

$$\min_{\hat{f}:V\to\mathbb{C}} \{\mathcal{R}_{\mathcal{G}}(\hat{f}) : \hat{f}(0, 0) = 0\} \geqslant \frac{1}{8} \min_{U:(0,0)\notin U} h_{\mathcal{G}}(U)^2 \geqslant \frac{1}{8} ,$$

where the final inequality is exactly the content of Lemma 2.2. Thus by Lemma 2.4 for some $\varepsilon > 0$ and every $n \geqslant 2$, we have $\lambda_2(G_n) \geqslant \varepsilon\lambda_2(\mathbb{T}^2_{S,T}) \geqslant \frac{\varepsilon}{8}$. This completes the proof of Theorem 2.3.

3 The general correspondence

We now perform the steps of the preceding section is somewhat greater generality. Consider $S, T \in GL_2(\mathbb{Z})$. We will write $\hat{G}^{S,T}$ to denote $G^{S^\top, T^\top}$. The main result of this section is a connection between the expansion of $\{G_n^{S,T}\}$ and $\hat{G}^{S,T}$.

Theorem 3.1. *For every $S, T \in GL_2(\mathbb{Z})$, if $h(\hat{G}^{S,T}) > 0$, then $\{G_n^{S,T}\}$ forms an expander family.*

Define the quantity

$$\lambda_2(\mathbb{T}^2_{S,T}) \stackrel{\text{def}}{=} \min_{f \in L^2(\mathbb{T}^2)} \left\{ \frac{\|f - f \circ S\|_{L^2}^2 + \|f - f \circ T\|_{L^2}2}{\|f\|_{L^2}} : \int_{\mathbb{T}^2} f = 0 \right\}.$$

The following result requires a bit more delicacy than Lemma 2.4.

Lemma 3.2. *There is an $\varepsilon > 0$ such that for every $S, T \in GL_2(\mathbb{Z})$ and $n \geqslant 2$, we have*

$$\lambda_2(G_n^{S,T}) \geqslant \frac{\varepsilon}{\|S\|_1^2 + \|T\|_1^2} \lambda_2(\mathbb{T}^2_{S,T}).$$

Proof. We will use the notion of canonical squares from Lemma 2.4. Suppose we have a map $f : V_n \to \mathbb{C}$ satisfying $\sum_{u \in V_n} f(u) = 0$. Define the extension $\tilde{f} : \mathbb{T}^2 \to \mathbb{C}$ as in (2.2). The fact that $\int_{\mathbb{T}^2} \tilde{f} = 0$ and $\int_{\mathbb{T}^2} |\tilde{f}|^2 \geqslant \frac{c}{n^2} \sum_{u \in V_n} |f(u)|^2$ for some absolute constant $c > 0$ is proved in Lemma 2.4. We are thus left to prove that for some $c > 0$,

$$\|\tilde{f} - \tilde{f} \circ S\|_{L^2}^2 + \|\tilde{f} - \tilde{f} \circ T\|_{L^2}^2 \leqslant c \frac{\|S\|_1^2 + \|T\|_1^2}{n^2} \sum_{\{u,v\} \in E_n^{S,T}} |f(u) - f(v)|^2 . \tag{3.1}$$

To this end, suppose $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and consider a point $z \in \square_1$ and $Sz \in \square_2$, where $\square_1$ and $\square_2$ are canonical squares whose corners are vertices from V_n (it is possible that $\square_1 = \square_2$). Since $\tilde{f}(z)$ is a convex combination of the values of f at the corners of $\square_1$ and similarly for $\tilde{f}(S(z))$ and $\square_2$, we have

$$|\tilde{f}(z) - \tilde{f}(Sz)|^2 \leqslant \max_{u,v \in C} |f(u) - f(v)|^2 , \tag{3.2}$$

where C contains the (at most) eight corners of $\square_1$ and $\square_2$.

Unlike in Lemma 2.4, the members of C can no longer be connected by paths of length $O(1)$ in G_n^S. However, it is elementary to see that they can be connected by paths of length at most $\|S\|_1 + 1 = |a| + |b| + |c| + |d| + 1$. We simply need to choose the paths in a consistent way in order to conclude that (3.1) holds. This will be a bit technical, but the underlying idea is very simple.

We will now specify canonical paths between the members of C. Let us write $E'_n \subseteq E_n^{S,T}$ for the set of edges connecting (x, y) to $(x \pm 1, y)$ or $(x, y \pm 1)$. Call an edge of E'_n *horizontal* if it changes the x coordinate and *vertical* if it changes the y coordinate.

Let $(x, y) \in [0,1)^2$ denote the lower-left corner of $\square_1$ and let $(x', y') \in [0,1)^2$ denote the lower-left corner of $\square_2$. We may assume that $z = (x+\alpha, y+\beta)$ for some $\alpha, \beta \in (0, 1/n)$, and

$$Sz = S(x,y) + S(\alpha,\beta) = S(x,y) + (a\alpha + b\beta, c\alpha + d\beta)\,.$$

We specify a path from (x, y) to (x', y'). Our path P_z in G_n^S will first follow the edge $\{(x,y), S(x,y)\}$ then move along edges of E_n' in the x direction for $\lfloor a\alpha + b\beta \rfloor$ steps, then move along edges of E_n' in the y direction for $\lfloor c\alpha + d\beta \rfloor$ steps. This will arrive at some corner of $\square_2$ (e.g., the lower-left corner if all the entries of S are positive). Our path then moves to (x', y') using at most two additional edges of $\square_2$. For any other pair $u, v \in C$: If they are in the same square, move along the edges of the square in some canonical way using a path of length at most two. Otherwise, if u is a corner of $\square_1$ and v is a corner of $\square_2$, first from u to (x, y) along edges of $\square_1$, then to (x', y') using P_z, then from (x', y') to v using edges of $\square_2$. Let P_{uv}^z denote the specified path between $u, v \in C$. Note that the length of P_{uv}^z is $O(\|S\|_1)$.

The main points of this construction are as follows. First, for every pair of horizontal (respectively, vertical) edges $e, e' \in E_n'$, we have

$$\int_{\mathbb{T}^2} \mathbf{1}_{\{e \in P_z\}} dz = \int_{\mathbb{T}^2} \mathbf{1}_{\{e' \in P_z\}} dz\,. \tag{3.3}$$

The second is that, combining (3.2) with Cauchy-Schwarz yields

$$|\tilde{f}(z) - \tilde{f}(S(z))|^2 \leqslant O(\|S\|_1) \sum_{u,v \in C} \sum_{\{r,s\} \in P_{uv}^z} |f(r) - f(s)|^2\,. \tag{3.4}$$

Using the equitable property (3.3) and the fact that every edge of the form $\{(x,y), S(x,y)\}$ appears on the right-hand side of (3.4) only when $z \in \square_1$, we can integrate (3.4) to yield

$$\int_{\mathbb{T}^2} |\tilde{f}(z) - \tilde{f}(S(z))|^2 dz \leqslant O\left(\frac{\|S\|_1^2}{n^2}\right) \sum_{\{u,v\} \in E_n^{S,T}} |f(u) - f(v)|^2\,.$$

An identical analysis holds for T, allowing us to verify (3.1). □

Lemma 3.3. *For any $S, T \in GL_2(\mathbb{Z})$, we have*

$$\lambda_2(\mathbb{T}^2_{S,T}) = \min_{\hat{f} \in \ell^2(\mathbb{Z}^2)} \{\mathcal{R}_{\hat{G}^{S,T}}(\hat{f}) : \hat{f}(0,0) = 0\}\,.$$

Proof. Note that if $f \in L^2(\mathbb{T}^2)$, then

$$\begin{aligned} \widehat{f \circ S} &= \sum_{m,n} \hat{f}(m,n) \chi_{am+cn, bm+dn} \\ &= \sum_{m,n} \hat{f}(m,n) \chi_{S^\top(m,n)} \\ &= \sum_{m,n} \hat{f}(S^{-\top}(m,n)) \chi_{m,n} \\ &= \hat{f} \circ S^{-\top}\,. \end{aligned}$$

Similarly, $\widehat{f \circ T} = \hat{f} \circ T^{-\top}$. Using the fact that the Fourier transform is a linear isometry from $L^2(\mathbb{T}^2)$ to $\ell^2(\mathbb{Z}^2)$ and $\hat{f}(0,0) = \int_{\mathbb{T}^2} f$, we have

$$\begin{aligned} \lambda_2(\mathbb{T}^2_{S,T}) &= \min_{\hat{f} \in \ell^2(\mathbb{Z}^2)} \left\{ \frac{\sum_{z \in \mathbb{Z}^2} |\hat{f}(z) - \hat{f}(S^{-\top} z)|^2 + |\hat{f}(z) - \hat{f}(T^{-\top} z)|^2}{\sum_{z \in \mathbb{Z}^2} |\hat{f}(z)|^2} : \hat{f}(0,0) = 0 \right\} \\ &= \min_{\hat{f} \in \ell^2(\mathbb{Z}^2)} \{ \mathcal{R}_{\hat{G}^{S,T}}(\hat{f}) : \hat{f}(0,0) = 0 \}, \end{aligned}$$

completing the proof. □

Combining Lemma 3.3 with the discrete Cheeger inequality (Lemma 2.1) yields the following.

Corollary 3.4. *For any $S, T \in GL_2(\mathbb{Z})$, $\lambda_2(\mathbb{T}^2_{S,T}) \geqslant \frac{1}{8} h(\hat{G}^{S,T})^2$.*

Finally, combining this corollary with Lemma 3.2 yields Theorem 3.1.

4 Expansion analysis

For ease of notation, we will write $G_n^S \stackrel{\text{def}}{=} G_n^{S,S^\top}$ and $G^S \stackrel{\text{def}}{=} G^{S,S^\top}$.

Theorem 4.1. *For any $S \in GL_2(\mathbb{Z})$, it holds that $h(G^S) > 0$ if and only if $S \neq S^\top$ and $\operatorname{tr}(S) \neq 0$.*

Combining the preceding result with Theorem 3.1, we can prove the following.

Theorem 4.2. *For any $S \in GL_2(\mathbb{Z})$, it holds that $\{G_n^S\}$ is an expander family if and only if $S \neq S^\top$ and $\operatorname{tr}(S) \neq 0$.*

Proof. Since $G^S = \hat{G}^{S,S^\top}$ and $h(G^S) > 0$ by Theorem 4.1, we can use Theorem 3.1 to conclude that $\{G_n^S\}$ is an expander family. On the other hand, if $S = S^\top$, then Lemma 4.15 shows that $\{G_n^S\}$ is not an expander family. If $\operatorname{tr}(S) = 0$ then $S^4 = I = (S^\top)^4$ and Lemma 4.16 shows that $\{G_n^S\}$ is not an expander family. □

To prove Theorem 4.1, we will first analyze the case when $\det(S) = 1$ and S has all non-negative entries. This is essentially the main technical lemma of the section; we will show that all other cases can be reduced to this one.

Lemma 4.3. *If $S \in GL_2(\mathbb{Z})$ has all non-negative entries, $\det(S) = 1$, and $S \neq S^\top$, then*

$$\begin{aligned} S(Q_1) \cap S^\top(Q_1) &= \emptyset \\ S(Q_3) \cap S^\top(Q_3) &= \emptyset \\ S^{-1}(Q_2) \cap S^{-\top}(Q_2) &= \emptyset \\ S^{-1}(Q_4) \cap S^{-\top}(Q_4) &= \emptyset \end{aligned}$$

Proof. Let $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for some $a, b, c, d \geqslant 0$ and let $T = S^\top$. Since $\det(S) = 1$, we can write:

$$S^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) \qquad T = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \qquad T^{-1} = \left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right) \tag{4.1}$$

We need only prove that $S(Q_1) \cap T(Q_1) = \emptyset$. Since $Q_3 = -Q_1$, this immediately yields $S(Q_3) \cap T(Q_3) = \emptyset$. Consider the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ that maps Q_1 bijectively to Q_2. Then

$$|S^{-1}(Q_2) \cap T^{-1}(Q_2)| = |A^{-1}S^{-1}A(Q_1) \cap A^{-1}T^{-1}A(Q_1)| = |T(Q_1) \cap S(Q_1)| = 0 \,.$$

Similarly, since $Q_2 = -Q_4$, this yields $S^{-1}(Q_4) \cap T^{-1}(Q_4) = \emptyset$ as well.

Now suppose that $S(Q_1) \cap T(Q_1) \neq \emptyset$. We will derive a contradiction. Restating our assumption, there exists $(x, y) \in Q_1$ with $S^{-1}T(x, y) \in Q_1$. This implies that

$$\begin{aligned} (ad - b^2)x + d(c - b)y &> 0 && (4.2) \\ a(b - c)x + (ad - c^2)y &\geqslant 0 \,. && (4.3) \end{aligned}$$

Note that $b \neq c$ since, by assumption, $S^\top \neq S$. Also, $ad \neq 0$, since in this case $bc = -1$, which is impossible under our assumption that $b, c \geqslant 0$.

If $ad = c^2$ then $1 = ad - bc = c(c - b)$ which implies that $c = 1$ and $b = 0$. This yields $-ax \geqslant 0$ in (4.3), which is impossible since $(x, y) \in Q_1 \implies x > 0$.

If $ad = b^2$ then $1 = ad - bc = b(b - c)$, which implies that $c = 0$ and $b = 1$. Altogether, in this case, we have $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Here we can conclude that $S(Q_1) \cap T(Q_1) = \emptyset$ because S maps points of Q_1 strictly below the line $y = x$ and T maps points of Q_1 above (or onto) the line $y = x$.

To summarize, we are left to deal with the case

$$b \neq c, \quad a > 0, \quad d > 0, \quad ad \neq b^2, \quad ad \neq c^2 \,.$$

If $b > c$ then $ad - b^2 < ad - bc = 1$ which implies $ad - b^2 < 0$ since $ad \neq b^2$. In this case, $d(c - b) < 0$ as well. Thus if (4.2) holds, then $x = y = 0$. Similarly, if $c > b$, then $ad - c^2 < ad - bc = 1$ hence $ad - c^2 < 0$ and $a(b - c) < 0$, implying $x = y = 0$. We conclude that $S(Q_1) \cap T(Q_1) = \emptyset$. ◻

Corollary 4.4. *If $S \in GL_2(\mathbb{Z})$ has all non-negative entries, $S \neq S^\top$, and $\det(S) = 1$, then for any subset $A \subseteq \mathbb{Z}^2 \setminus \{0\}$,*

$$|S(A) \cup S^\top(A) \cup S^{-1}(A) \cup S^{-\top}(A)| \geqslant 2|A| \,.$$

In particular, $h(G^S) > 0$.

Proof. In this case, we have $S(Q_1), S^\top(Q_1) \subseteq Q_1$, $S(Q_3), S^\top(Q_3) \subseteq Q_3$, $S^{-1}(Q_2), S^{-\top}(Q_2) \subseteq Q_2$, and $S^{-1}(Q_4), S^{-\top}(Q_4) \subseteq Q_4$. Thus Lemma 4.3 yields the desired result. ◻

To handle the case of general $S \in GL_2(\mathbb{Z})$, it will help to have the following well-known fact.

Lemma 4.5. *Consider two infinite graphs $G = (V, E)$ and $G' = (V, E')$ on the same countable index set V, both of which have uniformly bounded degree. Suppose there is a number $k \in \mathbb{N}$ such that that for every $\{x, y\} \in E$, there is a path of length at most k between x and y in G'. Then $h(G) > 0$ implies $h(G') > 0$.*

Proof. Let Δ be a uniform upper bound on the degree of vertices in G and G'. For a subset $U \subseteq V$ and $j \geqslant 1$, write $N^j_{G'}(U) \subseteq V$ for the set of vertices within distance j of the set U in G'.

Now, suppose that $h(G') = 0$. In that case, for every $\varepsilon > 0$, there exists a finite subset $U \subseteq V$ such that $|N^1_{G'}(U)| \leqslant (1 + \varepsilon)|U|$. In particular, this implies that $|N^k_{G'}(U)| \leqslant (1 + \varepsilon\Delta^k)|U|$. But, by our assumptions on G and G', this implies

$$|E(U, \bar{U})| \leqslant \Delta(|N^k_{G'}(U)| - |U|) \leqslant \varepsilon\Delta^{k+1}|U| \,.$$

Letting $\varepsilon \to 0$ shows that $h(G) = 0$ as well. ◻

The following two simple lemmas give conditions under which $G^{S,T}$ has Cheeger constant zero.

Lemma 4.6. *For any $S \in GL_2(\mathbb{Z})$, we have $h(G^{S,S^{-1}}) = h(G^{S,-S^{-1}}) = 0$.*

Proof. Let $G = G^{S,\pm S^{-1}}$ have edge set E. Consider the sets $\{U_k \subseteq \mathbb{Z}^2\}$ given by

$$U_k = \{(j,0), S(j,0), \ldots, S^k(j,0) : j \in \{-1,1\}\}\,.$$

If $\sup_k |U_k| < \infty$, then clearly $h_{G^S}(U_k) = 0$ for some k. Otherwise, since $|E(U_k)| \leqslant 4$, it must be that $h_{G^S}(U_k) \to 0$ as $k \to \infty$, implying that $h(G^S) = 0$. $\square$

Lemma 4.7. *Suppose $S, T \in GL_2(\mathbb{Z})$ satisfy $S^4 = T^4 = I$. Then $h(G^{S,T}) = 0$.*

Proof. First, an elementary calculation shows that if $A \in GL_2(\mathbb{Z})$ satisfies $\det(A) = 1$ and $A^2 = I$, then $A \in \{-I, I\}$. Thus $S^2, T^2 \in \{-I, I\}$. So for any $j_1, k_1, j_2, k_2, \ldots, j_m, k_m \in \mathbb{Z}$, we have

$$S^{j_1}T^{k_1}S^{j_2}T^{k_2}\cdots S^{j_m}T^{k_m} = (-1)^{i_0}T^{j_0}(ST)^j S^{k_0}\,.$$

for some $i_0, j_0, k_0 \in \{0,1\}$ and $j \in \mathbb{N} \cup \{0\}$. Consider now the sets

$$U_k = \left\{(-1)^{i_0}T^{j_0}(ST)^j S^{k_0}(1,0) : i_0, j_0, k_0 \in \{0,1\} \text{ and } 0 \leqslant j \leqslant k\right\}\,.$$

Letting $E^{S,T}$ denote the edge set of $G^{S,T}$, we have $|E^{S,T}(U_k, \bar{U}_k)| \leqslant 2 \cdot 8$ for every $k \geqslant 1$, and thus $h(G^{S,T}) = 0$. $\square$

Finally, we complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Suppose that $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$ satisfies $S \neq S^\top$ and $\mathrm{tr}(S) \neq 0$, i.e. $b \neq c$ and $a + d \neq 0$. Let $T = S^\top$. If S has all non-negative or all non-positive entries, then the matrix $S^2 = \left(\begin{smallmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{smallmatrix}\right)$ has all non-negative entries, $\det(S^2) = 1$, and $S^2 \neq (S^2)^\top$ by our initial assumptions. Therefore by Corollary 4.4, we have $h(G^{S^2}) > 0$. Now Lemma 4.5 implies $h(G^S) > 0$ as well.

If $ad > 0$ then $|\det(S)| = 1$ implies $bc \geqslant 0$. In this case, S^{-1} has all non-negative or all non-positive entries, hence $h(G^S) = h(G^{S^{-1}}) > 0$ by the preceding paragraph.

Thus we are left to deal with the case $ad \leqslant 0$. But now consider the matrix $ST^{-1} = \det(S)\left(\begin{smallmatrix} ad-b^2 & a(b-c) \\ d(c-b) & ad-c^2 \end{smallmatrix}\right)$. We have $\det(ST^{-1}) = 1$ and $ST^{-1} \neq (ST^{-1})^\top$, by our initial assumptions that $b \neq c$ and $a + d \neq 0$. Furthermore, the diagonal entries of ST^{-1} have the same sign, so our previous considerations yield $h(G^{ST^{-1}}) > 0$. By Lemma 4.5, this yields $h(G^S) > 0$ as well.

To finish the proof, we must now show that if S satisfies $S = S^\top$ or $\mathrm{tr}(S) = 0$ then $h(G^S) = 0$. In the former case, we can apply Lemma 4.6. If $\mathrm{tr}(S) = 0$, then $S^2 = \left(\begin{smallmatrix} a^2+bc & 0 \\ 0 & bc+d^2 \end{smallmatrix}\right) = \pm I$. Similarly, $T^2 = \pm I$. Thus $h(G^S) = 0$ by Lemma 4.7. $\square$

4.1 Conjugating by a reflection

To further exhibit the flexibility of our method, we analyze the expansion a different family of operators considered earlier by Cai [Cai03]. Let $R = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and for every $S \in GL_2(\mathbb{Z})$, consider the graph

$$G^{S,RSR} = \left(\mathbb{Z}^2 \setminus \{0\}, \left\{\{z, Sz\}, \{z, RSRz\} : z \in \mathbb{Z}^2 \setminus \{0\}\right\}\right).$$

Our goal is to prove the following analog of Theorem 4.1.

Theorem 4.8. *For any $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$, we have $h(G^{S,RSR}) > 0$ if and only if $(a+d)(b+c) \neq 0$.*

The next result follows from the preceding theorem and Theorem 3.1

Theorem 4.9. *For any $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$, $\{G_n^{S,RSR}\}$ is an expander family if and only if $(a+d)(b+c) \neq 0$.*

Proof. By Theorem 4.8, we have $h(G^{S^\top,RS^\top R}) > 0$. Now Theorem 3.1 implies that $\{G_n^{S,RSR}\}$ is an expander family, noting that $(RSR)^\top = RS^\top R$.

On the other hand, suppose that $a + d = 0$. Then $S^4 = I = RS^4R$ so Lemma 4.16 implies that $\{G_n^{S,RSR}\}$ is not an expander family. If $b + c = 0$ then $ST \in \{-I, I\}$, so Lemma 4.15 implies the same. ◻

To illustrate another method of expansion analysis, we recall the following result of [Cai03]. Gabber and Galil [GG81] proved this for $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Theorem 4.10. *Consider any $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$ such that $\det(S) = 1$ and $|a+d|, |b+c| \geqslant 2$ are satisfied. Then for any $z \in \mathbb{Z}^2 \setminus \{0\}$, one of the following two conclusions holds for the set*

$$\left\{\|Sz\|_\infty, \|S^{-1}z\|_\infty, \|RSRz\|_\infty, \|RS^{-1}Rz\|_\infty\right\}.$$

Either three of the elements are strictly greater than $\|z\|_\infty$ or at most two are equal to $\|z\|_\infty$ and the rest are strictly greater than $\|z\|_\infty$.

This rather immediately yields a positive Cheeger constant for $G^{S,RSR}$.

Theorem 4.11. *Suppose that S satisfies the assumptions of Theorem 4.10. Then $h(G^{S,RSR}) > 0$.*

Proof. For an edge $\{x, y\} \in E^{S,RSR}$, let

$$\Delta(x,y) = \begin{cases} 0 & \|x\|_\infty = \|y\|_\infty \\ 1 & \|x\|_\infty > \|y\|_\infty \\ -1 & \text{otherwise.} \end{cases}$$

Consider a finite set $U \subseteq \mathbb{Z}^2 \setminus \{0\}$. Then by Theorem 4.10,

$$\sum_{x \in U} \sum_{A \in \{S, RSR, S^{-1}, RS^{-1}R\}} \Delta(x, Ax) \geqslant 2|U|.$$

On the other hand, whenever x and Ax are both in U, the total contribution from the terms $\Delta(x, Ax)$ and $\Delta(Ax, x)$ is zero. Thus at least $|U|/2$ elements of U have a neighbor outside U. This implies that $h(G^{S,RSR}) > 0$. ◻

Remark 4.12. We observe that Theorem 4.10 appears to be a genuinely different reason for expansion, as an analysis akin to Lemma 4.3 does not appear to work in this setting when $ad \leqslant 0$. To illustrate this, suppose that $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $a, b > 0$ and $c, d < 0$. Setting $T = RSR$, one has $S(Q_1) \subseteq Q_2$, $S(Q_3) \subseteq Q_4$, $S^{-1}(Q_1) \subseteq Q_4$, $S^{-1}(Q_3) \subseteq Q_2$, $T(Q_1) \subseteq Q_4$, $T(Q_3) \subseteq Q_2$, $T^{-1}(Q_1) \subseteq Q_2$, $T^{-1}(Q_3) \subseteq Q_4$. Notice that unlike in the case of $T = S^\top$, one can only restrict the images to a single quadrant when the domain is Q_1 or Q_3. This seems to elude the simple counting argument of Lemma 4.3 and Corollary 4.4.

We can now prove our main theorem.

Proof of Theorem 4.8. Suppose first that $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$ satisfies $\det(S) = 1$ and $(a+d)(b+c) \neq 0$. Consider the matrix $S(RSR) = \begin{pmatrix} b^2+ad & a(b+c) \\ d(b+c) & c^2+ad \end{pmatrix}$. First, we have

$$\operatorname{tr}(SRSR) = b^2 + c^2 + 2ad = b^2 + c^2 + 2(1+bc) = (b+c)^2 + 2 > 2\,, \tag{4.4}$$

where we have used $ad - bc = 1$. Let $\left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right)$ denote $SRSR$ and note that (4.4) gives $u + x > 2$.

We have $(SRSR)^2 = \begin{pmatrix} u^2+vw & v(u+x) \\ w(u+x) & x^2+vw \end{pmatrix}$. The sum of the diagonal entries of this matrix is

$$u^2 + x^2 + 2vw = u^2 + x^2 + 2(ux-1) \geqslant (u+x)^2 - 2 > 2\,,$$

where we have used $1 = \det(SRSR) = ux - vw$ and $u + x \geqslant 2$. Furthermore, the sum of the off-diagonal entries satisfies

$$|(w+v)(u+x)| \geqslant 2|w+v| = 2|(a+d)(b+c)| \geqslant 2\,.$$

where we have used the assumption that $(a+d)(b+c) \neq 0$. Thus we can apply Theorem 4.11 to $(SRSR)^2$ to conclude that $h(G^{(SRSR)^2, R(SRSR)^2R}) > 0$. Noting that

$$R(SRSR)^2R = R(SRSR)(SRSR)R = (RSR)S(RSR)S$$

we can apply Lemma 4.5 to conclude that $h(G^{S,RSR}) > 0$ as well.

Finally, consider the case $\det(S) = -1$ and $(a+d)(b+c) \neq 0$. The matrix $S^2 = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}$ satisfies $\det(S^2) = 1$. The sum of the off-diagonal entries is $(b+c)(a+d) \neq 0$. The sum of the diagonal entries is $a^2 + d^2 + 2bc = a^2 + d^2 + 2(ad-1) = (a+d)^2 - 2 \neq 0$. Thus the preceding paragraph implies that $h(G^{S^2, RS^2R}) > 0$. Now Lemma 4.5 yields $h(G^{S,RSR}) > 0$ as well.

We now address the cases where the Cheeger constant is zero. Write $T = RSR$. If $a + d = 0$ then $S^2 = \pm I$ and $T^2 = \pm I$, so Lemma 4.7 yields $h(G^{S,T}) = 0$. If $b + c = 0$ then $ST = \begin{pmatrix} b^2+ad & 0 \\ 0 & b^2+ad \end{pmatrix} = \pm I$, so Lemma 4.6 yields $h(G^{S,T}) = 0$. ◻

4.2 Transformations for which $\{G_n^{S,T}\}$ is not an expander family

Here, we argue that if $T = S^{-1}$ or $S^4 = T^4 = I$, then the graphs $\{G_n^{S,T}\}$ do not form expander families. The arguments are related to Lemma 4.6 and Lemma 4.7, respectively, but we must also address the isoperimetric properties of boxes under linear transformations. To this end, we define for $L \geqslant 0$ the box $B_L = \{(x, y) \in \mathbb{R}^2 : -L \leqslant x \leqslant L, -L \leqslant y \leqslant L\}$. For a subset $\Omega \subseteq \mathbb{R}^2$, we write $[\Omega] = \Omega \cap \mathbb{Z}^2$. We also use $E_{\mathbb{Z}^2}$ to denote the edge set of the canonical graph on the integer lattice where $x, y \in \mathbb{Z}^2$ are connected by an edge if and only if $\|x - y\|_1 = 1$. The next lemma follows from elementary geometric considerations.

Lemma 4.13. *For every $S \in GL_2(\mathbb{Z})$, there is a constant $c > 0$ such that the following holds. For every $L \geqslant 0$, $S(B_L)$ is a parallelogram with area $4L^2$ and perimeter at most cL. Furthermore, we have $\liminf_{L\to\infty}[S(B_L)]/L^2 > 0$ and $\limsup_{L\to\infty} |E_{\mathbb{Z}^2}([S(B_L)])|/L \leqslant c$.*

We also have the following basic classification of matrices in $GL_2(\mathbb{Z})$; see, e.g. [Gun62, Ch. 1].

Lemma 4.14. *Every $S \in GL_2(\mathbb{Z})$ satisfies exactly one of the following.*

1. *S has order dividing 12.*
2. *S is conjugate in $GL_2(\mathbb{R})$ to $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ for some $\alpha \in \mathbb{R}$ with $|\alpha|, |\alpha^{-1}| \neq 1$.*
3. *S is conjugate in $GL_2(\mathbb{R})$ to $\pm 1 \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$ for some $\gamma \in \mathbb{R}$.*

The next lemma demonstrates our approach to proving non-expansion.

Lemma 4.15. *For any $S \in GL_2(\mathbb{Z})$, if $T \in \{S^{-1}, -S^{-1}\}$, it holds that $\{G_n^{S,T}\}$ is not an expander family.*

Proof. For $T \in \{S^{-1}, -S^{-1}\}$, let $\bar{G}$ have vertex set $\mathbb{Z}^2$ and edge set $E = E^{S,T} \cup E_{\mathbb{Z}^2}$. We will prove that $h(\bar{G}) = 0$. This is sufficient to show that $\{G_n^{S,T}\}$ is not an expander family. Indeed, if $\{U_k\}$ is a sequence of finite sets with $h_{\bar{G}}(U_k) \to 0$, then for each k one can choose the modulus n large enough to avoid "wrap around," yielding $h_{G_n^{S,T}}(U_k) = h_{\bar{G}}(U_k)$, where we consider U_k as a set of vertices in $G_n^{S,T}$ by reducing modulo n.

For $k \in \mathbb{N}$ and $L \geqslant 0$, consider the sets $\{U_k(L) \subseteq \mathbb{Z}^2\}$ given by

$$U_k(L) = [B_L] \cup [S(B_L)] \cup [S^2(B_L)] \cup \cdots \cup [S^k(B_L)] \, .$$

Observe that $B_L = -B_L$.

If we are in case (i) of Lemma 4.14, then $U_{k_0} = U_{k_0} + 1$ for some finite k_0. So by Lemma 4.13, we have $\liminf_{L\to\infty} |U_{k_0}(L)| \geqslant 4L^2$, while $E^{S,T}(U_{k_0}(L)) = \emptyset$ and $\limsup_{L\to\infty} |E_{\mathbb{Z}^2}(U_{k_0}(L))| \leqslant cL$, where c is some constant depending on S and k_0. Thus $\lim_{L\to\infty} |E(U_{k_0}(L))|/|U_{k_0}(L)| = 0$ and $h(\bar{G}) = 0$.

Now suppose that we are in case (ii) of Lemma 4.14 and, without loss of generality, $|\alpha| > 1$. In this case, for some constant $\varepsilon > 0$ (depending possibly on S) and every $k \in \mathbb{N}$, we have

$$\liminf_{L\to\infty} |U_k(L)|/L^2 \geqslant \varepsilon k \, . \tag{4.5}$$

This follows because the eccentricity of the parallelogram $S^k(B_L)$ grows exponentially fast; in fact, proportional to $|\alpha|^k$. Similarly, in case (iii) of Lemma 4.14, there is an $\varepsilon > 0$ (depending on both S) such that $\liminf_{L\to\infty} |U_k(L)|/L^2 \geqslant \varepsilon k$. To see this, it suffices to consider the case $\gamma = 1$ in (iii) (since ε can be depend on γ). In that case, the set $A_k = B_L \cup S(B_L) \cup \cdots \cup S^k(B_L)$ contains an isosceles triangle whose corners are $\{(0,0), (kL, L), (-kL, L)\}$, thus the volume of A_k is at least kL^2. Therefore (4.5) again holds.

On the other hand, from Lemma 4.13 it follows that for some constant $c > 0$ (depending on S and k), $\limsup_{L\to\infty} |E_{\mathbb{Z}^2}(U_k(L))|/L \leqslant c$ and $\limsup_{L\to\infty} |E^{S,T}(U_k(L))|/L^2 \leqslant c$. Therefore,

$$\limsup_{L\to\infty} \frac{|E(U_k(L))|}{|U_k(L)|} \leqslant \frac{c}{\varepsilon k} \, .$$

Taking $k \to \infty$ shows that $h(\bar{G}) = 0$.

Finally, suppose that S satisfies case (iii) of Lemma 4.14. $\square$

Lemma 4.16. *Suppose $S, T \in GL_2(\mathbb{Z})$ satisfy $S^4 = T^4 = I$. Then $\{G_n^{S,T}\}$ is not an expander family.*

Proof. Let $\bar{G}$ have vertex set $\mathbb{Z}^2$ and edge set $E = E^{S,T} \cup E_{\mathbb{Z}^2}$. As in Lemma 4.15, it will suffice to show that $h(\bar{G}) = 0$.

As in Lemma 4.7, an elementary calculation shows that if $A \in GL_2(\mathbb{Z})$ satisfies $\det(A) = 1$ and $A^2 = I$, then $A \in \{-I, I\}$. Thus $S^2, T^2 \in \{-I, I\}$. So for any $j_1, k_1, j_2, k_2, \ldots, j_m, k_m \in \mathbb{Z}$, we have

$$S^{j_1}T^{k_1}S^{j_2}T^{k_2}\cdots S^{j_m}T^{k_m} = (-1)^{i_0}T^{j_0}(ST)^j S^{k_0}.$$

for some $i_0, j_0, k_0 \in \{0, 1\}$ and $j \in \mathbb{N}$. Consider now the sets

$$U_k(L) = \left\{[T^{j_0}(ST)^j S^{k_0} B_L] : j_0, k_0 \in \{0,1\} \text{ and } 0 \leqslant j \leqslant k\right\}.$$

We can apply Lemma 4.14 to the matrix ST; the resulting case analysis is essentially the same as Lemma 4.15. ◻

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