Relations between scaling exponents in unimodular random graphs

James R. Lee*

Abstract

We investigate the validity of the "Einstein relations" in the general setting of unimodular random networks. These are equalities relating scaling exponents:

$$d_w = d_f + \tilde{\zeta},$$
$$d_s = 2d_f/d_w,$$

where d_w is the walk dimension, d_f is the fractal dimension, d_s is the spectral dimension, and $\tilde{\zeta}$ is the resistance exponent. Roughly speaking, this relates the mean displacement and return probability of a random walker to the density and conductivity of the underlying medium. We show that if d_f and $\tilde{\zeta} \geqslant 0$ exist, then d_w and d_s exist, and the aforementioned equalities hold. Moreover, our primary new estimate $d_w \geqslant d_f + \tilde{\zeta}$, is established for all $\tilde{\zeta} \in \mathbb{R}$.

For the uniform infinite planar triangulation (UIPT), this yields the consequence $d_w = 4$ using $d_f = 4$ (Angel 2003) and $\tilde{\zeta} = 0$ (established here as a consequence of the Liouville Quantum Gravity theory, following Gwynne-Miller 2017 and Ding-Gwynne 2020). The conclusion $d_w = 4$ had been previously established by Gwynne and Hutchcroft (2018) using more elaborate methods. A new consequence is that $d_w = d_f$ for the uniform infinite Schnyder-wood decorated triangulation, implying that the simple random walk is subdiffusive, since $d_f > 2$ (Ding and Gwynne 2020).

For the random walk on $\mathbb{Z}^2$ driven by conductances from an exponentiated Gaussian free field with exponent $\gamma > 0$, one has $d_f = d_f(\gamma)$ and $\tilde{\zeta} = 0$ (Biskup, Ding, and Goswami 2020). This yields $d_s = 2$ and $d_w = d_f$, confirming two predictions of those authors.

*University of Washington

Contents

1 Introduction

Consider an infinite, locally-finite graph $\mathcal{G}$ and a subgraph G of $\mathcal{G}$. For $x \in V(\mathcal{G})$, let $B^{\mathcal{G}}(x,R)$, denote the graph ball of radius R, and let $\tilde{B}(x,R) := B^{\mathcal{G}}(x,R) \cap V(G)$ denote this ball restricted to G. Let $d^{\mathcal{G}}(x,y)$ denote the path distance between a pair $x,y \in V(\mathcal{G})$. Denote by $\{X_n\}$ the simple random walk on G, and the discrete-time heat kernel

$$p_n^G(x,y) := \mathbb{P}[X_n = y \mid X_0 = x].$$

We write $\mathsf{R}_{\mathrm{eff}}^G(S \leftrightarrow T)$ for the effective resistance between two subsets $S,T \subseteq V(G)$.

For a variety of models arising in statistical physics, certain asymptotic geometric and spectral properties of the graph are known or conjectured to have scaling exponents:

$$\begin{aligned} |\tilde{B}(x,R)| &\sim R^{d_f} \\ \max_{1 \leqslant t \leqslant n} d^{\mathcal{G}}(X_0,X_t) &\sim n^{1/d_w} \\ \mathsf{R}_{\mathrm{eff}}^G\left(\tilde{B}(x,R) \leftrightarrow V(G) \setminus \tilde{B}(x,2R)\right) &\sim R^{\zeta} \\ p_{2n}^G(x,x) &\sim n^{-d_s/2}, \end{aligned} \tag{1.1}$$

where one takes $n,R \to \infty$, but we leave the meaning of "$\sim$" imprecise for a moment. These exponents are, respectively, referred to as the *fractal dimension*, *walk dimension*, *spectral dimension*, and *resistance exponent*. We refer to the extensive discussion in [BH00, Ch. 5–6].

Moreover, by modeling the subgraph G as a homogenous underlying substrate with density and conductivity prescribed by d_f and $\tilde{\zeta}$, one obtains the plausible relations

$$d_w = d_f + \tilde{\zeta} \tag{1.2}$$

$$d_s = \frac{2d_f}{d_w}. \tag{1.3}$$

In the regime $\tilde{\zeta} > 0$, these relations have been rigorously verified under somewhat stronger assumptions in the setting of *strongly recurrent graphs* (see [Tel90, Tel95]) and [Bar98, KM08, Kum14b]). In the latter set of works, the most significant departure from our assumptions is the stronger requirement for uniform control on pointwise effective resistances of the form

$$\max\left\{\mathsf{R}^G_{\text{eff}}(x,y) : y \in B^G(x,R)\right\} \leqslant R^{\tilde{\zeta}+o(1)}, \quad x \in V(G). \tag{1.4}$$

Such methods have been extended to the setting where (G, ρ) is a random rooted graph ([KM08, BJKS08]) under the statistical assumption that these relations hold sufficiently often for all sufficiently large scales, and only for balls around the root. Let us remark that the strongly recurrent theory can be modified to apply when $\tilde{\zeta} = 0$ (see Section 1.2.1 below), with a suitable modification of the assumptions, and the main new feature of our approach is that we don't require pointwise upper bounds as in (1.4).

Our main contribution is to establish (1.2) and (1.3) under somewhat less restrictive conditions, but using an additional feature of many such models: Unimodularity of the random rooted graph (G, ρ). When $\tilde{\zeta} \leqslant 0$ (equivalently, $d_s \geqslant 2$), it has been significantly more challenging to characterize situations where (1.2)–(1.3) hold; see, for instance, Open Problem III in [Kum14a]. Our main new estimate is the exit time relation $d_w \geqslant d_f + \tilde{\zeta}$, which is established for all $\tilde{\zeta} \in \mathbb{R}$. This is a non-trivial subdiffusive estimate whenever $d_f + \tilde{\zeta} > 2$, and applies equally well to models where the random walk is transient. We now highlight some notable settings in which the relations can be applied.

The IIC in high dimensions. As a prominent example, consider the resolution by Kozma and Nachmias [KN09] of the Alexander-Orbach conjecture for the incipient infinite cluster (IIC) of critical percolation on $\mathbb{Z}^d$, for d sufficiently large. If $(G, 0)$ denotes the IIC, then in our language, $\mathcal{G} = G$, as they consider the intrinsic graph metric, and establish that for every $\lambda > 1$ and $r \geqslant 1$, with probability at least $1 - p(\lambda)$, it holds that

$$\lambda^{-1} r^2 \leqslant |B^G(0,r)| \leqslant \lambda r^2, \tag{1.5}$$

$$\mathsf{R}^G_{\text{eff}}(0, \partial B^G(0,r)) \geqslant \lambda^{-1} r, \tag{1.6}$$

where $p(\lambda) \leqslant O(\lambda^{-q})$ for some $q > 1$. One should consider this a statistical verification that $d_f = 2$ and $\tilde{\zeta} = 1$, as in this setting, one gets the analog of (1.4) for free free from the trivial bound $\mathsf{R}^{\text{IIC}}_{\text{eff}}(0,x) \leqslant d^{\text{IIC}}(0,x)$.

Earlier, Barlow, Járai, Kumagai, and Slade [BJKS08] verified (1.2)–(1.3) under these assumptions, allowing Kozma and Nachmias to confirm the conjectured values $d_w = 3$ and $d_s = 4/3$. One can consult [Kum14a, §4.2.2] for several further examples where $\tilde{\zeta} > 0$ and (1.2)–(1.3) hold using the strongly recurrent theory.

The uniform infinite planar triangulation. Consider, on the other hand, the uniform infinite planar triangulation (UIPT) considered as a random rooted graph (G, ρ). In this case, Angel [Ang03] established that almost surely

$$\lim_{R\to\infty} \frac{\log |B^G(\rho, R)|}{\log R} = 4, \tag{1.7}$$

and Gwynne and Miller [GM17] showed that almost surely

$$\lim_{R\to\infty} \frac{\log \mathsf{R}^G_{\mathrm{eff}}(\rho \leftrightarrow V(G) \setminus B^G(\rho, R))}{\log R} = 0\,.$$

This falls short of verifying (1.1). Nevertheless, we show in Section 4.3 that $\tilde{\zeta} = 0$ is a consequence of the Liouville Quantum Gravity (LQG) estimates derived in [DMS14, GM17, GMS19, GHS17, DG20]. But while the known statistics of $|B^G(\rho, R)|$ are suitable to allow application of the strongly recurrent theory, this does not hold for the effective resistance bounds.

This is highlighted by Gwynne and Hutchcroft [GH18] who establish $d_w = 4$ using even finer aspects of the LQG theory. The authors state "while it may be possible in principle to prove $\beta \geqslant 4$ using electrical techniques, doing so appears to require matching upper and lower bounds for effective resistances [...] differing by at most a constant order multiplicative factor." Our methods show that, when leveraging unimodularity, even coarse estimates with subpolynomial errors suffice.

It is open whether $\tilde{\zeta} = 0$ or $d_w = 4$ for the uniform infinite planar *quadrangulation* (UIPQ), but our verification of (1.2) shows that only one such equality needs to be established.

Random planar maps in the γ-LQG universality class. More generally, we will establish in Section 4.3 that $\tilde{\zeta} = 0$ whenever a random planar map (G, ρ) can be coupled to a γ-mated-CRT map with $\gamma \in (0, 2)$. The connection between such maps and LQG was established in [DMS14].

This family includes the UIPT (where $\gamma = \sqrt{8/3}$). Ding and Gwynne [DG20] have shown that d_f exists for such maps, and Gwynne and Huthcroft [GH18] established that $d_w = d_f$ for most known examples, but not for the uniform infinite Schnyder-wood decorated triangulation [LSW17] (where $\gamma = 1$), for a technical reason underlying the construction of a certain coupling (see [GH18, Rem. 2.11]). We mention this primarily to emphasize the utility of a general theorem, since it is likely the technical obstacle could have been circumvented with sufficient effort.

Random walk driven by a Gaussian free field. Biskup, Ding, and Goswami [BDG20] study the model of random walk on $\mathcal{G} = \mathbb{Z}^2$ with random conductances $c^G(\{u, v\}) = e^{\gamma(\eta_v - \eta_u)}$, where $\gamma > 0$, and $\{\eta_v : v \in \mathbb{Z}^2\}$ is the discrete Gaussian free field (GFF) on $\mathbb{Z}^2$ grounded at the origin.

In this case, one has

$$d_f = \begin{cases} 2 + 2(\gamma/\gamma_c)^2 & \gamma \leqslant \gamma_c = \sqrt{\pi/2}, \\ 4\gamma/\gamma_c & \text{otherwise.} \end{cases}$$

(See below for the definition of d_f when the edges have conductances.)

In Section 4.4, we recall the model formally and observe that the paper [BDG20] contains estimates that establish $\tilde{\zeta} = 0$ for every $\gamma > 0$. Hence the relations (1.2)–(1.3) yield $d_f = d_w$ and $d_s = 2$, both of which were conjectured in [BDG20], though only annealed estimates were obtained. (See Section 1.3.1 for a brief discussion of why our approach yields two-sided quenched bounds.)

The IIC in dimension two. Consider the incipient infinite cluster for 2D critical percolation [Kes86], which can be realized as a unimodular random subgraph $(G, 0)$ of $\mathcal{G} = \mathbb{Z}^2$ [JÓ3]. It is known that $d_f = 91/48$ in the 2D hexagonal lattice [LSW02, Smi01], and the same value is conjectured to hold for all 2D lattices regardless of the local structure.

Existence of the exponent $\tilde{\zeta}$ is open for any lattice; experiments give the estimate $\tilde{\zeta} = 0.9825 \pm 0.0008$ [Gra99]. Indeed, the most precise experimental estimate for $d_w = 2.8784 \pm 0.0008$ is derived from estimates for $\tilde{\zeta}$, and our verfication of (1.2) puts this on rigorous footing (assuming, of course, that $\tilde{\zeta}$ is well-defined). Indeed, one motivation for our work was the question of whether the exponent d_w should be a conformal invariant of critical 2D percolation, and it is plausibly more tractable to establish this for $\tilde{\zeta}$.

1.1 Reversible random networks

We consider random rooted networks (G, ρ, c^G, ξ) where G is a locally-finite, connected graph, $\rho \in V(G)$, and $c^G : E(G) \to [0, \infty)$ are edge conductances. We allow $E(G)$ to contain self-loops $\{v, v\}$ for $v \in V(G)$. Here, $\xi : V(G) \cup E(G) \to \Xi$ is an auxiliary marking, where Ξ is some Polish mark space. Denote by $\{X_n\}$ the random walk on G with $X_0 = \rho$ and transition probabilities

$$p_n^G(u, v) := \mathbb{P}\left[X_{n+1} = v \mid X_n = u\right] = \frac{c^G(\{u, v\})}{c_u^G}, \tag{1.8}$$

where we denote $c_u^G := \sum_{v:\{u,v\}\in E(G)} c^G(\{u, v\})$. Say that (G, ρ, c^G, ξ) is a *reversible random network* if:

1. Almost surely $c_\rho^G > 0$.
2. (G, X_0, X_1, c^G, ξ) and (G, X_1, X_0, c^G, ξ) have the same law.

We will usually write a reversible random network as (G, ρ, ξ), allowing the conductances to remain implicit. Note that we allow the possibility $c^G(\{u, v\}) = 0$ when $\{u, v\} \in E(G)$. In this sense, random walks occur on the subnetwork G_+ with $V(G_+) = \{x \in V(G) : c_x^G > 0\}$ and $E(G_+) = \{\{x, y\} \in V(G) : c^G(\{x, y\}) > 0\}$, while distances are measured in the path metric d^G.

Throughout, we will make the following mild boundedness assumptions:

$$\begin{aligned} &\mathbb{E}[1/c_\rho^G] < \infty, \\ &\lim_{R\to\infty} \frac{\log |B^G(\rho, R)|}{R} = 0 \quad \text{almost surely.} \end{aligned} \tag{$\mathscr{B}$}$$

Note that the second equation asserts that the *cardinality* of graph balls grows subexponentially in the radius.

Unimodular random graphs are defined in Section 2.3 when we need to employ the Mass-Transport Principle. For now, it suffices to say that there is a one-to-one correspondence:

$$\begin{array}{ccc} (G, \rho, \xi) \text{ reversible} & \longleftrightarrow & (\tilde{G}, \tilde{\rho}, \tilde{\xi}) \text{ unimodular} \\ \mathbb{E}[1/c_\rho^G] < \infty & & \mathbb{E}[c_{\tilde{\rho}}^{\tilde{G}}] < \infty \end{array}$$

Indeed, if μ and $\tilde{\mu}$ are the respective measures, then the correspondence is given by a change of law

$$\frac{d\mu}{d\tilde{\mu}}(G_0, \rho_0, \xi_0) = \frac{c_{\rho_0}^{G_0}}{\mathbb{E}[c_{\tilde{\rho}}^{\tilde{G}}]}\,.$$

where $d\mu/d\tilde{\mu}$ is the Radon-Nikodym derivative. We refer to [AL07] for an extensive reference on unimodular random graphs, and to [BC12, Prop. 2.5] for the connection between unimodular and reversible random graphs.

1.2 Almost sure scaling exponents

Consider two sequences $\{A_n\}$ and $\{B_n\}$ of positive real-valued random variables. Write $A_n \lessapprox B_n$ if almost surely:

$$\limsup_{n\to\infty} \frac{\log A_n - \log B_n}{\log n} \leqslant 0,$$

and $A_n \approx B_n$ for the conjuction of $A_n \lessapprox B_n$ and $B_n \lessapprox A_n$. Note that $A_n \lessapprox n^d$ if and only if, for every $\delta > 0$, almost surely $A_n \leqslant n^{d+\delta}$ for n sufficiently large.

In what follows, we consider a reversible random network (G, ρ) (cf. Section 1.1). Define the random variables:

$$\begin{aligned} \sigma_R &:= \min\{n \geqslant 0 : d^G(X_0, X_n) > R\}, \\ \mathcal{M}_n &:= \max_{0\leqslant t\leqslant n} d^G(X_0, X_t), \end{aligned}$$

and define the walk exponents d_w and β by

$$\begin{aligned} \sigma_R &\approx R^{d_w} \\ \mathcal{M}_n &\approx n^{1/\beta}, \end{aligned}$$

assuming these limits exist. In that case we, we will use the language "d_w exists" or "β exists."[1]

Denote the volume function

$$\mathrm{vol}^G(x, R) := \sum_{y\in B^G(x,R)} c_y^G,$$

and define d_f as the asymptotic growth rate of the volume:

$$\mathrm{vol}^G(\rho, R) \approx R^{d_f},$$

Define the spectral dimension by

$$p_{2n}^G(\rho, \rho) \approx n^{-d_s/2}.$$

Let us define upper and lower resistance exponents. Define $\tilde{\zeta}$ and $\tilde{\zeta}_0$ as the largest and smallest values, respectively, such that, for every $\delta > 0$, almost surely, for all but finitely many $R \in \mathbb{N}$:

$$R^{\tilde{\zeta}} \leqslant \mathsf{R}_{\mathrm{eff}}^G \left(B^G(\rho, R^{1-\delta}) \leftrightarrow \bar{B}^G(\rho, R)\right) \leqslant \mathsf{R}_{\mathrm{eff}}^G \left(\rho \leftrightarrow \bar{B}^G(\rho, R)\right) \leqslant R^{\tilde{\zeta}_0+\delta}, \tag{1.9}$$

where we have denoted the complement of $B^G(\rho, R)$ in G by

$$\bar{B}^G(\rho, R) := V(G) \setminus B^G(\rho, R).$$

The exponents $\tilde{\zeta} \leqslant \tilde{\zeta}_0$ always exist and $\tilde{\zeta}_0 \geqslant 0$. The exponent $\tilde{\zeta}$ is referred to as the "resistance exponent" in the statistical physics literature (see [BH00, §5.3]); see Remark 1.2 below. We emphasize that all the exponents we define are not random variables, but functions of the law of (G, ρ). Our main theorem can then be stated as follows.

[1] In the next section, we control the annealed variants as well, where one takes expectations over the random walk.

Theorem 1.1. *Suppose that (G, ρ) is a reversible random network satisfying ($\mathcal{B}$). If d_f exists and $\tilde{\zeta} = \tilde{\zeta}_0$, then the exponents d_w, β, and d_s exist and it holds that*

$$\begin{aligned} d_w &= \beta = d_f + \tilde{\zeta}, \\ d_s &= \frac{2d_f}{d_w}. \end{aligned}$$

See Corollary 1.6 for further equalities involving annealed versions of d_w and β.

Remark 1.2 (The resistance exponents)**.** The resistance exponent is usually characterized heuristically as the value $\tilde{\zeta}$ such

$$\mathsf{R}_{\text{eff}}^{G}\left(B^G(\rho, R) \leftrightarrow \bar{B}^G(\rho, 2R)\right) \approx R^{\tilde{\zeta}}. \tag{1.10}$$

So the left-hand side of (1.9) would naturally be replaced by

$$\mathsf{R}_{\text{eff}}^{G}\left(B^G(\rho, R) \leftrightarrow \bar{B}^G(\rho, 2R)\right) \geqslant R^{\tilde{\zeta}-\delta}.$$

The lower bound we require is substantially weaker, allowing one to consider spatial fluctuations of magnitude $R^{o(1)}$. The upper bound in (1.9), on the other hand, is somewhat stronger than (1.10), and encodes a level of spectral regularity. For instance, if G satisfies an elliptic Harnack inequality and is "strongly recurrent" in the sense of [Tel06, Def. 2.1], then

$$\mathsf{R}_{\text{eff}}^{G}(B^G(\rho, R) \leftrightarrow \bar{B}^G(\rho, 2R)) \approx \mathsf{R}_{\text{eff}}^{G}(\rho \leftrightarrow \bar{B}^G(\rho, R)).$$

See [Tel06, Thm. 4.6] and Theorem 4.7.

1.2.1 Comparison to the strongly recurrent theory

Let us try to interpret the strongly recurrent theory (cf. Assumption 1.2 in [KM08]) in the setting of subpolynomial errors. The resistance assumptions would take the form: For every $\delta > 0$, almost surely, for R sufficiently large:

$$\max\left\{\mathsf{R}_{\text{eff}}^{G}(\rho \leftrightarrow x) : x \in B^G(\rho, R)\right\} \leqslant R^{\zeta+\delta}, \tag{1.11}$$

$$\mathsf{R}_{\text{eff}}^{G}\left(\rho \leftrightarrow \bar{B}^G(\rho, R)\right) \geqslant R^{\zeta-\delta}. \tag{1.12}$$

These assumptions imply that when $\zeta > 0$, it holds that $\tilde{\zeta} = \tilde{\zeta}_0 = \zeta$; this is proved in Theorem 4.7. Hence the theory we present (in the setting of unimodular random graphs) is more general, at least in terms of concluding the relations (1.2) and (1.3).

Under assumptions (1.11) and (1.12), one can uniformly lower bound the Green kernel $\mathsf{g}_{B^G(\rho,R')}(\rho, x)$ (see Section 4.2 for definitions) for all points $x \in B^G(\rho, R)$ and some $R' \gg R$. In other words, every point in $B^G(\rho, R)$ is visited often on average before the random walk exits $B^G(\rho, R')$. See, for instance, [BCK05, §3.2]. This yields a subdiffusive estimate on the speed of the random walk, specifically an almost sure lower bound on $\mathbb{E}[\sigma_R \mid (G, \rho)]$.

Instead of a pointwise bound, we use a lower bound on $\tilde{\zeta}$ to deform the graph metric d^G (see the next section). The effective resistance across an annulus being large is equivalent to its discrete extremal length being large (see Section 2.1). Thus in most scales and localities, we can extract a metric that locally "stretches" the space. By randomly covering the space with annuli at all scales, we obtain a "quasisymmetric" deformation (only in an asymptotic, statistical sense) that is bigger by a power than the graph metric. Finally, by applying Markov type theory, we bound the speed of the walk in the stretched metric, which leads to a stronger bound in the graph metric.

1.3 Upper and lower exponents

Even when scaling exponents do not exist, our arguments give inequalities between various superior and inferior limits. Given a sequence $\{\mathcal{E}_n : n \geqslant 1\}$ of events on some probability space, let us say that they occur *almost surely eventually (a.s.e.)* if $\mathbb{P}[\#\{n \geqslant 1 : \neg\mathcal{E}_n\} < \infty] = 1$.

For a family $\{A_n\}$ of random variables, we will define $\underline{d}$ and $\bar{d}$ to be the largest and smallest values, respectively, such that for every $\delta > 0$, almost surely eventually,

$$n^{\underline{d}+\delta} \leqslant A_n \leqslant n^{\bar{d}+\delta},$$

where we allow the exponents to take values $\{-\infty, +\infty\}$ if no such number exists. Note that $A_n \approxeq n^d$ (i.e., the exponent d "exists") if and only if $\bar{d} = \underline{d}$.

Let us consider the corresponding extremal exponents such that for every $\delta > 0$ the following relations hold almost surely eventually (with respect to $n, R \geqslant 1$):

$$\begin{aligned}
R^{\underline{d}_f-\delta} &\leqslant \mathrm{vol}^G(\rho, R) \leqslant R^{\bar{d}_f+\delta} \\
R^{\underline{d}_w-\delta} &\leqslant \sigma_R \leqslant R^{\bar{d}_w+\delta} \\
R^{\underline{d}_w^{\mathcal{A}}-\delta} &\leqslant \mathbb{E}[\sigma_R \mid (G, \rho)] \leqslant R^{\bar{d}_w^{\mathcal{A}}+\delta} \\
n^{-\delta+1/\bar{\beta}} &\leqslant \mathcal{M}_n \leqslant n^{\delta+1/\underline{\beta}} \\
n^{-\delta+2/\bar{\beta}^{\mathcal{A}}} &\leqslant \mathbb{E}[\mathcal{M}_n^2 \mid (G, \rho)] \leqslant n^{\delta+2/\underline{\beta}^{\mathcal{A}}} \\
n^{-\delta-\bar{d}_s/2} &\leqslant p_{2n}^G(\rho, \rho) \leqslant n^{\delta-\underline{d}_s/2},
\end{aligned}$$

We will establish the following chains of inequalities, which together prove Theorem 1.1.

Theorem 1.3. *Suppose that (G, ρ) is a reversible random network satisfying $(\mathcal{B})$. Then it holds that*

$$\begin{aligned}
2\underline{d}_f - \bar{d}_f + \tilde{\zeta} &\leqslant \underline{\beta}^{\mathcal{A}} && (1.13) \\
&\leqslant \underline{\beta} && (1.14) \\
&\leqslant \underline{d}_w \wedge \bar{\beta} && (1.15) \\
&\leqslant \underline{d}_w \vee \bar{\beta} && \\
&\leqslant \bar{d}_w && (1.16) \\
&\leqslant \bar{d}_w^{\mathcal{A}} && (1.17) \\
&\leqslant \bar{d}_f + \tilde{\zeta}_0, && (1.18)
\end{aligned}$$

and

$$2\left(1 - \frac{\tilde{\zeta}_0}{\underline{d}_w}\right) \leqslant \underline{d}_s \leqslant \bar{d}_s \leqslant \frac{2\bar{d}_f}{\underline{d}_w}. \qquad (1.19)$$

In fact, all the inequalities hold under the assumption $\mathbb{E}[1/c_\rho^G] < \infty$ with the exception of (1.13) which requires both conditions in $(\mathcal{B})$.

To see that this yields Theorem 1.1, simply note that when $\tilde{\zeta} = \tilde{\zeta}_0$ and $\underline{d}_f = \bar{d}_f$, then the upper and lower bounds in (1.18) and (1.13) match, and the upper and lower bounds in (1.19) are both equal to $2d_f/d_w$ because the first set of inequalities implies $d_w = d_f + \tilde{\zeta}$.

Remark 1.4 (Negative resistance exponent)**.** For $\tilde{\zeta} < 0$, Theorem 1.3 yields (assuming d_s, d_w exist):

$$\begin{aligned} d_w &\geqslant d_f + \tilde{\zeta} \\ 2 \leqslant d_s &\leqslant \frac{2d_f}{d_f + \tilde{\zeta}}. \end{aligned}$$

Without further assumptions, these equalities cannot be made tight. Indeed, for every $\varepsilon > 0$, there are unimodular random planar graphs of almost sure uniform polynomial growth and $\tilde{\zeta} \leqslant -1 + \varepsilon$ [EL20]. Yet these graphs must satisfy $d_s \leqslant 2$ [Lee17].

In the general setting of Dirichlet forms on metric measure spaces, the "resistance conjecture" [GHL15, pg. 1493] asserts conditions under which (1.2)–(1.3) might hold even for $\tilde{\zeta} < 0$. The primary additional condition is a Poincaré inequality with matching exponent. In our setting, the existence of d_f does not yield the "bounded covering" property, that almost surely every ball $B^G(\rho, R)$ can be covered by $O(1)$ balls of radius $R/2$. It seems likely that a variant of this condition should also be imposed to recover (1.2)–(1.3).

Let us give a brief outline of how Theorem 1.3 is proved. The unlabeled inequality is trivial. Both inequalities (1.14) and (1.17) are a straightforward consequence of Markov's inequality and the Borel-Cantelli lemma. The content of inequalities (1.15) and (1.16) lies in the relations $\underline{\beta} \leqslant \underline{d}_w$ and $\bar{\beta} \leqslant \bar{d}_w$. These follow from the elementary inequality

$$\mathcal{M}_n \geqslant \mathbb{1}_{\{\sigma_R \leqslant n\}} R, \tag{1.20}$$

which gives the implications

$$\begin{aligned} \mathcal{M}_n \leqslant n^{1/(\underline{\beta}-\delta)} \text{ a.s.e.} &\implies \sigma_R \geqslant R^{\underline{\beta}-\delta} \text{ a.s.e.}, \\ \sigma_R \leqslant R^{\bar{d}_w+\delta} \text{ a.s.e.} &\implies \mathcal{M}_n \geqslant n^{1/(\bar{d}_w+\delta)} \text{ a.s.e.} \end{aligned}$$

Since these hold for every $\delta > 0$, we conclude that $\bar{\beta} \leqslant \bar{d}_w$ and $\underline{d}_w \geqslant \bar{\beta}$, as desired. Inequalities (1.18) and (1.19) are proved in Section 4.2 using the standard relationships between effective resistance, the Green kernel, and return probabilities. That leaves (1.13), which relies on Markov type theory.

Reversible random weights. We consider a reversible random graph (G, ρ) and random edge weights $\omega : E(G) \to \mathbb{R}_+$. Denote by dist_ω^G the ω-weighted path metric in G.[2] When (G, ρ, ω) is a reversible random network and (G, ρ) is clear from context, we will say simply that the weight ω is *reversible*. Define the balls, for $x \in V(G)$ and $R \geqslant 0$,

$$\mathcal{B}_\omega^G(x, R) := \left\{ y \in V(G) : \mathrm{dist}_\omega^G(x, y) \leqslant R \right\}.$$

The next theorem (proved in Section 3) is a variant of the approach pursued in [Lee17].

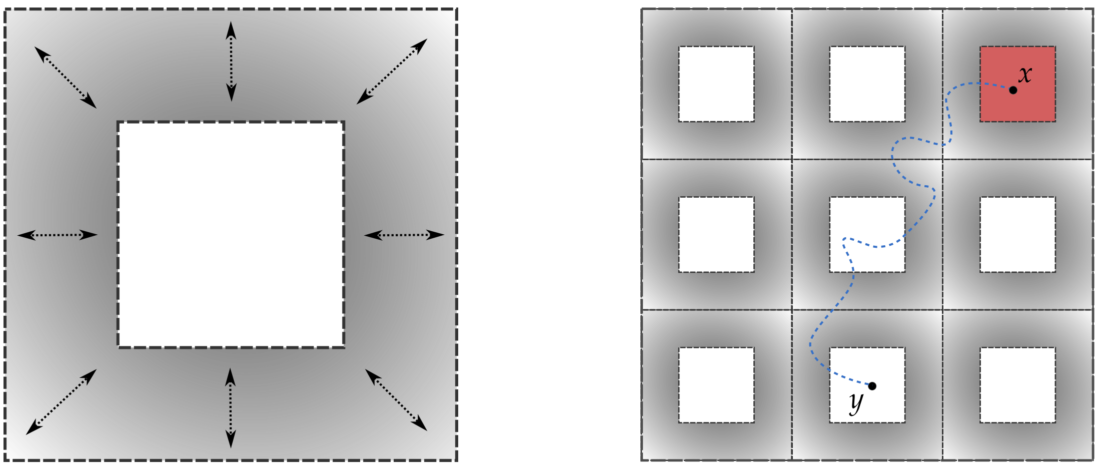
[2]Strictly speaking, since we allow ω to take the value 0, this is only a pseudometric, but that will not present any difficulty.

Theorem 1.5. *Suppose (G, ρ, ω) is a reversible random network for which ($\mathcal{B}$) holds, and additionally the random weights $\omega : E(G) \to \mathbb{R}_+$ satisfy*

$$\mathbb{E}\left[\omega(X_0, X_1)^2\right] < \infty, \tag{1.21}$$

where $\{X_n\}$ is random walk on G started from $X_0 = \rho$. Then it holds that

$$\mathbb{E}\left[\max_{0 \leqslant t \leqslant n} \mathrm{dist}^G_\omega(X_0, X_t)^2 \mid (G, \rho, \omega)\right] \lessapprox n. \tag{1.22}$$



(a) Stretching an annulus (b) Tiling by annuli

Figure 1: Streching the graph at a fixed scale

Given this theorem, let us now sketch the proof of (1.13). Consider a graph annulus

$$\mathcal{A} := \{x \in V(G) : R \leqslant d^G(\rho, x) \leqslant R^{1+\delta}\}.$$

If the effective resistance across $\mathcal{A}$ is at least $R^{\tilde{\zeta}}$, then by the duality between effective resistance and discrete extremal length (see Section 2.1), there is a length functional $L : E(G[\mathcal{A}]) \to \mathbb{R}_+$ satisfying

$$\sum_{\{x,y\} \in E(G[\mathcal{A}])} c^G(\{x, y\}) L(x, y)^2 \leqslant R^{-\tilde{\zeta}}$$
$$\mathrm{dist}_L^{G[\mathcal{A}]}\left(B^G(x, R), \bar{B}^G(x, R^{1+\delta})\right) \geqslant 1,$$

where $G[\mathcal{A}]$ is the subgraph induced on $\mathcal{A}$.

Let us suppose that the total volume in $\mathcal{A}$ satisfies

$$V_{\mathcal{A}} := \sum_{\{u,v\} \in E(G[\mathcal{A}])} c_u^G \approx R^{d_f},$$

and we normalize L to have expectation squared $\leqslant 1$ under the measure $c^G(\{x, y\})/V_{\mathcal{A}}$ on $E(G[\mathcal{A}])$:

$$\hat{L} := R^{(\tilde{\zeta}+d_f)/2} L \approx \left(\frac{R^{\tilde{\zeta}}}{V_{\mathcal{A}}}\right)^{1/2} L.$$

This yields:

$$\mathrm{dist}_{\hat{L}}^{G[\mathcal{A}]}\left(B^G(x,R), \bar{B}^G(x,R^{1+\delta})\right) \geqslant R^{(\tilde{\zeta}+d_f)/2},$$

meaning that, with normalized unit area, $\hat{L}$ "stretches" the graph annulus by a positive power when $\tilde{\zeta} + d_f > 2$ (see Figure 1(a)).

If G is sufficiently regular (e.g., a lattice), then we could tile annuli at this scale (as in Figure 1(b)) so that if we define ω_R as the sum of the length functionals over the tiled annuli, then for any pair $x, y \in V(G)$ with $d^G(x,y) \geqslant R^{1+\delta}$ and at least one of x or y in the center of an annulus, we would have $\mathrm{dist}_{\omega_R}^G(x,y) \geqslant R^{(\tilde{\zeta}+d_f)/2}$. In a finite-dimensional lattice, a bounded number of shifts of the tiling is sufficient for every vertex to reside in the center of some annulus.

By combining length functionals over all scales, and replacing the regular tiling by a suitable random family of annuli, we obtain, for every $\delta > 0$, a reversible random weight $\omega : E(G) \to \mathbb{R}_+$ satisfying (1.21) (intuitively, because of the unit area normalization), and such that almost surely eventually

$$\mathrm{dist}_{\omega}^G\left(\rho, \bar{B}^G(\rho, R)\right) \geqslant R^{(d-\delta)/2}, \tag{1.23}$$

where $d := d_f + \tilde{\zeta}$. In other words, distances in dist_ω^G are (asymptotically) increased by power $(d-\delta)/2$.

Thus (1.22) gives for every $\delta > 0$, eventually almost surely

$$\mathbb{E}\left[\mathcal{M}_n^2 \mid (G,\rho)\right] \leqslant n^{2(1+\delta)/(d-\delta)}.$$

Taking $\delta \to 0$ yields $\underline{\beta}^{\mathcal{A}} \geqslant d$. This is carried out formally in Section 4.1.

1.3.1 Annealed vs. quenched subdiffusivity

One can express $\mathbb{E}[\sigma_R \mid (G,\rho)]$ in terms of electrical potentials. Doing so, it is natural to arive at two-sided annealed estimates:

$$R^{d-o(1)} \leqslant \mathbb{E}[\sigma_R] \leqslant R^{d+o(1)} \quad \text{as } R \to \infty,$$

where expectation is taken over both the walk and the random network (G,ρ). Then a standard application of Borel-Cantelli gives that almost surely $\sigma_R \leqslant R^{d+o(1)}$, but not an almost sure lower bound. On the other hand,

$$\mathbb{E}[\mathcal{M}_n^2] \leqslant n^{2/d+o(1)} \quad \text{as } n \to \infty$$

provides that $\mathcal{M}_n \leqslant n^{1/d+o(1)}$ almost surely, which entails $\sigma_R \geqslant R^{d-o(1)}$ almost surely.

In this way, the two exponents β and d_w are complementary, allowing one to obtain two-sided quenched estimates from two-sided annealed estimates. This is crucial for establishing $d_s = 2d_f/d_w$, as the upper bound in (1.19) uses the fully quenched exponent $\underline{d}_w$ which, in the setting of Theorem 1.1, arises from the lower bound (1.13) on the (partially) *annealed* exponent $\underline{\beta}^{\mathcal{A}}$. We remark on the following strengthening of Theorem 1.1.

Corollary 1.6. *Under the assumptions of Theorem 1.1, it additionally holds that $\beta = \beta^{\mathcal{A}}$ and $d_w = d_w^{\mathcal{A}}$.*

Proof. We may assume that d_w and β exist, and $d_w = \beta$. From Theorem 1.3 we obtain:

$$\underline{\beta}^{\mathcal{A}} = \beta = \bar{d}_w^{\mathcal{A}}.$$

The relations $\underline{\beta} \leqslant \underline{d}_w^{\mathcal{A}}$ and $\bar{\beta}^{\mathcal{A}} \leqslant \bar{d}_w$ follow from (1.20), yielding

$$\beta \geqslant \bar{\beta}^{\mathcal{A}} \geqslant \underline{\beta}^{\mathcal{A}} = \beta,$$
$$\beta \leqslant \underline{d}_w^{\mathcal{A}} \leqslant \bar{d}_w^{\mathcal{A}} = \beta. \qquad \square$$

2 Reversible random weights

Throughout this section, (G, ρ) is a reversible random network satisfying $\mathbb{E}[1/c_\rho^G] < \infty$.

2.1 Modulus and effective resistance

For a network G and two finite subsets $S, T \subseteq V(G)$, define the *modulus*

$$\mathsf{Mod}^G(S \leftrightarrow T) := \min \left\{ \|\omega\|^2_{\ell^2(c^G)} : \mathrm{dist}^G_\omega(S, T) \geqslant 1 \right\},$$

where the minimum is over all weights $\omega : E(G) \to \mathbb{R}_+$, and

$$\|\omega\|^2_{\ell^2(c^G)} = \sum_{e \in E(G)} c^G(e) |\omega(e)|^2.$$

For $x \in V(G)$ and $0 < r < R$, define the annular modulus:

$$\mathsf{M}^G(x, r, R) := \mathsf{Mod}^G \left(B^G(x, r) \leftrightarrow \bar{B}^G(x, R) \right).$$

Denote by $\omega_{(G,x,r,R)}$ the unique weight achieving the value $\mathsf{M}^G(x, r, R)$. The standard duality between effective resistance and discrete extremal length [Duf62] gives an alternate characterization of $\mathsf{M}^G(x, r, R)$, as follows.

Lemma 2.1. *For any graph G and subsets $S, T \subseteq V(G)$, it holds that*

$$\mathsf{Mod}^G(S \leftrightarrow T) = \left(\mathsf{R}^G_{\mathrm{eff}}(S \leftrightarrow T) \right)^{-1}, \tag{2.1}$$

hence for all $x \in V(G)$ and $0 \leqslant r \leqslant R$,

$$\mathsf{M}^G(x, r, R) = \left(\mathsf{R}^G_{\mathrm{eff}} \left(B^G(x, r) \leftrightarrow \bar{B}^G(x, R) \right) \right)^{-1}.$$

For a function $g : V(G) \to \mathbb{R}$, we denote the Dirichlet energy

$$\mathscr{E}^G(g) := \sum_{\{x,y\} \in E(G)} c^G(\{x, y\}) |g(x) - g(y)|^2.$$

We will make use of the Dirichlet principle (see [LP16, Ch. 2]):

$$\mathsf{R}^G_{\mathrm{eff}}(S \leftrightarrow T) = \left(\min \left\{ \mathscr{E}^G(g) : g|_S \equiv 0, g|_T \equiv 1 \right\} \right)^{-1}, \tag{2.2}$$

and when G is connected, the minimizer of (2.2) is the unique function harmonic on $V(G) \setminus (S \cup T)$ with the given boundary values.

It will be helpul to note that, by the series law for effective resistances, we can equivalently characterize $\tilde{\zeta}$ (recall (1.9)) as the largest value such that for every $\delta > 0$, almost surely eventually

$$\mathsf{R}^G_{\mathrm{eff}} \left(B^G(\rho, R) \leftrightarrow \bar{B}^G(\rho, R^{1+\delta}) \right) \geqslant R^{\tilde{\zeta} - \delta}. \tag{2.3}$$

2.2 Approximate nets

Fix $R' \geqslant R \geqslant 1$ and $\lambda \geqslant 1$. For an edge $e \in E(G)$, define

$$\gamma_{R,R'}(e) := \max\left\{\mathrm{vol}^G(y,R) : d^G(e,y) \leqslant 2R'\right\}.$$

Let $\{\boldsymbol{u}_e \in \{0,1\} : e \in E(G)\}$ be an independent family of Bernoulli random variables where

$$\mathbb{P}\left(\boldsymbol{u}_e = 1\right) = \min\left(1, \lambda \frac{c^G(e)}{\gamma_{R,R'}(e)}\right).$$

Define $\boldsymbol{U}_{R,R'}(\lambda) := \{x \in V(G) : x \in e \text{ for some } \boldsymbol{u}_e = 1\}$. Observe the inequalities, valid for every $x \in V(G)$ and $1 \leqslant r \leqslant R$:

$$\begin{aligned}
\mathbb{P}[x \in \boldsymbol{U}_{R,R'}(\lambda)] &\leqslant \sum_{y:\{x,y\}\in E(G)} c^G(\{x,y\})\gamma_{R,R'}(\{x,y\}) \\
&\leqslant \frac{\lambda c_x^G}{\max\{\mathrm{vol}^G(y,R) : y \in B^G(x,R')\}}, \qquad (2.4)
\end{aligned}$$

$$\mathbb{P}[d^G(x,\boldsymbol{U}_{R,R'}(\lambda)) > r] \leqslant \prod_{e\in E^G(B^G(x,r))}\left(1 - \frac{\lambda c^G(e)}{\gamma_{R,R'}(e)}\right)_+ \leqslant \exp\left(-\lambda\frac{\mathrm{vol}^G(x,r)}{\mathrm{vol}^G(x,3R')}\right), \qquad (2.5)$$

where we use $E^G(B^G(x,R))$ to denote the set of edges in G incident to at least one vertex of $B^G(x,R)$.

The idea here is that, by (2.5), the balls $\{B^G(u,R) : u \in \boldsymbol{U}_{R,R'}(\lambda)\}$ tend to cover vertices $x \in V(G)$ for which $\mathrm{vol}^G(x,R) \approx \mathrm{vol}^G(x,3R')$, as long as λ is chosen sufficiently large. On the other hand, (2.4) will allow us to bound $\mathbb{E}\,|B^G(\rho,2R') \cap \boldsymbol{U}_{R,R'}(\lambda)|$. Referring to the argument sketched at the end of Section 1.3, we will center an annulus at every $x \in \boldsymbol{U}_{R,R'}(\lambda)$, and thus we need to control the average covering multiplicity to keep $\mathbb{E}[\omega(X_0,X_1)^2]$ finite.

Since the law of $\boldsymbol{U}_R(\lambda)$ does not depend on the root, we have the following.

Lemma 2.2. *The quadruple $(G,\rho,\boldsymbol{U}_R(\lambda))$ is a reversible random network.*

2.3 The Mass-Transport Principle

Let $\mathscr{G}_\bullet$ denote the collection of isomorphism classes of rooted, connected, locally-finite networks, and let $\mathscr{G}_{\bullet\bullet}$ denote the collection of isomorphism classes of doubly-rooted, connected, locally-finite networks. We will consider functionals $F : \mathscr{G}_{\bullet\bullet} \to [0,\infty)$. Equivalently, these are functionals $F(G_0,x_0,y_0,\xi_0)$ that are invariant under automorphisms of ψ of G_0: $F(G_0,x_0,y_0,\xi_0) = F(\psi(G_0),\psi(x_0),\psi(y_0),\xi_0 \circ \psi^{-1})$.

The *mass-transport principle (MTP)* for a random rooted network (G,ρ,ξ) asserts that for any nonnegative Borel $F : \mathscr{G}_{\bullet\bullet} \to [0,\infty)$, it holds that

$$\mathbb{E}\left[\sum_{x\in V(G)} F(G,\rho,x,\xi)\right] = \mathbb{E}\left[\sum_{x\in V(G)} F(G,x,\rho,\xi)\right].$$

Unimodular random networks are precisely those that satisfy the MTP (see [AL07]).

Using the fact that biasing the law of a reversible random network (G,ρ,ξ) with $\mathbb{E}[1/c_\rho^G] < \infty$ by $1/c_\rho^G$ (see [BC12, Prop. 2.5]) yields a unimodular random network, one arrives at the following biased MTP.

Lemma 2.3. *If (G,ρ,ξ) is a reversible random network with $\mathbb{E}[1/c^G_\rho]<\infty$, then for any nonnegative Borel functional $F:\mathcal{G}_{\bullet\bullet}\to[0,\infty)$, it holds that*

$$\mathbb{E}\left[\frac{1}{c^G_\rho}\sum_{x\in V(G)}F(G,\rho,x,\xi)\right]=\mathbb{E}\left[\frac{1}{c^G_\rho}\sum_{x\in V(G)}F(G,x,\rho,\xi)\right]. \tag{2.6}$$

2.4 Construction of the weights

Recall that (G,ρ) is a reversible random network satisfying $\mathbb{E}[1/c^G_\rho]<\infty$. Denote $d_* := 2\underline{d}_f-\bar{d}_f+\tilde{\zeta}$. Our goal is to prove the following.

Theorem 2.4. *For every $\delta>0$, there is a reversible random weight $\omega:E(G)\to\mathbb{R}_+$ such that $\mathbb{E}[\omega(X_0,X_1)^2]<\infty$, and almost surely eventually*

$$\mathrm{dist}^G_\omega\left(\rho,\bar{B}^G(\rho,R)\right)\geqslant R^{(d_*-\delta)/2}. \tag{2.7}$$

This this end, for $\varepsilon\in(0,1)$, define the set of networks with controlled geometry at scale R:

$$\mathcal{S}(\varepsilon,R):=\left\{(G,x):\frac{1+\mathrm{vol}^G(x,5R^{1+\varepsilon})}{\mathrm{vol}^G(x,R)^2}\mathsf{M}^G(x,2R,R^{1+\varepsilon})\leqslant R^{-d_*+2\varepsilon}\right.$$
$$\left.\text{and }\frac{\mathrm{vol}^G(x,R-1)}{\mathrm{vol}^G(x,15R^{1+\varepsilon})}\geqslant d_*R^{-2\varepsilon}\log R\right\}$$

Lemma 2.5. *For every $\varepsilon>0$, there is a reversible random weight $\omega_R:E(G)\to\mathbb{R}_+$ such that*

$$\mathbb{E}\left[\omega_R(X_0,X_1)^2\right]\leqslant 2R^{-d_*+4\varepsilon}, \tag{2.8}$$

and if $x\in V(G)$ satisfies $d^G(\rho,x)\geqslant 3R^{1+\varepsilon}$, then

$$\mathrm{dist}^G_{\omega_R}(\rho,x)\geqslant \mathbb{1}_{\mathcal{S}(\varepsilon,R)}(G,\rho). \tag{2.9}$$

Before proving the lemma, let us see that it establishes Theorem 2.4.

Proof of Theorem 2.4. Clearly we may assume $d_*>0$. Fix a value $\varepsilon\in(0,d_*)$, and define the sets

$$\mathcal{S}_{R_0}(\varepsilon):=\bigcap_{R\geqslant R_0}\mathcal{S}(\varepsilon,R)\,,$$
$$\mathcal{S}(\varepsilon):=\bigcup_{R_0\geqslant 1}\mathcal{S}_{R_0}(\varepsilon)\,.$$

Lemma 2.6. *Almost surely $(G,\rho)\in\mathcal{S}(\varepsilon)$.*

Proof. By assumption, for every $\delta>0$, it holds that almost surely eventually $\mathsf{M}^G(\rho,R,R^{1+\delta})\leqslant R^{\zeta+\delta}$ (recall Lemma 2.1) and and $R^{\underline{d}_f-\delta}\leqslant \mathrm{vol}^G(\rho,R)\leqslant R^{\bar{d}_f+\delta}$. □

For $k \geqslant 1$, let ω_{2^k} be the weight guaranteed by Lemma 2.5, and define the random weight

$$\omega := \left(\sum_{k \geqslant 1} \frac{2^{k(d_*-4\varepsilon)}}{k^2} \omega_{2^k}^2 \right)^{1/2},$$

so that

$$\mathbb{E}\left[\omega(X_0, X_1)^2\right] \overset{(2.8)}{\leqslant} 2 \sum_{k \geqslant 1} k^{-2} \leqslant O(1).$$

Moreover, for any $k \geqslant 1$ and $x \in V(G)$, if $d^G(\rho, x) \geqslant 3 \cdot 2^{k(1+\varepsilon)}$, then (2.9) gives

$$\mathrm{dist}_\omega^G(\rho, x) \geqslant k^{-1} 2^{k(d_*-4\varepsilon)/2}\, \mathrm{dist}_{\omega_{2^k}}^G(\rho, x) \geqslant k^{-1} 2^{k(d_*-4\varepsilon)/2} \mathbb{1}_{\mathcal{S}_{2^k}(\varepsilon)}(G, \rho),$$

hence for all $x \in V(G)$,

$$\mathrm{dist}_\omega^G(\rho, x) \geqslant \frac{\left(d^G(\rho, x)/3 - 3 \cdot 2^{k(1+\varepsilon)}\right)_+^{(d_*-4\varepsilon)/(2(1+\varepsilon))}}{2 \log d^G(\rho, x)} \mathbb{1}_{\mathcal{S}_{2^k}(\varepsilon)}(G, \rho).$$

Now by Lemma 2.6, this shows that almost surely eventually (with respect to $R \geqslant 1$):

$$d^G(\rho, x) \geqslant R \implies \mathrm{dist}_\omega^G(\rho, x) \geqslant d^G(\rho, x)^{(d_*-5\varepsilon)/(2(1+\varepsilon))}.$$

Since we can take $\varepsilon > 0$ arbitrarily small, the desired result follows. □

Let us now prove the lemma.

Proof of Lemma 2.5. Fix $R \geqslant 1$, and define

$$\mathcal{S}'(\varepsilon, R) := \left\{ z \in V(G) : \frac{1 + \mathrm{vol}^G(z, 4R^{1+\varepsilon})}{\left(\max\{\mathrm{vol}^G(y, R) : y \in B^G(z, R)\}\right)^2} \mathsf{M}^G(z, R, 2R^{1+\varepsilon}) \leqslant R^{-d_*+2\varepsilon} \right\}.$$

Lemma 2.7. *If $(G, \rho) \in \mathcal{S}(\varepsilon, R)$ and $d^G(\rho, z) \leqslant R$, then $z \in \mathcal{S}'(\varepsilon, R)$.*

Proof. Note that $d^G(\rho, z) \leqslant R$ gives

$$\mathsf{M}^G(z, R, 2R^{1+\varepsilon}) \leqslant \mathsf{M}^G(\rho, 2R, R^{1+\varepsilon}).$$

Similarly, we have $\mathrm{vol}^G(z, 4R^{1+\varepsilon}) \leqslant \mathrm{vol}^G(\rho, 5R^{1+\varepsilon})$, and

$$\max\left\{\mathrm{vol}^G(y, R) : y \in B^G(z, R)\right\} \geqslant \mathrm{vol}^G(\rho, R).$$ □

Denote $R' := 5R^{1+\varepsilon}$ and, recalling Section 2.1,

$$\omega^{(z)} := \omega_{(G,z,R,2R^{1+\varepsilon})} \mathbb{1}_{\mathcal{S}'(\varepsilon,R)}(z)\,.$$

Then define: $\omega_R : E(G) \to \mathbb{R}_+$ by

$$\begin{aligned} \hat{\omega} &:= \sum_{z \in \boldsymbol{U}_{R,R'}(\lambda)} \omega^{(z)}\,, \\ \tilde{\omega}(\{x, y\}) &:= \begin{cases} 1 & \{x, y\} \not\subseteq B^G(\boldsymbol{U}_{R,R'}(\lambda), R) \text{ and } \{(G, x), (G, y)\} \cap \mathcal{S}(\varepsilon, R) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \\ \omega_R &:= \hat{\omega} + \tilde{\omega}. \end{aligned}$$

Lemma 2.8. *If $x \in V(G)$ satisfies $d^G(\rho, x) \geqslant 3R^{1+\varepsilon}$, then $\mathrm{dist}^G_{\omega_R}(\rho, x) \geqslant \mathbb{1}_{\mathcal{S}(\varepsilon,R)}(G, \rho)$.*

Proof. If $d^G(\rho, \boldsymbol{U}_{R,R'}(\lambda)) > R$ and $(G, \rho) \in \mathcal{S}(\varepsilon, R)$, then $\tilde{\omega}(\{\rho, y\}) \geqslant 1$ for every $\{\rho, y\} \in E(G)$, implying $\mathrm{dist}^G_{\tilde{\omega}}(\rho, x) \geqslant 1$.

So suppose that $z \in \boldsymbol{U}_{R,R'}(\lambda)$ satisfies $d^G(\rho, z) \leqslant R$. By Lemma 2.7, we have $z \in \mathcal{S}'(\varepsilon, R)$, and therefore $\hat{\omega} \geqslant \omega_{(G,z,R,2R^{1+\varepsilon})}$. Thus by definition,

$$\mathrm{dist}^G_{\omega_R}(\rho, x) \geqslant \mathrm{dist}^G_{\omega_{(G,z,R,2R^{1+\varepsilon})}}\left(B^G(z, R), \bar{B}^G(z, 2R^{1+\varepsilon})\right) \geqslant 1,$$

since $\rho \in B^G(z, R)$, and $x \notin B^G(z, 2R^{1+\varepsilon})$. $\square$

So we are left only to evaluate $\mathbb{E}[\omega_R(X_0, X_1)^2]$. Use Cauchy-Schwarz to bound

$$\begin{aligned}
\mathbb{E}\left[\hat{\omega}(X_0, X_1)^2\right] &= \mathbb{E}\left[\left(\sum_{z \in \boldsymbol{U}_{R,R'}(\lambda)} \omega^{(z)}(X_0, X_1)\right)^2\right] \\
&\leqslant \mathbb{E}\left[\left|B^G(\rho, 2R^{1+\varepsilon}) \cap \boldsymbol{U}_{R,R'}(\lambda)\right| \sum_{z \in B^G(X_0, 2R^{1+\varepsilon})} \mathbb{1}_{\boldsymbol{U}_{R,R'}(\lambda)}(z)\, \omega^{(z)}(X_0, X_1)^2\right], \qquad (2.10)
\end{aligned}$$

where we have used the fact that $\omega^{(z)}$ is supported on edges e such that $e \subseteq B^G(z, 2R^{1+\varepsilon})$.

Apply the biased Mass-Transport Principle (2.6) with the functional

$$F(G, y, z, U_{R,R'}(\lambda)) = c^G_y \left|B^G(y, 2R^{1+\varepsilon}) \cap U_{R,R'}(\lambda)\right| \mathbb{1}_{U_{R,R'}(\lambda)}(z)\, \mathbb{E}[\omega^{(z)}(X_0, X_1)^2 \mid X_0 = y]$$

to conclude from (2.10) that

$$\begin{aligned}
\mathbb{E}\left[\hat{\omega}(X_0, X_1)^2\right] &\leqslant \mathbb{E}\left[\frac{\mathbb{1}_{\boldsymbol{U}_{R,R'}(\lambda)}(\rho)}{c^G_\rho} \sum_{z \in B^G(\rho, 2R^{1+\varepsilon})} \left|B^G(z, 2R^{1+\varepsilon}) \cap \boldsymbol{U}_{R,R'}(\lambda)\right| c^G(z)\, \mathbb{E}\left[\omega^{(\rho)}(X_0, X_1)^2 \mid X_0 = z\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_{\boldsymbol{U}_{R,R'}(\lambda)}(\rho)}{c^G_\rho} \sum_{z \in B^G(\rho, 2R^{1+\varepsilon})} \left|B^G(z, 2R^{1+\varepsilon}) \cap \boldsymbol{U}_{R,R'}(\lambda)\right| \sum_{y : \{y,z\} \in E(G)} c^G(\{y, z\}) \omega^{(\rho)}(y, z)^2\right] \\
&\leqslant \mathbb{E}\left[\frac{\mathbb{1}_{\boldsymbol{U}_{R,R'}(\lambda)}(\rho)}{c^G_\rho} |B^G(\rho, 4R^{1+\varepsilon}) \cap \boldsymbol{U}_{R,R'}(\lambda)| \sum_{\{y,z\} \in E(G)} c^G(\{y, z\}) \omega^{(\rho)}(y, z)^2\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_{\boldsymbol{U}_{R,R'}(\lambda)}(\rho)}{c^G_\rho} \left|B^G(\rho, 4R^{1+\varepsilon}) \cap \boldsymbol{U}_{R,R'}(\lambda)\right| \mathsf{M}^G(\rho, R, 2R^{1+\varepsilon}) \mathbb{1}_{\mathcal{S}'(\varepsilon,R)}(\rho)\right].
\end{aligned}$$

Now (2.4) gives, for every $x \in B^G(\rho, 4R^{1+\varepsilon})$,

$$\mathbb{P}[x \in \boldsymbol{U}_{R,R'}(\lambda) \mid (G, \rho)] \leqslant \frac{\lambda c^G_x}{\max\{\mathrm{vol}^G(y, R) : y \in B^G(x, R')\}} \leqslant \frac{\lambda c^G_x}{\max\{\mathrm{vol}^G(y, R) : y \in B^G(\rho, R)\}},$$

where we have used $R' = 5R^{1+\varepsilon} \geqslant 4R^{1+\varepsilon} + R$. Along with independence of the sampling procedure, this yields

$$\mathbb{E}\left[\mathbb{1}_{\boldsymbol{U}_{R,R'}(\lambda)}(\rho)\left|B^G(\rho, 4R^{1+\varepsilon}) \cap \boldsymbol{U}_{R,R'}(\lambda)\right| \mid (G,\rho)\right] \leqslant \lambda^2 c_\rho^G \frac{1 + \mathrm{vol}^G(\rho, 4R^{1+\varepsilon})}{\left(\max\{\mathrm{vol}^G(y,R) : y \in B^G(\rho,R)\}\right)^2}$$

Therefore

$$\mathbb{E}\left[\hat{\omega}(X_0,X_1)^2\right] \leqslant \lambda^2\, \mathbb{E}\left[\mathbb{1}_{\mathcal{S}'(\varepsilon,R)}(\rho)\frac{1 + \mathrm{vol}^G(\rho, 4R^{1+\varepsilon})}{\left(\max\{\mathrm{vol}^G(y,R) : y \in B^G(\rho,R)\}\right)^2}\mathsf{M}^G(\rho, R, 2R^{1+\varepsilon})\right] \leqslant \lambda^2 R^{-d_*+\varepsilon},$$

by definition of $\mathcal{S}'(\varepsilon, R)$.

Let us use (2.5) with $r = R - 1$ to bound

$$\begin{aligned}
\mathbb{E}[\tilde{\omega}(X_0,X_1)^2] &\leqslant \mathbb{P}\left[d^G(\rho, \boldsymbol{U}_{R,R'}(\lambda)) \geqslant R \mid \rho \in \mathcal{S}(\varepsilon,R)\right] \\
&\leqslant \mathbb{E}\left[\exp\left(-\lambda\frac{\mathrm{vol}^G(\rho,R-1)}{\mathrm{vol}^G(\rho,15R^{1+\varepsilon})}\right) \mid \rho \in \mathcal{S}(\varepsilon,R)\right] \\
&\leqslant \exp(-\lambda d_* R^{-2\varepsilon}\log R),
\end{aligned}$$

where the last line follows from the definition of $\mathcal{S}(\varepsilon, R)$. Now choose $\lambda := R^{2\varepsilon}$, yielding

$$\mathbb{E}\left[\omega_R(X_0,X_1)^2\right] \leqslant 2\left(\mathbb{E}[\hat{\omega}(X_0,X_1)^2 + \tilde{\omega}(X_0,X_1)^2]\right) \leqslant 2R^{-d_*+4\varepsilon}. \qquad \square$$

3 Markov type and the rate of escape

Our goal now is to prove Theorem 1.5. It is essentially a consequence of the fact that every N-point metric space has maximal Markov type 2 with constant $O(\log N)$ (see Section 3.2 below), and that the random walk on a reversible random graph with almost sure subexponential growth can be approximated, quantitatively, by a limit of random walks restricted to finite subgraphs.

3.1 Restricted walks on clusters

Definition 3.1 (Restricted random walk)**.** Consider a network $G = (V, E, c^G)$ and a finite subset $S \subseteq V$. Let

$$N_G(x) = \{y \in V : \{x,y\} \in E\}$$

denote the neighborhood of a vertex $x \in V$.

Define a measure π_S on S by

$$\pi_S(x) := \frac{c_x^G}{c^G(E^G(S))}\mathbb{1}_S(x)\,, \tag{3.1}$$

where $E^G(S) := \left\{\{x,y\} \in E(G) : \{x,y\} \cap S \neq \emptyset\right\}$ is the set of edges incident on S.

We define *the random walk restricted to S* as the following process $\{Z_t\}$: For $t \geqslant 0$, put

$$\mathbb{P}(Z_{t+1} = y \mid Z_t = x) = \begin{cases} \frac{c^G(E^G(x, V\setminus S))}{c_x^G} & y = x \\ \frac{c^G(\{x,y\})}{c_x^G} & y \in N_G(x) \cap S \\ 0 & \text{otherwise,} \end{cases}$$

where we have used the notation $E^G(x, U) := \big\{\{x, y\} \in E : y \in U\big\}$. It is straightforward to check that $\{Z_t\}$ is a reversible Markov chain on S with stationary measure π_S. If Z_0 has law π_S, we say that $\{Z_t\}$ is the *stationary random walk restricted to S.*

A *bond percolation on G* is a mapping $\xi : E(G) \to \{0, 1\}$. For a vertex $v \in V(G)$ and a bond percolation ξ, we let $K_\xi(v)$ denote the connected component of v in the subgraph of G given by $\xi^{-1}(1)$. Say that a bond percolation $\xi : E(G) \to \{0, 1\}$ is *finitary* if $K_\xi(\rho)$ is almost surely finite.

Lemma 3.2. *Suppose (G, ρ, ξ) is a reversible random network and ξ is finitary. Let $\hat{\rho} \in V(G)$ be chosen according to the measure $\pi_{K_\xi(\rho)}$ from Definition 3.1. Then (G, ρ) and $(G, \hat{\rho})$ have the same law.*

Proof. Define the transport

$$F(G, x, y, \xi) := c_x^G \frac{c_y^G}{c^G(E^G(K_\xi(x)))} \mathbb{1}_{K_\xi(x)}(y) \mathbb{1}_{\mathcal{S}}(G, x),$$

where $\mathcal{S}$ denotes some Borel measurable subset of $\mathscr{G}_\bullet$ (recall the definition from Section 2.3). Then the biased mass-transport principle (2.6) gives

$$\begin{aligned} \mathbb{P}[(G, \rho) \in S] &= \mathbb{E}\left[\frac{1}{c_\rho^G} \sum_{x \in V(G)} F(G, \rho, x, \xi)\right] \\ &= \mathbb{E}\left[\frac{1}{c_\rho^G} \sum_{x \in V(G)} F(G, x, \rho, \xi)\right] = \mathbb{E}\left[\sum_{x \in K_\xi(\rho)} \frac{c_x^G}{c^G(E^G(K_\xi(\rho)))} \mathbb{1}_S(G, x)\right], \end{aligned}$$

and

$$\mathbb{P}[(G, \hat{\rho}) \in S] = \mathbb{E}\left[\sum_{x \in K_\xi(\rho)} \pi_{K_\xi(\rho)}(x) \mathbb{1}_{\mathcal{S}}(G, x)\right] = \mathbb{E}\left[\sum_{x \in K_\xi(\rho)} \frac{c_x^G}{c^G(E^G(K_\xi(\rho)))} \mathbb{1}_{\mathcal{S}}(G, x)\right]. \qquad \square$$

3.2 Maximal Markov type

A metric space $(\mathcal{X}, d_\mathcal{X})$ has *maximal Markov type* 2 *with constant K* if it holds that for every finite state space Ω, every map $f : \Omega \to X$, and every stationary, reversible Markov chain $\{Z_n\}$ on Ω,

$$\mathbb{E}\left[\max_{0 \leqslant t \leqslant n} d_\mathcal{X}(Z_0, Z_t)^2\right] \leqslant K^2 n\, \mathbb{E}\left[d_\mathcal{X}(Z_0, Z_1)^2\right], \quad \forall n \geqslant 1.$$

This is a maximal variant of K. Ball's Markov type [Bal92]. Note that every Hilbert space has maximal Markov type 2 with constant K for some universal K (independent of the Hilbert space); see, e.g., [NPSS06, §8]. Bourgain's embedding theorem [Bou85] asserts that every N-point metric space embeds into a Hilbert space with bilipschitz distortion $O(\log N)$, yielding the following.

Lemma 3.3. *If $(\mathcal{X}, d_{\mathcal{X}})$ is a finite metric space with $N = |\mathcal{X}|$, then for every stationary, reversible Markov chain $\{Z_n\}$ on $\mathcal{X}$, it holds that*

$$\mathbb{E}\left[\max_{0\leqslant t\leqslant n} d_{\mathcal{X}}(Z_0, Z_t)^2\right] \leqslant O(n(\log N)^2)\,\mathbb{E}\left[d_{\mathcal{X}}(Z_0, Z_1)^2\right], \quad \forall n \geqslant 1.$$

3.3 Proof of Theorem 1.5

We are ready to prove Theorem 1.5. Consider a finitary bond peroclation $\xi : E(G) \to \{0,1\}$ such that (G, ρ, ξ) is a reversible random network, and let $\{X_n^\xi\}$ be simple random walk on $K_\xi(\rho)$, where X_0^ξ has law $\pi_{K_\xi(\rho)}$. By Lemma 3.2, (G, ρ) and (G, X_0^ξ) have the same law. Let $\{X_n\}$ denote simple random walk on G started from ρ.

Thus there is a natural coupling of $\{X_n\}$ and $\{X_n^\xi\}$ such that

$$\{X_0, X_1, \ldots, X_\tau\} = \{X_0^\xi, X_1^\xi, \ldots, X_\tau^\xi\},$$

where $\tau := \min\{t \geqslant 0 : X_t^\xi \in \partial_G K_\xi(\rho)\}$, and for $S \subseteq V(G)$, we write

$$\partial_G S := \{x \in S : \{x, y\} \in E(G) \text{ for some } y \notin S\}.$$

In particular, there is a coupling under which

$$B^G(\rho, n) \subseteq K_\xi(\rho) \implies \{X_0, X_1, \ldots, X_n\} = \{X_0^\xi, X_1^\xi, \ldots, X_n^\xi\}. \tag{3.2}$$

We will use the notation $\mathrm{diam}_\omega^G(S) := \max\{\mathrm{dist}_\omega^G(x, y) : x, y \in S\}$.

Lemma 3.4. *Suppose (G, ρ, ω, ξ) is a reversible random network, where ξ is bond percolation such that $\mathrm{diam}_\omega^G(K_\xi(\rho)) \leqslant \Delta$ almost surely, and $\omega : E(G) \to \mathbb{R}_+$ satisfies $\mathbb{E}[\omega(X_0, X_1)^2] < \infty$ and almost surely $\omega \geqslant 1$. Then almost surely*

$$\mathbb{E}\left[\mathbb{1}_{\{\mathcal{B}_\omega^G(\rho,n)\subseteq K_\xi(\rho)\}} \max_{0\leqslant t\leqslant n} \mathrm{dist}_\omega^G(X_0, X_t)^2 \mid (G, \rho, \omega, \xi)\right]$$
$$\leqslant O(n)\left(\log|\mathcal{B}_\omega^G(\rho, \Delta)|\right)^2 \mathbb{E}\left[\omega\left(X_0^\xi, X_1^\xi\right)^2 \mid (G, \rho, \omega, \xi)\right].$$

Proof. Consider the stationary, reversible Markov chain $\{X_n^\xi\}$ on $K_\xi(\rho)$. Applying Lemma 3.3, we obtain that almost surely over the choice of (G, ρ, ω, ξ),

$$\begin{aligned}
\mathbb{E}\left[\max_{0\leqslant t\leqslant n} \mathrm{dist}_\omega^{K_\xi(\rho)}(X_0^\xi, X_n^\xi)^2 \mid (G, \rho, \omega, \xi)\right] &\leqslant O(n)\left(\log|K_\xi(\rho)|\right)^2 \mathbb{E}\left[\omega(X_0^\xi, X_1^\xi)^2 \mid (G, \rho, \omega, \xi)\right] \\
&\leqslant O(n)\left(\log|K_\xi(\rho)|\right)^2 \mathbb{E}\left[\omega(X_0^\xi, X_1^\xi)^2 \mid (G, \rho, \omega, \xi)\right] \\
&\leqslant O(n)\left(\log|\mathcal{B}_\omega^G(\rho, \Delta)|\right)^2 \mathbb{E}\left[\omega(X_0^\xi, X_1^\xi)^2 \mid (G, \rho, \omega, \xi)\right],
\end{aligned}$$

where the last inequality uses our assumption that $\mathrm{diam}_\omega^G(K_\xi(\rho)) \leqslant \Delta$ almost surely.

To conclude, we use the coupling that gives (3.2), along with the fact that $\mathrm{dist}_\omega^G \leqslant \mathrm{dist}_\omega^{K_\xi(\rho)}$ for all $x, y \in V(K_\xi(\rho))$ to arrive at

$$\mathbb{E}\left[\mathbb{1}_{\{B^G(\rho,n)\subseteq K_\xi(\rho)\}} \max_{0\leqslant t\leqslant n} \mathrm{dist}_\omega^G(X_0, X_t)^2 \mid (G,\rho,\omega,\xi)\right] \leqslant \mathbb{E}\left[\max_{0\leqslant t\leqslant n} \mathrm{dist}_\omega^{K_\xi(\rho)}(X_0^\xi, X_t^\xi)^2 \mid (G,\rho,\omega,\xi)\right].$$

Finally, we observe that since $\omega \geqslant 1$ almost surely, we have $\mathrm{dist}_\omega^G \geqslant d^G$, and therefore $\mathcal{B}_\omega^G(\rho, n) \subseteq B^G(\rho, n)$, so the preceding inequality implies the statement of the lemma. □

We need a unimodular random partitioning scheme that adapts to the volume measure. Here we state it for any unimodular vertex measure. This argument employs a unimodular variation on the method and analysis from [CKR01], adapted to an arbitrary underlying measure as in [KLMN05].

Lemma 3.5. *Suppose that (G,ρ,ω,μ) is a unimodular random network, where $\omega : E(G) \to \mathbb{R}_+$ and $\mu : V(G) \to \mathbb{R}_+$ satisfy almost surely:*

(i) $\mu(\rho) > 0$, *and*

(ii) $\mathcal{B}_\omega^G(\rho, \Delta)$ *is finite.*

Then for every $\Delta > 0$, there is a bond percolation $\xi_\Delta : E(G) \to \{0,1\}$ such that

1. $(G,\rho,\omega,\xi_\Delta)$ *is a unimodular random network.*
2. *Almost surely* $\mathrm{diam}_\omega^G(K_{\xi_j}(\rho)) \leqslant \Delta$.
3. *For every $r \geqslant 0$, it holds that almost surely*

$$\mathbb{P}\left[\mathcal{B}_\omega^G(\rho, r) \nsubseteq K_{\xi_j}(\rho) \mid (G,\rho)\right] \leqslant \frac{16r}{\Delta}\left(1 + \log\left(\frac{\mu\left(\mathcal{B}_\omega^G(\rho, \frac{5}{8}\Delta)\right)}{\mu\left(\mathcal{B}_\omega^G(\rho, \frac{1}{8}\Delta)\right)}\right)\right),$$

where we use the notation $\mu(S) := \sum_{x\in S} \mu(x)$ for $S \subseteq V(G)$.

Proof. By assumption (ii), the ball $\mathcal{B}_\omega^G(\rho, R)$ is almot surely finite. Thus we may assume that $\mu(x) > 0$ for all $x \in V(G)$ as follows: Define $\hat{\mu}(x) = \mu(x)$ if $\mu(x) > 0$ and $\hat{\mu}(x) = 1$ otherwise. We may then prove the lemma for $\hat{\mu}$, and observe that because properties (2) and (3) only refer to finite neighborhoods of the root, μ and $\hat{\mu}$ are identical on these neighborhoods, except for a set of measure zero.

Let $\{\beta_x : x \in V(G)\}$ be a sequence of independent random variables where β_x is an exponential with rate $\mu(x)$. Let $R \in [\frac{\Delta}{4}, \frac{\Delta}{2})$ be independent and chosen uniformly random. We need the following elementary lemma:

Lemma 3.6. *For any finite subset $S \subseteq V(G)$, it holds that*

$$\mathbb{P}\left[\beta_x = \min\{\beta_v : v \in S\} \mid (G,\mu)\right] = \frac{\mu(x)}{\mu(S)}, \qquad \forall x \in S.$$

Proof. A straightforward calculation shows that $\min\{\beta_v : v \in S \setminus \{x\}\}$ is exponential with rate $\mu(S \setminus \{x\})$. Moreover, if β and β' are independent exponentials with rates λ and λ', respectively, then

$$\mathbb{P}[\beta = \min(\beta, \beta')] = \frac{\lambda}{\lambda + \lambda'}.$$ □

Define a labeling $\ell : V(G) \to V(G)$, where $\ell(x) \in \mathcal{B}^G_\omega(x, R)$ is such that

$$\beta_{\ell(x)} = \min \left\{ \beta_y : y \in \mathcal{B}^G_\omega(x, R) \right\}.$$

Define the bond percolation ξ_Δ by

$$\xi_\Delta(\{x, y\}) := \mathbb{1}_{\{\ell(x)=\ell(y)\}}, \quad \{x, y\} \in E(G).$$

In other words, we remove edges whose endpoints receive different labels.

Since the law of ξ_Δ does not depend on ρ, it follows that $(G, \rho, \omega, \xi_\Delta)$ is unimodular, yielding claim (1). Moreover, since $\ell(x) = z$ implies that $\mathrm{dist}^G_\omega(x, z) \leqslant R \leqslant \Delta$, it holds that almost surely

$$\mathrm{diam}^G_\omega(K_{\xi_\Delta}(\rho)) = \mathrm{diam}^G_\omega(\ell^{-1}(\ell(\rho))) \leqslant \Delta,$$

yielding claim (2).

Since the statement of the lemma is vacuous for $r > \Delta/8$, consider some $r \in [0, \Delta/8]$. Let $x^* \in \mathcal{B}^G_\omega(\rho, r + R)$ be such that

$$\beta_{x^*} = \min \left\{ \beta_x : x \in \mathcal{B}^G_\omega(\rho, R + r) \right\}.$$

Then we have

$$\mathbb{P}\left[\mathcal{B}^G_\omega(\rho, r) \nsubseteq K_{\xi_\Delta}(\rho) \right] \leqslant \mathbb{P}\left[\mathrm{dist}^G_\omega(\rho, x^*) \geqslant R - r \right]. \tag{3.3}$$

For $x \in \mathcal{B}^G_\omega(\rho, 2\Delta)$, define the interval $I(x) := [\mathrm{dist}^G_\omega(\rho, x) - r, \mathrm{dist}^G_\omega(\rho, x) + r]$. Note that the bad event $\{\mathrm{dist}^G_\omega(\rho, x^*) \geqslant R - r\}$ coincides with the event $\{R \in I(x^*)\}$. Order the points of $\mathcal{B}^G_\omega(\rho, 2\Delta)$ in non-decreasing order from ρ: $x_0 = \rho, x_1, x_2, \ldots, x_N$. Then (3.3) yields

$$\begin{aligned}
\mathbb{P}\left[\mathcal{B}^G_\omega(\rho, r) \nsubseteq K_{\xi_\Delta}(\rho) \right] &\leqslant \mathbb{P}[R \in I(x^*)] \\
&= \sum_{j=1}^{N} \mathbb{P}[R \in I(x_j)] \cdot \mathbb{P}[x_j = x^* \mid R \in I(x_j)] \\
&\leqslant \frac{2r}{\Delta/8} \sum_{j=1}^{N} \mathbb{P}[x_j = x^* \mid R \in I(x_j)].
\end{aligned} \tag{3.4}$$

Note that since $R \geqslant \Delta/4$ and $r \leqslant \Delta/8$,

$$R \in I(x_j) \implies x_j \in \mathcal{B}^G_\omega(\rho, \tfrac{5}{8}\Delta) \setminus \mathcal{B}^G_\omega(\rho, \tfrac{1}{8}\Delta).$$

Observe, moreover, that $R \in I(x_j)$ implies $x_1, x_2, \ldots, x_j \in \mathcal{B}^G_\omega(\rho, R + r)$, hence

$$\mathbb{P}[x_j = x^* \mid R \in I(x_j)] = \mathbb{P}\left[\beta_{x_j} = \min \left\{ \beta_x : x \in \mathcal{B}^G_\omega(\rho, R + r) \right\} \mid R \in I(x_j) \right] \leqslant \frac{\mu(x_j)}{\mu(\{x_1, x_2, \ldots, x_j\})},$$

where the last inequality follows from Lemma 3.6.

Plugging these bounds into (3.4) gives

$$\mathbb{P}\left[\mathcal{B}^G_\omega(\rho, r) \nsubseteq K_{\xi_\Delta}(\rho) \right] \leqslant \frac{16r}{\Delta} \sum_{j=|\mathcal{B}^G_\omega(\rho, \frac{1}{8}\Delta)|+1}^{|\mathcal{B}^G_\omega(\rho, \frac{5}{8}\Delta)|} \frac{\mu(x_j)}{\mu(x_1) + \cdots + \mu(x_j)}.$$

Finally, observe that for any $a_0, a_1, a_2, \ldots, a_m > 0$,

$$\sum_{j=1}^{m} \frac{a_j}{a_0 + a_1 + a_2 + \cdots + a_j} = \sum_{j=1}^{m} \frac{a_j/a_0}{1 + a_1/a_0 + \cdots + a_j/a_0}$$
$$\leqslant \int_0^{(a_1+\cdots+a_j)/a_0} \frac{1}{t+1}\,dt = \log\left(1 + \frac{a_1 + \cdots + a_j}{a_0}\right),$$

and therefore

$$\mathbb{P}\left[\mathcal{B}_\omega^G(\rho, r) \not\subseteq K_{\xi_\Delta}(\rho)\right] \leqslant \frac{16r}{\Delta} \log\left(1 + \frac{\mu(\mathcal{B}_\omega^G(\rho, \frac{5}{8}\Delta))}{\mu(\mathcal{B}_\omega^G(\rho, \frac{1}{8}\Delta))}\right),$$

as desired (noting that $\log(1+y) \leqslant 1 + \log(y)$ for $y \geqslant 1$). □

With this in hand, we can proceed to our goal of proving Theorem 1.5.

Proof of Theorem 1.5. Recall that (G, ρ, ω) is a reversible random network and $\{X_n\}$ is the random walk on G started from $X_0 = \rho$. We may replace ω by $1 + \omega$ so that $\omega \geqslant 1$ almost surely. Note that (1.21) is still satisfied, and the conclusion under the new weight is only stronger. In particular, this guarantees that $\mathcal{B}_\omega^G(\rho, R)$ is almost surely finite for every $R > 0$.

Define $j_k := 2^{2k}$ for $k \geqslant 1$ and $\mu \equiv \mathbb{1}$. Let $(G, \rho, \omega, \langle \xi_{j_k} : k \geqslant 1\rangle)$ be the reversible random network provided by applying Lemma 3.5 with μ and $\Delta = j_k$ for each $k \geqslant 1$ Denote

$$\boldsymbol{n}_k := \frac{j_k}{16k^2\left(1 + \log\left|\mathcal{B}_\omega^G(\rho, j_k)\right|\right)}$$

so that for $k \geqslant 1$,

$$\mathbb{P}[\mathcal{B}_\omega^G(\rho, \boldsymbol{n}_k) \not\subseteq K_{\xi_{j_k}}(\rho) \mid (G, \rho, \omega)] \leqslant O(k^{-2}).$$

By the Borel-Cantelli lemma, it holds thats almost surely over the choice of (G, ρ, ω),

$$\#\left\{k \geqslant 1 : \mathcal{B}_\omega^G(\rho, \boldsymbol{n}_k) \not\subseteq K_{\xi_{j_k}}(\rho)\right\} < \infty,$$

and therefore Lemma 3.4 shows that almost surely for all but finitely many k,

$$\mathbb{E}\left[\max_{0 \leqslant t \leqslant \boldsymbol{n}_k} \mathrm{dist}_\omega^G(X_0, X_t)^2 \mid (G, \rho, \omega)\right] \leqslant O(n)\left(\log|\mathcal{B}_\omega^G(\rho, 2^{2k})|\right)^2 \mathbb{E}\left[\omega(X_0^{\xi_{j_k}}, X_1^{\xi_{j_k}})^2 \mid (G, \rho, \omega)\right],$$

where we have used that $\mathrm{diam}_\omega^G(K_{\xi_{j_k}}(\rho)) \leqslant j_k = 2^{2k}$ almost surely.

Now the second condition in ($\mathcal{B}$) gives $\log|B^G(\rho, R)| \lessapprox 1$. Since $\omega \geqslant 1$, we have $\mathrm{dist}_\omega^G \geqslant d^G$, hence $\log|\mathcal{B}_\omega^G(\rho, R)| \lessapprox 1$ holds as well. This gives almost surely

$$\varlimsup_{n\to\infty} \frac{\log \mathbb{E}\left[\max_{0\leqslant t\leqslant n} \mathrm{dist}_\omega^G(X_0, X_t)^2 \mid (G, \rho)\right]}{\log n} \leqslant 1 + \varlimsup_{k\to\infty} \frac{\log \mathbb{E}[\omega(X_0^{\xi_{j_k}}, X_1^{\xi_{j_k}})^2 \mid (G, \rho)]}{\log n}. \tag{3.5}$$

Recalling Lemma 3.2, for every k we have $\mathbb{E}[\omega(X_0^{\xi_{j_k}}, X_1^{\xi_{j_k}})^2] \leqslant \mathbb{E}[\omega(X_0, X_1)^2] < \infty$, and therefore

$$\mathbb{P}\left(\mathbb{E}[\omega(X_0^{\xi_{j_k}}, X_1^{\xi_{j_k}})^2 \mid (G, \rho)] > k^2\, \mathbb{E}[\omega(\rho)^2]\right) \leqslant k^{-2},$$

so again Borel-Cantelli tells us that almost surely

$$\#\left\{k \geqslant 1 : \mathbb{E}\left[\omega(X_0^{\xi_{j_k}}, X_1^{\xi_{j_k}})^2\right] > k^2\, \mathbb{E}\left[\omega(X_0, X_1)^2\right]\right\} < \infty$$

Plugging this into (3.5) yields that almost surely

$$\lim_{n\to\infty} \frac{\log \mathbb{E}\left[\max_{0\leqslant t\leqslant n} \mathrm{dist}_\omega^G(X_0, X_t)^2 \mid (G, \rho, \omega)\right]}{\log n} \leqslant 1,$$

completing the proof. □

4 Exponent relations

Let us now prove the nontrivial inequalities in Theorem 1.3.

4.1 The speed upper bound

Theorem 4.1. *If (G, ρ) is a reversible random network satisfying ($\mathcal{B}$), then $\underline{\beta}^{\mathcal{A}} \geqslant 2\underline{d}_f - \bar{d}_f + \tilde{\zeta}$.*

Proof. Recall that $\{X_n\}$ is the random walk on G (cf. (1.8)) started from $X_0 = \rho$. Let us denote $d_* := 2\underline{d}_f - \bar{d}_f + \tilde{\zeta}$. If $d_* \leqslant 2$, we can use the weight $\omega \equiv 1$ for which $\mathrm{dist}_\omega^G = d^G$, and (1.22) yields $\underline{\beta}^{\mathcal{A}} \geqslant 2$. Consider now $d_* > 2$ and fix $\delta \in (0, d^* - 2)$. Apply Theorem 2.4 to arrive at a reversible random weight $\omega : E(G) \to \mathbb{R}_+$ such that $\mathbb{E}[\omega(X_0, X_1)^2] < \infty$ and almost surely eventually,

$$\mathrm{dist}_\omega^G(\rho, \bar{B}^G(\rho, R)) \geqslant R^{(d_*-\delta)/2}. \tag{4.1}$$

Apply Theorem 1.5 to (G, ρ, ω) yielding: Almost surely eventually (with respect to n),

$$\mathbb{E}\left[\max_{0\leqslant t\leqslant n} \mathrm{dist}_\omega^G(X_0, X_t)^2 \mid (G, \rho, \omega)\right] \leqslant n^{1+\delta}.$$

Combining this with (4.1) yields almost surely eventually

$$\mathbb{E}\left[\max_{0\leqslant t\leqslant n} d^G(X_0, X_t)^{d_*-\delta} \mid (G, \rho, \omega)\right] \leqslant n^{1+\delta}.$$

Now since $d_* - \delta > 2$, convexity of $y \mapsto y^{(d_*-\delta)/2}$ gives

$$\mathbb{E}\left[\max_{0\leqslant t\leqslant n} d^G(X_0, X_t)^2 \mid (G, \rho, \omega)\right] \leqslant n^{2(1+\delta)/(d_*-\delta)}.$$

Since we can take $\delta > 0$ arbitrarily small, this yields $\underline{\beta}^{\mathcal{A}} \geqslant d_*$, completing the proof. □

4.2 Effective resistance and the Green kernel

Assume again that (G,ρ) is a reversible random graph.

Definition 4.2 (Green kernels)**.** For $S \subseteq V(G)$, let $\tau_S := \min\{n \geqslant 0 : X_n \in S\}$, and define the Green kernel killed off S by

$$\mathsf{g}_S^G(x,y) := \mathbb{E}\left[\sum_{t<\tau_{V(G)\setminus S}} \mathbb{1}_{\{X_t=y\}} \;\middle|\; X_0 = x\right].$$

For $n \geqslant 1$, define

$$\mathsf{Gr}_n^G(x,y) := \mathbb{E}\left[\sum_{t\leqslant n} \mathbb{1}_{\{X_t=y\}} \;\middle|\; X_0 = x\right].$$

It is well-known (see [LP16, Ch. 2]) that for any $x \in V(G)$ and $S \subseteq V(G)$:

$$c_x^G\, \mathsf{R}_{\mathrm{eff}}^G(x \leftrightarrow V(G)\setminus S) = \mathsf{g}_S^G(x,x)\,. \tag{4.2}$$

We recall the standard relationship between effective resistances and commute times [CRR+97] gives the following.

Lemma 4.3. *For any $R \geqslant 1$, almost surely:*

$$\mathbb{E}[\sigma_R \mid (G,\rho), X_0=\rho] \leqslant \mathsf{R}_{\mathrm{eff}}^G(\rho \leftrightarrow \bar{B}^G(\rho,R))\ \mathrm{vol}^G(\rho,R).$$

This immediately yields (1.18):

Theorem 4.4. *It holds that $\bar{d}_w^{\mathcal{A}} \leqslant \bar{d}_f + \tilde{\zeta}_0$.*

Similarly standard arguments yields the upper and lower bounds in (1.19), as follows.

Theorem 4.5. *It holds that*

$$\underline{d}_s \geqslant 2\left(1 - \frac{\tilde{\zeta}_0}{\underline{d}_w}\right)$$

Proof. Since the even return times are non-increasing (see, e.g., [LPW09, Prop. 10.18]), we have

$$p_{2n}^G(\rho,\rho) \leqslant \frac{1}{n}\sum_{j=1}^n p_{2j}^G(\rho,\rho) \leqslant \frac{1}{n}\mathsf{Gr}_{2n}^G(\rho,\rho). \tag{4.3}$$

By definition, for any $\delta > 0$, we have that almost surely eventually

$$\sigma_R > R^{\underline{d}_w-\delta}.$$

Therefore almost surely eventually

$$\mathsf{Gr}_n^G(\rho,\rho) \leqslant \mathsf{g}^G_{B^G(\rho,n^{1/(\underline{d}_w-\delta)})}(\rho,\rho) \overset{(4.2)}{=} c_\rho^G\, \mathsf{R}_{\mathrm{eff}}^G\left(\rho \leftrightarrow \bar{B}^G(\rho, n^{1/(\underline{d}_w-\delta)})\right) \leqslant c_\rho^G\, n^{(\tilde{\zeta}_0+\delta)/(\underline{d}_w-\delta)}.$$

Combined with (4.3), this gives almost surely eventually

$$p_{2n}^G(\rho,\rho) \leqslant 2c_\rho^G(2n)^{(\tilde{\zeta}_0+\delta)/(\underline{d}_w-\delta)-1},$$

and since this holds for all $\delta > 0$, we obtain $\underline{d}_s \geqslant 2(1-\tilde{\zeta}_0/\underline{d}_w)$. ∎

Theorem 4.6. *It holds that*

$$\bar{d}_s \leqslant \frac{2\bar{d}_f}{\underline{d}_w}.$$

Proof. Using reversibility, we have almost surely

$$p_{2n}^G(\rho,\rho) \geqslant \sum_{x\in B^G(\rho,R)} p_n^G(\rho,x)p_n^G(x,\rho) = c_\rho^G \sum_{x\in B^G(\rho,R)} \frac{p_n^G(\rho,x)^2}{c_x^G}.$$

Thus applying Cauchy-Schwarz yields

$$\frac{p_{2n}^G(\rho,\rho)}{c_\rho^G} \geqslant \frac{\left(\sum_{x\in B^G(\rho,R)} p_n^G(\rho,x)\right)^2}{\mathrm{vol}^G(\rho,R)} \geqslant \frac{\left(\mathbb{P}[X_n \in B^G(\rho,R) \mid (G,\rho)]\right)^2}{\mathrm{vol}^G(\rho,R)}. \tag{4.4}$$

Observe that

$$\mathbb{P}[X_n \in B^G(\rho,R) \mid (G,\rho)] \geqslant \mathbb{P}[\sigma_R \geqslant n \mid (G,\rho)]. \tag{4.5}$$

By definition, for every $\delta > 0$, almost surely eventually $\sigma_R \geqslant R^{\underline{d}_w-\delta}$ and $\mathrm{vol}^G(\rho,R) \leqslant R^{\bar{d}_f+\delta}$. Combining these with (4.4) and (4.5) gives almost surely eventually

$$\frac{p_{2n}^G(\rho,\rho)}{c_\rho^G} \geqslant \left(\mathrm{vol}^G\left(\rho, n^{1/(\underline{d}_w-\delta)}\right)\right)^{-1} \geqslant n^{-(\bar{d}_f+\delta)/(\underline{d}_w-\delta)}.$$

As this holds for every $\delta > 0$, it yields the claimed inequality. □

Finally, let us prove that the assumptions (1.11) and (1.12) imply $\tilde{\zeta} = \tilde{\zeta}_0$ in the case $\zeta > 0$. The first part of the argument follows [BCK05, §3.2]. The second part uses methods similar to those employed by Telcs [Tel89].

Theorem 4.7. *If* (1.11) *and* (1.12) *hold for some* $\zeta > 0$, *then* $\tilde{\zeta} = \tilde{\zeta}_0 = \zeta$.

Proof. First note that if $d^G(\rho,x) = R+1$, then

$$\mathsf{R}_{\mathrm{eff}}^G(\rho \leftrightarrow \bar{B}^G(\rho,R)) \leqslant \mathsf{R}_{\mathrm{eff}}^G(\rho \leftrightarrow x) \overset{(1.11)}{\leqslant} (R+1)^{\tilde{\zeta}+\delta}, \tag{4.6}$$

hence (1.11) yields

$$\tilde{\zeta}_0 \leqslant \zeta. \tag{4.7}$$

Thus we are left to prove that $\tilde{\zeta} \geqslant \zeta$.

For $y \in V(G)$ and $R \geqslant 1$, define

$$Q_\rho^R(y) := \mathbb{P}\left[\tau_{\{\rho\}} < \tau_{\bar{B}^G(\rho,R)} \mid X_0 = y\right] = \frac{c_\rho^G}{c_y^G}\frac{\mathsf{g}_{B^G(\rho,R)}(\rho,y)}{\mathsf{g}_{B^G(\rho,R)}(\rho,\rho)}, \tag{4.8}$$

where the second equality arises because both Q_ρ^R and the function $y \mapsto \mathsf{g}_{B^G(\rho,R)}(\rho,y)/c_y^G$ are harmonic on $B^G(\rho,R) \setminus \{\rho\}$. Moreover, Q_ρ^R and the right-hand side vanish on $\bar{B}^G(\rho,R)$ and are equal to 1 at ρ.

Hence, the Dirichlet principle (2.2) yields

$$\mathcal{E}^G(Q_\rho^G) = \frac{1}{\mathsf{R}_{\text{eff}}^G(\rho \leftrightarrow \bar{B}^G(\rho, R))}. \tag{4.9}$$

In particular, we have

$$\left|1 - Q_\rho^R(y)\right|^2 = \left|Q_\rho^R(\rho) - Q_\rho^R(y)\right|^2 \leqslant \mathsf{R}_{\text{eff}}^G(\rho \leftrightarrow y)\,\mathcal{E}^G(Q_\rho^R) = \frac{\mathsf{R}_{\text{eff}}^G(\rho \leftrightarrow y)}{\mathsf{R}_{\text{eff}}^G(\rho \leftrightarrow \bar{B}^G(\rho, R))}, \tag{4.10}$$

where the inequality is another application of the Dirchlet principle (2.2).

Assume now that $\zeta > 0$, and fix $\delta \in (0, \zeta)$. Denote $R' := R^{(\zeta+2\delta)/(\zeta-\delta)}$ and $Q_\rho := Q_\rho^{R'}$. Using (1.11) and (1.12), we have almost surely eventually

$$\max\{\mathsf{R}_{\text{eff}}^G(\rho \leftrightarrow x) : x \in B^G(\rho, R)\} \leqslant R^{\zeta+\delta}, \tag{4.11}$$

$$\mathsf{R}_{\text{eff}}^G\left(\rho \leftrightarrow \bar{B}^G(\rho, R')\right) \geqslant R^{\zeta+2\delta}. \tag{4.12}$$

So by (4.10), almost surely eventually

$$\min\left\{Q_\rho(y) : y \in B^G(\rho, R)\right\} \geqslant 1 - R^{-\delta/2} > \frac{1}{2}. \tag{4.13}$$

Remark 4.8. Here one notes that this conclusion cannot be reached for $\zeta = 0$ because we cannot choose R' large enough with respect to R so as to create a gap between the respective upper and lower bounds in (4.11) and (4.12). Indeed, it is this sort of gap that Telcs defines as "strongly recurrent" (see [Tel01, Def. 2.1]), although his quantitative notion (which requires a uniform multiplicative gap with $R' = O(R)$) is too strong for us, as it entails $\tilde{\zeta} > 0$.

Let us assume that R is such that (4.13) holds. Denote by H the induced graph on $G[B^G(\rho, 2R')]$, and consider the sets

$$V_{1/2} := \left\{x \in B^G(\rho, R') : Q_\rho(x) = 1/2\right\},$$
$$E_{1/2} := \left\{\{x, y\} \in E(H) : Q_\rho(x) < 1/2 \leqslant Q_\rho(y)\right\}.$$

Define a new graph $\tilde{H}$ where each edge $e = \{x, y\} \in E_{1/2}$ is replaced by a pair of edges $e_x = \{x, v_{xy}\}, e_y = \{v_{xy}, y\}$ with conductances satisfying the system

$$\begin{aligned} \frac{1}{c^{\tilde{H}}(e_x)} + \frac{1}{c^{\tilde{H}}(e_y)} &= \frac{1}{c^H(e)}, \\ \frac{c^{\tilde{H}}(e_x) + c^{\tilde{H}}(e_y)}{2} &= c^{\tilde{H}}(e_x)Q_\rho(x) + c^{\tilde{H}}(e_y)Q_\rho(y) \end{aligned} \tag{4.14}$$

and $c^{\tilde{H}}(e) = c^H(e)$ for the remaining original edges $\{e \in E(\tilde{H}) : e \subseteq V(H)\}$.

Denote $\tilde{V}_{1/2} := V_{1/2} \cup \{v_{xy} : \{x, y\} \in E_{1/2}\}$, and extend Q_ρ to the new vertices so that $\tilde{Q}_\rho(v) = 1/2$ for $v \in \tilde{V}_{1/2}$. Then:

1. $\tilde{Q}_\rho(\rho) = 1$, $\tilde{Q}_\rho$ is harmonic on $(B^G(\rho, R') \cup \tilde{V}_{1/2}) \setminus \{\rho\}$,

2. $\tilde{Q}_\rho$ vanishes elsewhere on $V(\tilde{H})$, and

3. $\mathcal{E}^{\tilde{H}}(\tilde{Q}_\rho) = \mathcal{E}^H(Q_\rho)$.

Since $\tilde{Q}_\rho(\tilde{V}_{1/2}) = 1/2$ and $\tilde{Q}_\rho(\bar{B}^G(\rho, R')) = 0$, we conclude from the Dirichlet principle and (1)–(3) that

$$\mathsf{R}_{\text{eff}}^{\tilde{H}}(\tilde{V}_{1/2} \leftrightarrow \bar{B}^G(\rho, R')) = \frac{1}{4\mathcal{E}^H(Q_\rho)} = \frac{1}{4\mathcal{E}^G(Q_\rho)} = \frac{\mathsf{R}_{\text{eff}}^G(\rho \leftrightarrow \bar{B}^G(\rho, R))}{4},$$

where the last equality is (4.9). Moreover, by (4.13), it holds that $\tilde{V}_{1/2}$ separates $B^G(\rho, R)$ from $\bar{B}^G(\rho, R')$ in $\tilde{H}$, and thus

$$\mathsf{R}_{\text{eff}}^{\tilde{H}}(B^G(\rho, R) \leftrightarrow \bar{B}^G(\rho, R')) \geqslant \mathsf{R}_{\text{eff}}^{\tilde{H}}(\tilde{V}_{1/2} \leftrightarrow \bar{B}^G(\rho, R')) \geqslant \frac{1}{4}\mathsf{R}_{\text{eff}}^G(\rho \leftrightarrow \bar{B}^G(\rho, R)) \geqslant \frac{1}{4}R^{\zeta-\delta},$$

where the last inequality follows from (1.12) and holds almost surely eventually. Finally, observe that by the series law for conductances, (4.14) does not change the effective conductance across subdivided edges, hence

$$\mathsf{R}_{\text{eff}}^G(B^G(\rho, R) \leftrightarrow \bar{B}^G(\rho, R')) = \mathsf{R}_{\text{eff}}^{\tilde{H}}(B^G(\rho, R) \leftrightarrow \bar{B}^G(\rho, R')) \geqslant \frac{1}{4}R^{\zeta-\delta}.$$

Since this holds for any $\delta > 0$, we conclude that $\tilde{\zeta} \geqslant \zeta$, as required. □

4.3 Resistance exponent for planar maps coupled to a mated-CRT

We first establish that $\tilde{\zeta} = 0$ for the γ-mated-CRT with $\gamma \in (0, 2)$. It is known that $\tilde{\zeta}_0 = 0$ [GM17, Prop. 3.1]. While the following argument is somewhat technical and, to our knowledge, does not appear elsewhere, we stress that it is an easy consequence of [GMS19, DG20].

Fix some $\gamma \in (0, 2)$ and for $\varepsilon > 0$, let $\mathcal{G}^\varepsilon$ be the γ-mated-CRT with increment ε. See, for instance, the description in [GMS19]. For our purposes, we may consider this as a random planar multigraph. When needed, we can replace multiple edges by appropriate conductances.

From [DMS14, Thm. 1.9], one can identify $V(\mathcal{G}^\varepsilon) = \varepsilon\mathbb{Z}$ and there is a space-filling SLE curve $\eta : \mathbb{R} \to \mathbb{C}$ parameterized by the LQG mass of the γ-quantum cone, with $\eta(0) = 0$ and such that $\{a, b\} \in E(\mathcal{G}^\varepsilon)$ are connected by an edge if and only if the corresponding cells $\eta([a - \varepsilon, a])$ and $\eta([b - \varepsilon, b])$ share a non-trivial connected boundary arc. Thus we can envision η as an embedding of $V(\mathcal{G}^\varepsilon)$ into the complex plane, where a vertex $v \in V(\mathcal{G}^\varepsilon)$ is sent to $\eta(v)$. Let us denote the Euclidean ball $B^{\mathbb{C}}(z, r) := \{y \in \mathbb{C} : |y - z| \leqslant r\}$.

The underlying idea is simple: We will arrange that, with high probability, the image of a graph annulus under η contains a Euclidean annulus $\mathcal{A}$ of large width. Then we pull back a Lipschitz test functional from $\mathcal{A}$ to $\mathcal{G}^\varepsilon$, and use the Dirichlet principle (2.2) to lower bound the effective resistance across the annulus.

By [DG20, Prop. 4.6], there is a number $d_\gamma > 2$ such that the following holds: For every $\theta \in (0, 1)$ and $\delta > 0$, there is an $\alpha = \alpha(\delta, \gamma, \theta) > 0$ such that as $\varepsilon \to 0$,

$$\mathbb{P}\left[\eta\left(B^{\mathcal{G}^\varepsilon}(0, \varepsilon^{-1/(d_\gamma+\delta)})\right) \subseteq B^{\mathbb{C}}(0, \theta)\right] \geqslant 1 - O(\varepsilon^\alpha)$$

$$\mathbb{P}\left[\eta^{-1}\left(B^{\mathbb{C}}(0, \theta) \cap \eta(\varepsilon\mathbb{Z})\right) \subseteq B^{\mathcal{G}^\varepsilon}(0, \varepsilon^{-1/(d_\gamma-\delta)})\right] \geqslant 1 - O(\varepsilon^\alpha).$$

In particular, taking $\theta = 1/4$ and $\theta = 3/4$, respectively, yields, for some $\alpha = \alpha(\delta, \gamma) > 0$:

$$\mathbb{P}\left[\eta\left(B^{\mathcal{G}^\varepsilon}(0, \varepsilon^{-1/(d_\gamma+\delta)})\right) \subseteq B^{\mathbb{C}}(0, 1/4) \cap \eta(\varepsilon\mathbb{Z}) \right.$$
$$\left. \subseteq B^{\mathbb{C}}(0, 3/4) \cap \eta(\varepsilon\mathbb{Z}) \subseteq \eta\left(B^{\mathcal{G}^\varepsilon}(0, \varepsilon^{-1/(d_\gamma-\delta)})\right)\right] \geqslant 1 - O(\varepsilon^\alpha). \tag{4.15}$$

For a subset $D \subseteq \mathbb{C}$, denote

$$\mathcal{VG}^\varepsilon(D) := \{x \in \varepsilon\mathbb{Z} : \eta([x-\varepsilon, x]) \cap D \neq \emptyset\},$$

and let $\mathcal{G}^\varepsilon(D)$ be the subgraph of $\mathcal{G}^\varepsilon$ induced on $\mathcal{VG}^\varepsilon(D)$. For a function $f : \overline{D} \to \mathbb{R}$, define $f^\varepsilon : \mathcal{VG}^\varepsilon(D) \to \mathbb{R}$ by

$$f^\varepsilon(z) := \begin{cases} f(\eta(z)) & z \in \mathcal{VG}^\varepsilon(D) \setminus \mathcal{VG}^\varepsilon(\partial D) \\ \sup_{x \in \eta([z-\varepsilon,z]) \cap \partial D} f(z) & z \in \mathcal{VG}^\varepsilon(\partial D). \end{cases}$$

Take now $D := B^{\mathbb{C}}(0,1)$ and define $f : D \to \mathbb{R}$ by $f(z) := \min(1, 4\,(|z| - 3/8)_+)$, which is a 4-Lipschitz function satisfying

$$f|_{B^{\mathbb{C}}(0,3/8)} \equiv 0, \qquad f|_{B^{\mathbb{C}}(0,1) \setminus B^{\mathbb{C}}(0,5/8)} \equiv 1. \tag{4.16}$$

Let $\{f_n\}$ be a sequence of continuously differentiable, uniformly Lipschitz functions such that $f_n \to f$ uniformly on D. Then we may apply [GMS19, Lem. 3.3] to each f_n to obtain, for every $n \geqslant 1$,

$$\mathbb{P}\left(\mathcal{E}^{\mathcal{G}^\varepsilon(D)}(f_n^\varepsilon) \leqslant \varepsilon^\alpha + A \int_D |\nabla f_n(z)|^2\, dz\right) \geqslant 1 - O(\varepsilon^\alpha),$$

where $A = A(\gamma), \alpha(\gamma) > 0$. We conclude that with probability at least $1 - O(\varepsilon^\alpha)$, the Dirichlet energy of f_n^ε is uniformly (in n) bounded. Taking $f^\varepsilon = \lim_{n\to\infty} f_n^\varepsilon$, we obtain the following in conjuction with (4.15) and (4.16).

Lemma 4.9. *For every $\gamma \in (0,2)$ and $\delta > 0$, there are numbers $\alpha, A > 0$ such that for every $\varepsilon > 0$, with probability at least $1 - O(\varepsilon^\alpha)$, there is a function $f^\varepsilon : V(\mathcal{G}^\varepsilon) \to \mathbb{R}$ such that*

1. *f^ε vanishes on $B^{\mathcal{G}^\varepsilon}(0, \varepsilon^{-1/(d_\gamma+\delta)})$,*
2. *f^ε is identically 1 on $\partial_{\mathcal{G}^\varepsilon} B^{\mathcal{G}^\varepsilon}(0, \varepsilon^{-1/(d_\gamma-\delta)})$.*
3. *$\mathcal{E}^{\mathcal{G}^\varepsilon}(f^\varepsilon) \leqslant A$.*

In particular, the Dirichlet principle (2.2) gives, with probability at least $1 - O(\varepsilon^\alpha)$,

$$\mathsf{R}_{\mathrm{eff}}^{\mathcal{G}^\varepsilon}\left(\partial_{\mathcal{G}^\varepsilon} B^{\mathcal{G}^\varepsilon}(0, \varepsilon^{-1/(d_\gamma+\delta)})) \leftrightarrow \partial_{\mathcal{G}^\varepsilon} B^{\mathcal{G}^\varepsilon}(0, \varepsilon^{-1/(d_\gamma-\delta)}))\right) \geqslant 1/A.$$

Note that the law of $\mathcal{G}^\varepsilon$ is independent of $\varepsilon > 0$, and therefore denoting its law by $\mathcal{G}$ and taking $R := 1/\varepsilon$, we arrive at the following.

Corollary 4.10. *Let $\mathcal{G}$ denote the γ-mated-CRT for $\gamma \in (0,2)$. Then for every $\delta > 0$, there are numbers $\alpha, \kappa > 0$ such that with probability at least $1 - O(R^{-\alpha})$*

$$\mathsf{R}_{\text{eff}}^{\mathcal{G}}\left(\partial_{\mathcal{G}} B^{\mathcal{G}}(0,R) \leftrightarrow \partial_{\mathcal{G}} B^{\mathcal{G}}(0,R^{1+\delta})\right) \geqslant \kappa. \tag{4.17}$$

In particular, it holds that for every $\delta > 0$, almost surely eventually

$$\mathsf{R}_{\text{eff}}^{\mathcal{G}}\left(\partial_{\mathcal{G}} B^{\mathcal{G}}(0,R) \leftrightarrow \partial_{\mathcal{G}} B^{\mathcal{G}}(0,R^{1+\delta})\right) \geqslant \kappa.$$

Since this holds for every $\delta > 0$, and $(\mathcal{G}, 0)$ is a unimodular random network, we have $\tilde{\zeta} = 0$.

Proof. (4.17) follows immediately from Lemma 4.9. The other conclusion is a standard consequence: The Borel-Cantelli lemma implies that almost surely, for all but finitely many $k \in \mathbb{N}$, we have

$$\mathsf{R}_{\text{eff}}^{\mathcal{G}}\left(\partial_{\mathcal{G}} B^{\mathcal{G}}(0,2^k) \leftrightarrow \partial_{\mathcal{G}} B^{\mathcal{G}}(0,2^{(1+\delta)k})\right) \geqslant \kappa,$$

so by the series law for effective resistances, it holds that almost surely eventually

$$\mathsf{R}_{\text{eff}}^{\mathcal{G}}\left(\partial_{\mathcal{G}} B^{\mathcal{G}}(0,R) \leftrightarrow \partial_{\mathcal{G}} B^{\mathcal{G}}(0,2R^{1+\delta})\right) \geqslant \mathsf{R}_{\text{eff}}^{\mathcal{G}}\left(\partial_{\mathcal{G}} B^{\mathcal{G}}(0,2^{\lfloor \log_2 R \rfloor}) \leftrightarrow \partial_{\mathcal{G}} B^{\mathcal{G}}(0,2^{\lceil \log_2(R^{1+\delta})\rceil})\right) \geqslant \kappa,$$

and thus for any $\delta' > \delta$, almost surely eventually $\mathsf{R}_{\text{eff}}^{\mathcal{G}}\left(\partial_{\mathcal{G}} B^{\mathcal{G}}(0,R) \leftrightarrow \partial_{\mathcal{G}} B^{\mathcal{G}}(0,R^{1+\delta'})\right) \geqslant \kappa$. □

Note that since $\tilde{\zeta} = \tilde{\zeta}_0 = 0$ and d_f exists [DG20], it follows from Theorem 1.1 that $d_w = d_f$ and $d_s = 2$. Both equalities were known previously: $d_s \leqslant 2$ from [Lee17], $d_w \leqslant d_f$ and $d_s \geqslant 2$ from [GM17], and and $d_w \geqslant d_f$ from [GH18]. Let us remark that the preceding argument requires somewhat less detailed information about $\mathcal{G}$ than that of [GH18]. In particular, bounding $\tilde{\zeta}$ only requires control of one scale at a time.

4.3.1 Other planar maps

We consider now the case of random planar maps that can be appropriately coupled to a γ-mated CRT for some $\gamma \in (0,2)$; we refer to [GHS17] for a discussion of such examples, including the UIPT, and random planar maps whose law is biased by the number of different spanning trees ($\gamma = \sqrt{2}$), bipolar orientations ($\gamma = \sqrt{4/3}$), or Schynder woods ($\gamma = 1$).

Our goal is to prove that $\tilde{\zeta} = 0$ for each of these random planar maps (M, ρ). We employ the same approach as in the preceding section, arguing that an annulus in (M, ρ) can be mapped into $\mathcal{G}$ so that its image contains an annulus of large width, and that the Dirichlet energy of functionals in $\mathcal{G}$ is controlled when pulling them back to M.

Fix $\gamma \in (0,2)$ and let $\mathcal{G}$ be the γ-mated-CRT with increment 1. Let $\mathcal{G}_n$ be the subgraph of $\mathcal{G}$ induced on the vertices $[-n, n] \cap \mathbb{Z}$. Parts (1)–(3) in the following theorem are the conjunction of Lemma 1.11 and Theorem 1.9 in [GHS17]. Part (4) is [GM17, Lem. 4.3].

Theorem 4.11. *For each model considered in [GHS17], the following holds. There is a coupling of (M, ρ) and $(\mathcal{G}, 0)$, and a family of random rooted graphs $\{(M_n, \rho_n) : n \geqslant 1\}$ and numbers $\alpha, K, q > 0$ such that for every $n \geqslant 1$, with probability at least $1 - O(n^{-\alpha})$:*

1. $B^{\mathcal{G}}(0, n^{1/K}) \subseteq V(\mathcal{G}_n)$,

2. *The induced, rooted subnetworks $B^M(\rho, n^{1/K})$ and $B^{M_n}(\rho_n, n^{1/K})$ are isomorphic.*

3. *There is a mapping $\phi_n : V(M_n) \to V(\mathcal{G}_n)$ with $\phi_n(\rho_n) = 0$, and for all $3 \leqslant r \leqslant R$,*

$$\phi_n\left(B^{M_n}\left(\rho_n, (K\log n)^{-q}(r-2)\right)\right) \subseteq B^{\mathcal{G}_n}(0, r)$$
$$\phi_n\left(V(M_n) \setminus B^{M_n}\left(\rho_n, (K\log n)^{q}R - 1\right)\right) \subseteq V(\mathcal{G}_n) \setminus B^{\mathcal{G}_n}(0, R).$$

4. *For every $f : V(\mathcal{G}_n) \to \mathbb{R}$, it holds that*

$$\mathscr{E}^{M_n}(f \circ \phi_n) \leqslant K(\log n)^q \mathscr{E}^{\mathcal{G}_n}(f).$$

Corollary 4.12. *For any model considered in [GHS17], it holds that $\tilde{\zeta} = 0$.*

We prove this momentarily, but first note the following consequence. Since $d_f > 2$ for each of these models [DG20, Prop. 4.7], and $\tilde{\zeta}_0 = 0$ by [GM17, Prop. 4.4], Theorem 1.1 yields:

Theorem 4.13. *For any model considered in [GHS17], it holds that $d_w = d_f > 2$ and $d_s = 2$.*

Remark 4.14. We remark that the lower bound $d_s \geqslant 2$ is established in [GM17], and the upper bound $d_s \leqslant 2$ follows for any unimodular random planar graph where the degree of the root has superpolynomial tails [Lee17] (which is true for each of these models; see [GM17, §1.3]). The consequence $d_w = d_f$ is proved in [GH18] for every model except the uniform infinite Schynder-wood decorated triangulation. This is for a technical reason underlying the identification of $V(M_n)$ with a subset of $V(M)$ used in the proof of [GHS17, Lem. 1.11] (see [GHS17, Rem. 1.3] and [GH18, Rem. 2.11]).

Proof of Corollary 4.12. Fix $\delta > 0$ and $R \geqslant 2$. Denote

$$\tilde{r} := (K\log n)^{-q}(R-2),$$
$$\tilde{R} := (K\log n)^{q}R^{1+\delta},$$
$$n := \lceil \tilde{R}^K \rceil,$$

and let $\mathcal{E}_n$ be an event on which Theorem 4.11(1)–(4) and (4.17) hold. Note that we can take $\mathbb{P}(\mathcal{E}_n) \geqslant 1 - O(R^{-\alpha'})$ for some $\alpha' = \alpha'(\delta, K) > 0$.

Assume now that $\mathcal{E}_n$ holds. Then (4.17) and the Dirichlet principle (2.2) give a test function $f : V(\mathcal{G}) \to \mathbb{R}$ such that

$$f(B^{\mathcal{G}}(0,R)) = 0, \qquad f(\partial_{\mathcal{G}} B^{\mathcal{G}}(0, R^{1+\delta})) = 1, \qquad \mathscr{E}^{\mathcal{G}}(f) \leqslant 1/\kappa.$$

Theorem 4.11(1) asserts that the restriction of f to $B^{\mathcal{G}}(0, R^{1+\delta})$ gives a function $\tilde{f} : V(\mathcal{G}_n) \to \mathbb{R}$ on which

$$\tilde{f}(B^{\mathcal{G}_n}(0,R)) = 0, \qquad \tilde{f}(\partial_{\mathcal{G}} B^{\mathcal{G}_n}(0, R^{1+\delta})) = 1, \qquad \mathscr{E}^{\mathcal{G}_n}(\tilde{f}) \leqslant 1/\kappa.$$

Without increasing the energy of $\tilde{f}$, we may assume that $\tilde{f}(V(\mathcal{G}_n) \setminus B^{\mathcal{G}_n}(0, R^{1+\delta})) = 1$ as well.

By our choice of $\tilde{r}$ and $\tilde{R}$, Theorem 4.11(3) implies that

$$\tilde{f} \circ \phi_n(B^{M_n}(\rho, \tilde{r})) = 0, \qquad \tilde{f} \circ \phi_n(\partial_{M_n} B^{M_n}(\rho, \tilde{R})) = 1, \qquad \mathscr{E}^{M_n}(\tilde{f} \circ \phi_n) \leqslant K'(\log R)/\kappa,$$

where the last inequality is from Theorem 4.11(4), and $K' = K'(K, q, \delta)$. Now the Dirichlet principle (2.2) yields

$$\mathsf{R}_{\mathrm{eff}}^{M_n}\left(\partial_{M_n} B^{M_n}(\rho_n, \tilde{r}) \leftrightarrow \partial_{M_n} B^{M_n}(\rho_n, \tilde{R})\right) \geqslant \frac{1}{K'(\log R)/\kappa},$$

and from the graph isomorphism Theorem 4.11(2) and the fact that $n^{1/K} \geqslant \tilde{R}$, we conclude that

$$\mathsf{R}_{\mathrm{eff}}^{M}\left(\partial_{M} B^{M}(\rho, \tilde{r}) \leftrightarrow \partial_{M} B^{M}(\rho, \tilde{R})\right) \geqslant \frac{1}{K'(\log R)/\kappa}.$$

Since this conclusion holds with probability at least $1 - O(R^{-\alpha'})$, we conclude (using Borel-Cantelli as in the proof of Corollary 4.10) that for every $\delta > 0$, almost surely eventually

$$\mathsf{R}_{\mathrm{eff}}^{M}\left(\partial_{M} B^{M}(\rho, R) \leftrightarrow \partial_{M} B^{M}(\rho, R^{1+\delta})\right) \geqslant R^{-\delta}.$$

This yields $\tilde{\zeta} = 0$, recalling the characterization of $\tilde{\zeta}$ in (2.3). □

4.4 Random walk driven by the GFF

Denote by $\boldsymbol{\eta} = \{\eta_v : v \in \mathbb{Z}^2\}$ the centered Gaussian process with $\eta_0 = 0$ and covariances $\mathbb{E}[\eta_u \eta_v] = \mathsf{g}_{\mathbb{Z}^2 \setminus \{0\}}^{\mathbb{Z}^2}(u, v)$ for all $u, v \in \mathbb{Z}^2$, where we recall the Green kernel from Section 4.2.

Fix $\gamma > 0$, and define $G = \mathbb{Z}^2$ with $E(G) = \big\{\{u, v\} \subseteq \mathbb{Z}^2 : \|u - v\|_1 = 1\big\}$, and the conductances

$$c^G(\{u, v\}) := e^{\gamma(\eta_u - \eta_v)}, \qquad \{u, v\} \in E(G).$$

Since the edge conductances only depend on the differences $\eta_u - \eta_v$, the law of the conductances is translation invariant, and thus $(G, 0, c^G)$ is a reversible random network. Moreover, $\mathbb{E}[1/c_0^G] < \infty$ follows from the fact that $\eta_u - \eta_v$ is a Gaussian of variance of bounded variance for $\{u, v\} \in E(G)$.

Theorem 4.15. *For the reversible random network $(G, 0, c^G)$, it holds that*

$$\begin{aligned} d_f &= \begin{cases} 2 + 2(\gamma/\gamma_c)^2 & \gamma \leqslant \gamma_c = \sqrt{\pi/2}, \\ 4\gamma/\gamma_c & \textit{otherwise}, \end{cases} \\ \tilde{\zeta} &= 0, \\ \tilde{\zeta}_0 &= 0, \\ d_w &= d_f, \\ d_s &= 2. \end{aligned}$$

Since the value of d_f is elementary to calculate (see [BDG20] for details), and $\tilde{\zeta}_0 = 0$ is the content of Theorem 1.4(1.11) in [BDG20], we can apply Theorem 1.1, and what remains is to verify $\tilde{\zeta} = 0$. To this end, denote $\mathcal{S}(N) := [-N, N]^2 \cap \mathbb{Z}^2$. Theorem 1.4 in [BDG20] (specifically, equation (1.12)) establishes that

$$\liminf_{N \to \infty} \frac{\log \mathsf{R}_{\mathrm{eff}}^{G}\left(0 \leftrightarrow \mathbb{Z}^2 \setminus \mathcal{S}(N)\right)}{(\log N / \log\log\log N)^{1/2}} > 0. \tag{4.18}$$

In a moment, we will observe the following consequence of their argument.

Lemma 4.16. *There is some $c = c(\gamma) > 0$ such that for every $N \geqslant 8$ sufficiently large, the following holds. For $1 \leqslant k \leqslant n-1$, where $n = \lfloor \log_8(N) \rfloor$, let $\mathcal{E}_k$ denote the event*

$$\mathsf{R}^G_{\mathrm{eff}}\left(\mathcal{S}(2 \cdot 8^{n-k}N) \leftrightarrow \mathbb{Z}^2 \setminus \mathcal{S}(4 \cdot 8^{n-k}N)\right) \geqslant e^{-(1/c)\log\log(N)}$$

Then $\mathbb{P}(\mathcal{E}_k) > c$ for each $k \in \{1, 2 \ldots, n-1\}$.

Proof of Theorem 4.15. Fix some $\delta > 0$, and define $n' := \lceil \delta \log(N) \rceil$. Since the events $\{\mathcal{E}_k : 1 \leqslant k \leqslant n-1\}$ involve disjoint sets of edges, they are independent, and we have

$$\mathbb{P}\left(\mathcal{E}_1 \vee \mathcal{E}_2 \vee \cdots \vee \mathcal{E}_{n'}\right) \geqslant 1 - (1-c)^{n'} \geqslant 1 - N^{-\delta c}.$$

Moreover, the series law for effective resistances gives

$$\mathsf{R}^G_{\mathrm{eff}}\left(\mathcal{S}(2 \cdot 8^{n-1}N) \leftrightarrow \mathbb{Z}^2 \setminus \mathcal{S}(4 \cdot 8^{n-n'}N)\right) \geqslant \mathbb{1}_{\{\mathcal{E}_1 \vee \mathcal{E}_2 \vee \cdots \vee \mathcal{E}_{n'}\}} e^{-(1/c)\log\log(N)}.$$

Thus an application of Borel-Cantelli yields: Almost surely eventually,

$$\mathsf{R}^G_{\mathrm{eff}}\left(\mathcal{S}(R) \leftrightarrow \mathbb{Z}^2 \setminus \mathcal{S}(R^{1+\delta})\right) \geqslant R^{-\delta}.$$

Since $\delta > 0$ could be chosen arbitrarily small, recalling (2.3), we conclude that $\tilde{\zeta} = 0$. $\square$

Let us finally indicate how Lemma 4.16 follows from the arguments in [BDG20]. The authors define in equation (5.79) an event F_k such that $\mathbb{P}(F_k) > c_0 > 0$, and on F_k it holds that

$$\mathsf{R}^G_{\mathrm{eff}}\left(\mathcal{S}(2 \cdot 8^{n-k}N) \leftrightarrow \mathbb{Z}^2 \setminus \mathcal{S}(4 \cdot 8^{n-k}N)\right) \geqslant e^{-2\gamma(\Delta_k - S_k)} e^{-3\hat{\mathfrak{c}}\log\log(N)},$$

where S_k, Δ_k, and $\mathbb{1}_{F_k}$ are mutually independent random variables, $c_0 > 0$ is a universal constant, and $\hat{\mathfrak{c}} = \hat{\mathfrak{c}}(\gamma) > 0$. We remark that in [BDG20], the exponent is given as $-3\hat{\mathfrak{c}}\log(b^k)$ (where $b = 8$), but the correct value (as stated in [BDG20, Lem. 4.13]) is $-3\hat{\mathfrak{c}}\log\log(b^k)$. (And, indeed, the correct quantitative dependence is needed to conclude (4.18).)

Moreover, it holds that the law of S_k is symmetric, and (5.74) in [BDG20] asserts that for some constants $c_1, c_2 > 0$ and all $t \geqslant 0$,

$$\mathbb{P}(\Delta_k \geqslant c_1 + t) \leqslant e^{-c_2 t^2}.$$

We conclude that for some number $C > 0$ chosen sufficiently large,

$$\mathbb{P}\left(\mathsf{R}^G_{\mathrm{eff}}\left(\mathcal{S}(2 \cdot 8^{n-k}N) \leftrightarrow \mathbb{Z}^2 \setminus \mathcal{S}(4 \cdot 8^{n-k}N)\right) \geqslant e^{-2C\gamma} e^{-3\hat{\mathfrak{c}}\log\log(N)}\right) \geqslant \frac{c_0}{4} > 0,$$

thereby verifying Lemma 4.16.

Acknowledgements

The author thanks Jian Ding, Ewain Gwynne, Mathav Murugan, and Asaf Nachmias for useful feedback and pointers to the literature, and Jian for suggesting that our main theorem is applicable to the model in [BDG20]. This work was partially funded by a Simons Investigator Award.

References

[AL07] David Aldous and Russell Lyons. Processes on unimodular random networks. *Electron. J. Probab.*, 12:no. 54, 1454–1508, 2007. 5, 13

[Ang03] O. Angel. Growth and percolation on the uniform infinite planar triangulation. *Geom. Funct. Anal.*, 13(5):935–974, 2003. 3

[Bal92] K. Ball. Markov chains, Riesz transforms and Lipschitz maps. *Geom. Funct. Anal.*, 2(2):137–172, 1992. 18

[Bar98] Martin T. Barlow. Diffusions on fractals. In *Lectures on probability theory and statistics (Saint-Flour, 1995)*, volume 1690 of *Lecture Notes in Math.*, pages 1–121. Springer, Berlin, 1998. 3

[BC12] Itai Benjamini and Nicolas Curien. Ergodic theory on stationary random graphs. *Electron. J. Probab.*, 17:no. 93, 20 pp., 2012. 5, 13

[BCK05] Martin T. Barlow, Thierry Coulhon, and Takashi Kumagai. Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs. *Comm. Pure Appl. Math.*, 58(12):1642–1677, 2005. 7, 25

[BDG20] Marek Biskup, Jian Ding, and Subhajit Goswami. Return probability and recurrence for the random walk driven by two-dimensional Gaussian free field. *Comm. Math. Phys.*, 373(1):45–106, 2020. 4, 31, 32

[BH00] Daniel Ben-Avraham and Shlomo Havlin. *Diffusion and reactions in fractals and disordered systems*. Cambridge University Press, Cambridge, 2000. 2, 6

[BJKS08] Martin T. Barlow, Antal A. Járai, Takashi Kumagai, and Gordon Slade. Random walk on the incipient infinite cluster for oriented percolation in high dimensions. *Comm. Math. Phys.*, 278(2):385–431, 2008. 3

[Bou85] J. Bourgain. On Lipschitz embedding of finite metric spaces in Hilbert space. *Israel J. Math.*, 52(1-2):46–52, 1985. 18

[CKR01] Gruia Calinescu, Howard Karloff, and Yuval Rabani. Approximation algorithms for the 0-extension problem. In *Proceedings of the 12th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 8–16, Philadelphia, PA, 2001. SIAM. 20

[CRR+97] Ashok K. Chandra, Prabhakar Raghavan, Walter L. Ruzzo, Roman Smolensky, and Prasoon Tiwari. The electrical resistance of a graph captures its commute and cover times. *Comput. Complexity*, 6(4):312–340, 1996/97. 24

[DG20] Jian Ding and Ewain Gwynne. The Fractal Dimension of Liouville Quantum Gravity: Universality, Monotonicity, and Bounds. *Comm. Math. Phys.*, 374(3):1877–1934, 2020. 4, 27, 29, 30

[DMS14] Bertrand Duplantier, Jason Miller, and Scott Sheffield. Liouville quantum gravity as a mating of trees. Preprint at arXiv:math/1409.7055, 2014. 4, 27

[Duf62] R. J. Duffin. The extremal length of a network. *J. Math. Anal. Appl.*, 5:200–215, 1962. 12

[EL20] F. Ebrahimnejad and J. R. Lee. On planar graphs of uniform polynomial growth. Preprint at arXiv:math/2005.03139, 2020. 9

[GH18] E. Gwynne and T. Hutchcroft. Anomalous diffusion of random walk on random planar maps. Preprint at arXiv:math/1807.01512, 2018. 4, 29, 30

[GHL15] Alexander Grigor'yan, Jiaxin Hu, and Ka-Sing Lau. Generalized capacity, Harnack inequality and heat kernels of Dirichlet forms on metric measure spaces. *J. Math. Soc. Japan*, 67(4):1485–1549, 2015. 9

[GHS17] Ewain Gwynne, Nina Holden, and Xin Sun. A mating-of-trees approach for graph distances in random planar maps, 2017. 4, 29, 30

[GM17] E. Gwynne and J. Miller. Random walk on random planar maps: spectral dimension, resistance, and displacement. Preprint at arXiv:math/1701.00836, 2017. 4, 27, 29, 30

[GMS19] Ewain Gwynne, Jason Miller, and Scott Sheffield. Harmonic functions on mated-CRT maps. *Electron. J. Probab.*, 24:Paper No. 58, 55, 2019. 4, 27, 28

[Gra99] Peter Grassberger. Conductivity exponent and backbone dimension in 2-d percolation. *Physica A: Statistical Mechanics and its Applications*, 262(3):251 – 263, 1999. 5

[J03́] Antal A. Járai. Incipient infinite percolation clusters in 2D. *Ann. Probab.*, 31(1):444–485, 2003. 4

[Kes86] Harry Kesten. The incipient infinite cluster in two-dimensional percolation. *Probab. Theory Related Fields*, 73(3):369–394, 1986. 4

[KLMN05] R. Krauthgamer, J. R. Lee, M. Mendel, and A. Naor. Measured descent: A new embedding method for finite metrics. *Geom. Funct. Anal.*, 15(4):839–858, 2005. 20

[KM08] Takashi Kumagai and Jun Misumi. Heat kernel estimates for strongly recurrent random walk on random media. *J. Theoret. Probab.*, 21(4):910–935, 2008. 3, 7

[KN09] Gady Kozma and Asaf Nachmias. The Alexander-Orbach conjecture holds in high dimensions. *Invent. Math.*, 178(3):635–654, 2009. 3

[Kum14a] Takashi Kumagai. Anomalous random walks and diffusions: from fractals to random media. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. IV*, pages 75–94. Kyung Moon Sa, Seoul, 2014. 3

[Kum14b] Takashi Kumagai. *Random walks on disordered media and their scaling limits*, volume 2101 of *Lecture Notes in Mathematics*. Springer, Cham, 2014. Lecture notes from the 40th Probability Summer School held in Saint-Flour, 2010, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School]. 3

[Lee17] James R. Lee. Conformal growth rates and spectral geometry on distributional limits of graphs. Preprint at arXiv:math/1701.01598, 2017. 9, 29, 30

[LP16] Russell Lyons and Yuval Peres. *Probability on Trees and Networks*. Cambridge University Press, New York, 2016. Preprint at http://pages.iu.edu/~rdlyons/. 12, 24

[LPW09] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov chains and mixing times*. American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson. 24

[LSW02] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. One-arm exponent for critical 2D percolation. *Electron. J. Probab.*, 7:no. 2, 13, 2002. 4

[LSW17] Y. Li, X. Sun, and S. S. Watson. Schnyder woods, SLE(16), and Liouville quantum gravity. Preprint at arXiv:math/1705.03573, 2017. 4

[NPSS06] Assaf Naor, Yuval Peres, Oded Schramm, and Scott Sheffield. Markov chains in smooth Banach spaces and Gromov-hyperbolic metric spaces. *Duke Math. J.*, 134(1):165–197, 2006. 18

[Smi01] Stanislav Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(3):239–244, 2001. 4

[Tel89] A. Telcs. Random walks on graphs, electric networks and fractals. *Probab. Theory Related Fields*, 82(3):435–449, 1989. 25

[Tel90] András Telcs. Spectra of graphs and fractal dimensions. I. *Probab. Theory Related Fields*, 85(4):489–497, 1990. 3

[Tel95] András Telcs. Spectra of graphs and fractal dimensions. II. *J. Theoret. Probab.*, 8(1):77–96, 1995. 3

[Tel01] András Telcs. Local sub-Gaussian estimates on graphs: the strongly recurrent case. *Electron. J. Probab.*, 6:no. 22, 33, 2001. 26

[Tel06] András Telcs. The Einstein relation for random walks on graphs. *J. Stat. Phys.*, 122(4):617–645, 2006. 7