embeddings, flow, and cuts: an introduction

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**Flow network:** Graph $G$ and non-negative capacities on edges

**Max-flow Min-Cut Theorem:** Value of maximum flow $= \text{value of minimum s-t cut}$

[Menger’27, Elias-Feinstein-Shannon’56, Ford-Fulkerson’56]:

$max$-flow $min$-cut theorem
**max-flow min-cut theorem**

**Flow network:** Graph $G$ and non-negative capacities on edges

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**2-commodity MFMC Theorem:** $\max$ concurrent flow $=$ min-cut [Hu’69]

**Implied by:** Every 4-point metric space embeds isometrically in the Euclidean plane equipped with the $L_1$ norm
For a subset $S \subseteq V$ of vertices, we define its sparsity by

$$\Phi(S) = \frac{\text{capacity of edges across } S}{\text{number of demand pairs separated}}$$

Is there a Multi-flow/Sparse-cut theorem?  NO, even for 3 demand pairs
multi-commodity flows

For a subset $S \subseteq V$ of vertices, we define its sparsity by

$$\Phi(S) = \frac{\text{capacity of edges across } S}{\text{number of demand pairs separated}}$$

Define $\text{gap}(G) =$ worst possible ratio between min sparse cut and max multi-flow over all instances on $G$

[instance = capacities + demand pairs]
Consider an arbitrary undirected graph $G=(V, E)$ as a **topological template**.

Every length on edges $\text{len} : E \to [0, \infty]$ induces a **shortest-path geometry** on $G$.

We consider the set of all (pseudo-)metrics induced on a given $G$:

- Every metric on a **path** embeds isometrically into $\mathbb{R}$.
- Every metric on a **tree** embeds isometrically into $L_1$.
- The **complete graph** on $n$ nodes supports every metric on $n$ points.
Let $L_1 = L_1([0,1])$ (can also think of $\mathbb{R}^n$ equipped with the $L_1$ norm).

For a metric space $(X, d)$, a mapping $f : X \to L_1$ is a $D$-embedding if

$$d(x, y) \leq \|f(x) - f(y)\|_1 \leq D \cdot d(x,y) \quad \text{for all } x, y \in X$$

For a metric space $X$:

$$c_1(X, d) = \inf \{ D : (X, d) \text{ admits a } D\text{-embedding into } L_1 \}$$

For a graph $G$:

$$c_1(G) = \sup \{ c_1(X, d) : (X, d) \text{ is supported on } G \}$$

**Theorem** [Linial-London-Rabinovich’92, Gupta-Newman-Rabinovich-Sinclair’99]:

For every graph $G$, $c_1(G) = \text{gap}(G)$
Theorem [Linial-London-Rabinovich, Aumann-Rabani]:

If $G$ has $n$ vertices, then $\text{gap}(G) = O(\log n)$
and this it tight (expander graphs)

For a graph family $\mathcal{F}$, define: $c_1(\mathcal{F}) = \sup \{ c_1(G) : G \in \mathcal{F} \}$

Which families admit a uniform multi-flow/cut gap,
i.e. when do we have $c_1(\mathcal{F}) < \infty$ ?

If $\mathcal{F}$ supports all finite metric space, then $c_1(\mathcal{F}) = \infty$.

Conjecture [GNRS 1999]:

$c_1(\mathcal{F}) < \infty$ if and only if $\mathcal{F}$ does not support all finite metric spaces.
the relation to graph minors

A graph \( H \) is a minor of \( G \) if it can be obtained from \( G \) by a sequence of edge deletions and contractions…

If \( H \) is a minor of \( G \), then \( G \) supports every metric space that \( H \) supports.

So it suffices to consider graph families \( \mathcal{F} \) which are closed under taking minors.
Conjecture [GNRS 1999]:

\[ c_1(\mathcal{F}) = \text{gap}(\mathcal{F}) \] is finite if and only if \( \mathcal{F} \) forbids some minor.

(infinite graphs) equivalent:

\[ c_1(G) \] is finite if and only if \( G \) forbids some finite minor.

Outline:
- Cuts and \( L_1 \) embeddings
- The (geometric) Okamura-Seymour theorem
- Planar graphs
- Sums of tetrahedra
- More general flow networks
Consider a finite set $X$ and a mapping $f : X \rightarrow L_1$.

**Fact:** There exists a measure $\mu$ on subsets $S \subseteq X$ such that:

$$\|f(x) - f(y)\|_1 = \sum_{S \subseteq X} \mu(S)|1_S(x) - 1_S(y)|$$
simple examples
If $G$ is a (weighted) planar graph and $F$ is the outer face, there is a 1-Lipschitz mapping $f : G \to L_1$ which is an isometry on $F$.

Proof: Assume $G$ is unit-weighted and bipartite.
the Okamura-Seymour theorem

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Proof: Assume $G$ is unit-weighted and bipartite.
Planar embedding conjecture (circa 1992):
\[ c_1(F) \text{ is finite when } F = \text{family of planar graphs} \]
(equivalent to \[ c_1(\mathbb{Z}^2) \text{ finite} \])

Theorem [Gupta-Newman-Rabinovich-Sinclair’99]:
\[ c_1(F) \text{ is finite when } F = \text{family of series-parallel graphs} \]

Theorem [L-Ragahvendra’07, Chakrabarti-L-Jaffe-Vincent’08]:
\[ c_1(\{\text{series-parallel graphs}\}) = 2 \]

Lower bound based on quantitative differentiation
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Open problem: What about summing tetrahedra along triangles?
(What about summing \((k+1)\)-cliques along \(k\) cliques?)

[L-Sidiropoulos’09]: Answer is positive (bounded distortion) if we only take sums along paths.
Consider a metric space \((X, d_X)\) and a random mapping \(F : X \to Y\), where \((Y, d_Y)\) is a random metric space. This is a \textbf{probabilistic} \(D\)-\textit{embedding} if

1. \(d_Y(F(x), F(y)) \geq d_X(x, y)\) with probability one
2. \(\mathbb{E} \left[ d_Y(F(x), F(y)) \right] \leq D \cdot d_X(x, y)\)

For two graph families \(\mathcal{F}\) and \(\mathcal{G}\), we write \(\mathcal{F} \rightsquigarrow \mathcal{G}\) if there exists a \(D\) such that every metric supported on \(\mathcal{F}\) probabilistically \(D\)-embeds into a metric supported on \(\mathcal{G}\).
random simplifying the topology

GNRS conjecture

\[ \sim \]
The k-sum conjecture:

For every $k \in \mathbb{N}$, $c_1(\bigoplus_k \mathcal{F})$ is finite whenever $c_1(\mathcal{F})$ is finite.

Take two graphs $G$ and $G'$ and fix a $k$-clique in each. Identify the cliques.

$\bigoplus_k \mathcal{F}$ is the closure of $\mathcal{F}$ under the operation of taking $k$-sums.

Example: $\bigoplus_1 \{\bullet \cdots \bullet\}$ is the family of all (connected) trees.

Conjecture is true for $k=1$: $c_1(\bigoplus_1 \mathcal{F}) = c_1(\mathcal{F})$ for any family $\mathcal{F}$. 
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What if we have capacities on nodes instead of edges?

- Okamura-Seymour analogue is now (trivially) false.
- But an $O(1)$-approximate version is true! [L-Mendel-Moharrami’13]
- Very different techniques from the edge case
  (Random Lipschitz maps into trees, random Whitney decompositions)
open questions