CSE 599: Algorithmic Spectral Graph Theory HW #2

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This homework is due in two weeks on Monday, May 14th. You make work in small groups of 2 or 3. You should reference any discussions with others, books, lecture notes, or other resources that you use in coming up with a solution.

Problems

1. Consider a finite group G with a symmetric generating set S, i.e. $s \in S \iff s^{-1} \in S$. The Cayley graph $\mathrm{Cay}(G; S)$ is the undirected graph with vertex set G and edges $\{\{g, gs\} : s \in S\}$. This is clearly an $|S|$-regular graph. Let dist denote the shortest-path distance in this graph, and define $\Delta := \mathsf{diam}(\mathrm{Cay}(G; S)) = \max_{h,g \in G} \mathsf{dist}(h, g)$.

   In this exercise, you will prove that if λ_2 is the smallest non-zero eigenvalue of the combinatoriali Laplacian on $\mathrm{Cay}(G; S)$, then
   $$\lambda_2 \gtrsim \frac{1}{\Delta^2}\,. \tag{1}$$
   Let $A \subseteq G$ be any subset with $|A| \leqslant |G|/2$.

   (a) First, prove that there exists a $g \in G$ such that $|A \cap Ag| \leqslant |A|^2/|G|$.

   (b) Show that, for this element g, we have $|Ag \setminus A| \geqslant |A|/2$.

   (c) Now write $g = s_1 s_2 \cdots s_k$ for some $k \leqslant \Delta$. Argue that there exists an i for which
   $$|As_i \setminus A| \geqslant \frac{|A|}{2 \cdot \Delta}\,.$$

   (d) Use Cheeger's inequality to prove (1).

   (e) Given an example of a graph (not a Cayley graph) such that the bound (1) is far from holding.

2. Your goal in this exercise is to prove that if P_n is the path graph on n vertices, then
   $$\lambda_k(P_n) \leqslant O\left(\frac{k^2}{n^2}\right)\,.$$
   You should do this without using an explicit formula for the eigenvalues. Instead, use the Courant-Fischer formula,
   $$\lambda_k = \min_{\mathsf{dim}(W)=k} \max\left\{R_G(f) : 0 \neq f \in W\right\}\,.$$
   Here, the minimum is over all subspaces $W \subseteq \{f : V \to \mathbb{R}\}$ of dimension k. Use k test functions of disjoint support to give the claimed upper bound λ_k.

3. Let G be any graph, and let $L = D - A$ be the combinatorial Laplacian, where D is the diagonal degree matrix and A is the adjacency matrix. You will prove that, for any $r \geqslant 1$, if L has at most $r + 1$ distinct eigenvalues, then the diameter of G is at most r. In particular, this implies that the only graph with exactly two distinct eigenvalues is the complete graph. Let Δ be the diameter of G.

(a) Prove that if p is a polynomial of degree m such that $p(A)$ has all non-zero entries, then $\Delta \leqslant m$.

(b) Prove the same thing for L instead of A.

(c) Now, let $0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ be the eigenvalues of L with corresponding orthogonal eigenvectors $u_1, u_2, \ldots, u_n$. Prove that for any polynomial p such that $p(0) = 1$, we have

$$p(L) = \frac{J}{n} + \sum_{i=2}^{n} p(\lambda_i) u_i u_i^T ,$$

where J is the all ones matrix.

(d) Using the fact that L has only $r + 1$ distinct eigenvalues, use the preceding two parts to show that $\Delta \leqslant r$.

4. For this problem, consider the d-dimensional discrete hypercube. This graph has vertex set $\{-1, 1\}^d$ and an edge between two vertices precisely when they differ in exactly one coordinate. Let L_d be the combinatorial Laplacian.

(a) Verify that the eigenvectors of L_d are the functions $u_S(x) = \prod_{i \in S} x_i$ for subsets $S \subseteq \{1, 2, \ldots, d\}$. Determine the corresponding eigenvalues.

(b) Determine the smallest non-zero eigenvalue of L_d. What is its multiplicity?

(c) This implies that the mixing time τ_∞ is $O(d^{3/2})$. Argue that $\tau_\infty \gtrsim d \log d$. (It turns out that this latter bound is the correct order.)

(d) Consider any function $f : \{-1, 1\}^d \to \{0, 1\}$. Let b_i be the vector with a -1 in the ith coordinate and zeros elsewhere, and for $x, y \in \{-1, 1\}^d$, write $x \cdot y$ for element-wise multiplication. Consider the quantity

$$as(f) = \mathbf{Pr}[f(x \cdot b_i) \neq f(x)],$$

where $x \in \{-1, 1\}^d$ and $i \in \{1, 2, \ldots, d\}$ are chosen uniformly at random. Write $as(f)$ in terms of the Laplacian L_d.

(e) Define $\hat{f}(S) = \mathbb{E}[f(x) u_S(x)]$ where the expectation is over a uniformly random $x \in \{-1, 1\}^d$. Use the preceding part to write $as(f)$ in terms of the "Fourier coefficients" $\{\hat{f}(S) : S \subseteq \{1, 2, \ldots, d\}\}$.