1. **Partitioning metric spaces with bounded growth rate.**

Recall in Lecture 9 that we showed how to construct random partitions of metric spaces with bounded diameter such that nearby points are not separated too often.

Suppose that $G = (V, E)$ is a connected (unweighted) graph and $d_G(x, y)$ is the shortest-path distance between two vertices $x, y \in V$. Let $B_G(x, R) = \{ y \in V : d_G(x, y) \leq R \}$. Suppose furthermore that for some number $k \geq 1$, it holds that $|B_G(x, R)| \leq O(R^k)$ for every $x \in X, R \geq 1$.

Show that for every $\Delta \geq 2$, there is a random partition $P$ of $V$ such that $\text{diam}(S) \leq \Delta$ for every $S \in P$, and for every $x, y \in V$,

$$\mathbb{P}[x \text{ and } y \text{ are separated in } P] \leq O(k) \frac{d_G(x, y)}{\Delta} \log \Delta.$$

2. **Random linear maps: Part I**

In this problem, you will show the following: There are constants $c, C \geq 1$ such that for every $n \geq 1$, with high probability a random map $A : \mathbb{R}^n \to \mathbb{R}^{[cn]}$ satisfies

$$\frac{\|x\|_2}{C} \leq \|Ax\|_2 \leq C\|x\|_2 \quad \forall x \in \mathbb{R}^n.$$

Let $N = [cn]$. The random map $A$ is an $N \times n$ matrix where the entries are $A_{ij} = \frac{\varepsilon_{ij}}{\sqrt{N}}$ and $\varepsilon_{ij} \in \{-1, 1\}$ is an independent, uniform random sign.

Such maps are very useful. For instance, suppose that $y = Ax$ and $\hat{y}$ is a corruption of $y$ that results from arbitrarily changing $\delta n$ coordinates for some small constant $\delta > 0$. Then one can uniquely recover $x$ from the corrupted vector $\hat{y}$, and moreover there is an efficient algorithm to do it. (On HW #5, you will design this recovery algorithm.)

**Fact 0.1.** You will need the following fact. Let $S^{n-1} := \{ x \in \mathbb{R}^n : \|x\|_2 = 1 \}$. For every $\theta > 0$, there is a subset $X \subseteq S^{n-1}$ with $|X| \leq \left( \frac{4}{\theta} \right)^n$ such that for every $x \in S^{n-1}$,

$$\min_{y \in X} \|x - y\|_2 \leq \theta.$$ 

(a) Consider fixed vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^N$ with $\|y\|_2 = 1, \|x\|_2 = 1$. Use Azuma’s inequality to prove that

$$\mathbb{P}[|\langle y, Ax \rangle| > \lambda] \leq 2 \exp \left( -\frac{\lambda^2 N}{2} \right).$$

(It will help to remember the more general version: Theorem 3.1 in Lecture 7.)
(b) Show that by choosing \( c \geq 1 \) large enough (but still \( c = O(1) \)),

\[
P \left[ \exists x \in \mathbb{R}^n : \|Ax\|_2 \geq 20\|x\|_2 \right] \leq \frac{1}{100}. \tag{0.1}
\]

i. To do this, you will need to choose \( \theta \) with \( 0 < \theta < 1 \) and two subsets \( X \subseteq S^{n-1} \) and \( X' \subseteq S^{N-1} \) using Fact 0.1. Then argue that

\[
P \left[ \exists x \in X, y \in X' : \|y, Ax\| \geq 10 \right] \leq \frac{1}{100}.
\]

ii. Now to achieve (0.1), you should use a method called chaining. First, prove that if \( X \) is your set chosen above (with parameter \( \theta > 0 \)), then every \( x \in S^n \) can be written

\[
x = \sum_{k \geq 0} \alpha_k x_k
\]

where \( x_0, x_1, \ldots \in X \) and \( |\alpha_k| \leq \theta^k \). (The same holds true for \( X' \).)

iii. Use the chaining and part (i) to argue that (0.1) holds.

3. Random linear maps: Part II

To complement (0.1), we need to show that for some constant \( C \geq 1 \),

\[
P \left[ \exists x \in \mathbb{R}^n : \|Ax\|_2 \leq \frac{\|x\|_2}{C} \right] \leq \frac{1}{10}. \tag{0.2}
\]

(a) Use the second moment method to argue that for any \( x \in \mathbb{R}^n \) with \( \|x\|_2 = 1 \), if \( \varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\} \) are i.i.d. uniform signs, then

\[
P \left[ \left| \sum_{i=1}^n \varepsilon_i x_i \right| > 0.01 \right] > 0.01.
\]

[Hint: See Corollary 1.4 in Lecture 3, but instead of arguing about \( X = 0 \), think about \( |X| \leq 0.01 \).]

(b) Use a Chernoff bound together with part (a) to show that for a fixed vector \( x \in \mathbb{R}^n \),

\[
P \left[ \|Ax\|_2 \leq \frac{\|x\|_2}{C} \right] \leq \exp(-c'n).
\]

Here, \( c' \) is some constant that will depend on your choice of \( C \) and \( c \).

(c) Again choose a number \( \theta > 0 \) and an appropriate set \( X \) using Fact 0.1, along with part (b) to argue that

\[
P \left[ \exists x \in X : \|Ax\|_2 \leq \frac{\|x\|_2}{C} \right] \leq \frac{1}{100}.
\]

If you choose \( \theta \) correctly, then in conjunction with (0.1), you should be able to argue that this is enough to achieve (0.2).