Midterm CSE 525, Fall 2016
Out: Monday, November 7th
Due: Before class on Monday, November 14th

Policy: You may consult textbooks and the class lecture notes, but no other sources. You should work on these problems and write up the solutions by yourself with no help from other students.

Please feel free to ask me for clarifications if something is unclear.

1. **Partitioning metric spaces with bounded growth rate.**

   Recall in Lecture 9 that we showed how to construct random partitions of metric spaces with bounded diameter such that nearby points are not separated too often.

   Suppose that $G = (V, E)$ is a connected (unweighted) graph and $d_G(x, y)$ is the shortest-path distance between two vertices $x, y \in V$. Let $B_G(x, R) = \{y \in V : d_G(x, y) \leqslant R\}$. Suppose furthermore that for some number $k \geqslant 1$, it holds that

   $$|B_G(x, R)| \leqslant O(R^k) \qquad \forall x \in X, R \geqslant 1 .$$

   Show that for every $\Delta \geqslant 2$, there is a random partition P of V such that $\text{diam}(S) \leqslant \Delta$ for every $S \in P$, and for every $x, y \in V$,

   $$\mathbb{P}\left[x \text{ and } y \text{ are separated in } P\right] \leqslant O(k)\frac{d_G(x, y)}{\Delta} \log \Delta .$$

2. **Random linear maps: Part I**

   In this problem, you will show the following: There are constants $c, C \geqslant 1$ such that for every $n \geqslant 1$, with high probability a random map $A : \mathbb{R}^n \to \mathbb{R}^{\lceil cn \rceil}$ satisfies

   $$\frac{\|x\|_2}{C} \leqslant \|Ax\|_2 \leqslant C\|x\|_2 \qquad \forall x \in \mathbb{R}^n .$$

   Let $N = \lceil cn \rceil$. The random map A is an $N \times n$ matrix where the entries are $A_{ij} = \frac{\varepsilon_{ij}}{\sqrt{N}}$ and $\varepsilon_{ij} \in \{-1, 1\}$ is an independent, uniform random sign.

   Such maps are very useful. For instance, suppose that $y = Ax$ and $\hat{y}$ is a corruption of y that results from arbitrarily changing δn coordinates for some small constant $\delta > 0$. Then one can uniquely recover x from the corrupted vector $\hat{y}$, and moreover there is an efficient algorithm to do it. (On HW #5, you will design this recovery algorithm.)

   Fact 0.1. *You will need the following fact. Let $S^{n-1} := \{x \in R^n : \|x\|_2 = 1\}$. For every $\theta > 0$, there is a subset $X \subseteq S^{n-1}$ with $|X| \leqslant \left(\frac{4}{\theta}\right)^n$ such that for every $x \in S^{n-1}$,*

   $$\min_{y \in X} \|x - y\|_2 \leqslant \theta .$$

   (a) Consider fixed vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^N$ with $\|y\|_2 = 1, \|x\|_2 = 1$. Use Azuma's inequality to prove that

   $$\mathbb{P}\left[|\langle y, Ax\rangle| > \lambda\right] \leqslant 2 \exp\left(\frac{-\lambda^2 N}{2}\right) .$$

   (It will help to remember the more general version: Theorem 3.1 in Lecture 7.)

(b) Show that by choosing $c \geqslant 1$ large enough (but still $c = O(1)$),

$$\mathbb{P}\left[\exists x \in \mathbb{R}^n : \|Ax\|_2 \geqslant 20\|x\|_2\right] \leqslant \frac{1}{100}\,. \tag{0.1}$$

i. To do this, you will need to choose θ with $0 < \theta < 1$ and two subsets $X \subseteq S^{n-1}$ and $X' \subseteq S^{N-1}$ using Fact 0.1. Then argue that

$$\mathbb{P}\left[\exists x \in X, y \in X' : |\langle y, Ax\rangle| \geqslant 10\right] \leqslant \frac{1}{100}\,.$$

ii. Now to achieve (0.1), you should use a method called *chaining*. First, prove that if X is your set chosen above (with parameter $\theta > 0$), then every $x \in S^{n-1}$ can be written

$$x = \sum_{k \geqslant 0} \alpha_k x_k$$

where $x_0, x_1, \ldots \in X$ and $|\alpha_k| \leqslant \theta^k$. (The same holds true for X'.)

iii. Use the chaining and part (i) to argue that (0.1) holds.

3. **Random linear maps: Part II**

To complement (0.1), we need to show that for some constant $C \geqslant 1$,

$$\mathbb{P}\left[\exists x \in \mathbb{R}^n : \|Ax\|_2 \leqslant \frac{\|x\|_2}{C}\right] \leqslant \frac{1}{10}\,. \tag{0.2}$$

(a) Use the second moment method to argue that for any $x \in \mathbb{R}^n$ with $\|x\|_2 = 1$, if $\varepsilon_1, \ldots \varepsilon_n \in \{-1, 1\}$ are i.i.d. uniform signs, then

$$\mathbb{P}\left[\left|\sum_{i=1}^{n} \varepsilon_i x_i\right| > 0.01\right] > 0.01\,.$$

[Hint: See Corollary 1.4 in Lecture 3, but instead of arguing about $X = 0$, think about $|X| \leqslant 0.01$.]

(b) Use a Chernoff bound together with part (a) to show that for a fixed vector $x \in \mathbb{R}^n$,

$$\mathbb{P}\left[\|Ax\|_2 \leqslant \frac{\|x\|_2}{C}\right] \leqslant \exp(-c'n)\,.$$

Here, c' is some constant that will depend on your choice of C and c.

(c) Again choose a number $\theta > 0$ and an appropriate set X using Fact 0.1, along with part (b) to argue that

$$\mathbb{P}\left[\exists x \in X : \|Ax\|_2 \leqslant \frac{\|x\|_2}{C}\right] \leqslant \frac{1}{100}\,.$$

If you choose θ correctly, then in conjunction with (0.1), you should be able to argue that this is enough to achieve (0.2).