Homework #1 CSE 525, Fall 2016
Out: Monday, 3-Oct
Due: Before class, Monday, 10-Oct

1. **2D pattern matching.** Suppose that X is an $n \times n$ $\{0,1\}$-matrix and Y is an $m \times m$ $\{0,1\}$-matrix for $m < n$. One wants to check if Y occurs as a block submatrix of X. Describe how to change the Karp-Rabin algorithm for 1D pattern matching to handle this. Your algorithm should run in $O(n^2)$ time. You may assume that arithmetic on $O(\log n)$-bit integers can be done in constant time. Make sure you justify why the the integers arising in your algorithm are of this size. You may also assume (as we did in lecture) that it's possible to choose a uniformly random prime p from a given interval.

   [Note that you will have to use some thought to get a running time of $O(n^2)$, since it's no longer clear how to update the hashes at every step.]

2. **The probabilistic method.** Consider an undirected graph $G = (V, E)$ with $n = |V|$. An *all pairs multi-flow* in G is way of sending one unit of flow between every pair of vertices.

   More precisely: For every pair $u, v \in V$, let $\mathcal{P}_{uv}$ denote the collection of undirected, simple u-v paths in G. Let $\mathcal{P} = \bigcup_{u,v \in V} \mathcal{P}_{uv}$. An *all-pairs multi-flow* is a mapping $\Lambda : \mathcal{P} \to [0, \infty)$ such that for every $u, v \in V$ with $u \neq v$,

   $$\sum_{\gamma \in \mathcal{P}_{u,v}} \Lambda(\gamma) = 1 \,.$$

   The flow is *integral* if for every $u, v \in V$ with $u \neq v$, it holds that $\Lambda(\gamma) = 1$ for precisely one path $\gamma \in \mathcal{P}_{u,v}$.

   For $v \in V$, define the amount of flow passing through vertex v by:

   $$\Lambda[v] = \sum_{\gamma \in \mathcal{P} : v \in \gamma} \Lambda(\gamma) \,.$$

   Finally, define the *energy of the flow* Λ as

   $$\mathcal{E}(\Lambda) = \sum_{v \in V} \Lambda[v]^2 \,.$$

   (a) Use the probabilistic method to show that given any all-pairs multi-flow Λ, there exists an integral all-pairs multi-flow Λ' such that

   $$\mathcal{E}(\Lambda') \leqslant \mathcal{E}(\Lambda) + n^3 \,.$$

   You will need to use the fact that if X and Y are independent random variables, then $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

   (b) Use part (a) and the crossing number inequality (proved in Lecture 2) to show the following: There is a constant $c > 0$ such that if G is a *planar graph*, then every all-pairs multi-flow Λ in G has

   $$\mathcal{E}(\Lambda) \geqslant cn^4 \,.$$

   [Hint: You should relate an integral all-pairs multi-flow in G to a drawing of the complete graph in the plane.]

3. **The second moment method.** Consider the following simple model for a social network: There are $2n$ users in two groups A and B with $|A| = |B| = n$. For every distinct pair of users i and j, the pair $\{i, j\}$ are *friends* independently with probability p if they are in the same group, and probability q if they are in different groups. A *loner* is a user i that has no friends.

   Let $\mathcal{L}$ denote the event that there exists a loner. Prove that if $p + q \gg \frac{\ln n}{n}$, then $\mathbb{P}[\mathcal{L}] \to 0$ and if $p + q \ll \frac{\ln n}{n}$, then $\mathbb{P}[\mathcal{L}] \to 1$. [Here the notation $f(n) \gg g(n)$ means that $\lim f(n)/g(n) = \infty$.]