For these problems, consider an undirected, connected graph $G = (V, E)$. Let $\{X_t\}$ denote the random walk on $G$.

1. (a) For a vertex $u \in V$, let $\hat{T}_u = \min\{t > 0 : X_t = u\}$. Define $\hat{H}_{u,u} = \mathbb{E}[\hat{T}_u | X_0 = u]$. Prove that
   \[\hat{H}_{u,u} = \frac{2 |E|}{\deg_G(u)}.\]
   (b) Prove that for any tree $T$ on $n$ vertices, it holds that the cover time of $T$ is at least $\Omega(n \log n)$. You can use Theorem 1.9 from Lecture 13.
   (c) If $P_n$ is the path graph on $\{1, 2, \ldots, n\}$, derive a formula for $H_{ij}$ for any $i, j \in \{1, 2, \ldots, n\}$.

2. (a) Recall from class that $R_{\text{eff}}(u, v) = \min_{F} \mathcal{E}(F)$ where the minimum is over all unit flows $F$ from $u$ to $v$ and
   \[\mathcal{E}(F) = \sum_{e \in E} F(e)^2.\]
   Use this to prove the following: For any two vertices $u, v \in V$, if $S_1, S_2, \ldots, S_k$ are disjoint sets of edges that separate $u$ and $v$ in $G$, then
   \[R_{\text{eff}}(u, v) \geq \sum_{i=1}^{k} \frac{1}{|S_i|}.\]
   (b) Let $G_n$ be the the $n \times n$ grid graph, i.e., the vertices are $V = \{1, 2, \ldots, n\}^2$ and there is an edge between $(a, b)$ and $(a', b')$ if $|a - a'| + |b - b'| = 1$. Use part (a) and Theorem 1.9 from Lecture 13 to show that the cover time of $G_n$ is $\Omega(n^2 (\log n)^2)$.