

Out: 16-Nov

Due: Before class, November 23rd

For these problems, consider an undirected, connected graph $G = (V, E)$. Let $\{X_t\}$ denote the random walk on G .

1. (a) For a vertex $u \in V$, let $\hat{T}_u = \min\{t > 0 : X_t = u\}$. Define $\hat{H}_{u,u} = \mathbb{E}[\hat{T}_u \mid X_0 = u]$. Prove that

$$\hat{H}_{u,u} = \frac{2|E|}{\deg_G(u)}.$$

- (b) Prove that for any tree T on n vertices, it holds that the cover time of T is at least $\Omega(n \log n)$. You can use Theorem 1.9 from Lecture 13.
- (c) If P_n is the path graph on $\{1, 2, \dots, n\}$, derive a formula for H_{ij} for any $i, j \in \{1, 2, \dots, n\}$.
2. (a) Recall from class that $R_{\text{eff}}(u, v) = \min_F \mathcal{E}(F)$ where the minimum is over all unit flows F from u to v and

$$\mathcal{E}(F) = \sum_{e \in E} F(e)^2.$$

Use this to prove the following: For any two vertices $u, v \in V$, if S_1, S_2, \dots, S_k are *disjoint* sets of edges that separate u and v in G , then

$$R_{\text{eff}}(u, v) \geq \sum_{i=1}^k \frac{1}{|S_i|}.$$

- (b) Let G_n be the $n \times n$ grid graph, i.e., the vertices are $V = \{1, 2, \dots, n\}^2$ and there is an edge between (a, b) and (a', b') if $|a - a'| + |b - b'| = 1$. Use part (a) and Theorem 1.9 from Lecture 13 to show that the cover time of G_n is $\Omega(n^2(\log n)^2)$.