

1 The curse (and blessing) of dimensionality

The “curse” of dimensionality refers to the fact that many algorithmic approaches to problems in \mathbb{R}^k become exponentially more difficult as k grows. This is essentially due to volume growth: The k -dimensional Euclidean ball of radius r grows exponentially in k : $\text{vol}_k(B(0, 2r)) = 2^k \text{vol}_k(B(0, r))$.

On the other hand, in high dimensions, the concentration of measure phenomenon can come to our aid: Sufficiently smooth functionals on \mathbb{R}^k are tightly concentrated around their expected value. A prototypical example is the Johnson-Lindenstrauss dimension reduction lemma.

1.1 Dimension reduction

Lemma 1.1. *For every $n \geq 1$ and every n -point subset $X \subseteq \mathbb{R}^n$, the following holds. For every $\varepsilon > 0$, there is a linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^k$ such that*

$$(1 - \varepsilon)^2 \|x - y\|_2^2 \leq \|A(x) - A(y)\|_2^2 \leq (1 + \varepsilon)^2 \|x - y\|_2^2 \quad \forall x, y \in X, \quad (1.1)$$

and $k \leq \frac{24 \ln n}{\varepsilon^2}$.

Proof. We may assume that $0 < \varepsilon < 1/2$. We will define random linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^k$ and then argue that A satisfies (1.1) with high probability. Let $\{X_i^{(j)} : i = 1, \dots, k, j = 1, \dots, n\}$ be a family of i.i.d. $N(0, 1)$ random variables, and define an $k \times n$ matrix A by $A_{ij} = \frac{1}{\sqrt{k}} X_i^{(j)}$.

Claim 1.2. For every $u \in \mathbb{R}^n$ with $\|u\|_2 = 1$, we have

$$\mathbb{P} [\|Au\|_2^2 \notin [1 - \varepsilon, 1 + \varepsilon]] \leq 2e^{-\varepsilon k/8}.$$

Let us finish the proof of Lemma 1.1 and then prove the claim. By setting $u = \frac{x-y}{\|x-y\|_2}$ for every $x, y \in X$ and taking a union bound over these n^2 different pairs, we have

$$\mathbb{P} ((1 - \varepsilon)\|x - y\|_2^2 \leq \|A(x - y)\|_2^2 \leq (1 + \varepsilon)\|x - y\|_2^2) \geq 1 - 2n^2 e^{-\varepsilon k/8}.$$

Taking $k = \frac{24 \ln n}{\varepsilon^2}$ the latter probability is at least $1 - \frac{2}{n}$, showing that the desired map A exists (and can be found by a randomized algorithm).

Proof of Claim 1.2. Note first that

$$\|Au\|_2^2 = \frac{1}{k} \sum_{i=1}^k \left(\sum_{j=1}^n u_j X_i^{(j)} \right)^2.$$

By the 2-stability property of independent $N(0, 1)$ random variables, we have $\sum_{j=1}^n u_j X_i^{(j)}$ again has distribution $N(0, 1)$. Thus we have

$$\|Au\|_2^2 = \frac{1}{k} \sum_{i=1}^k Y_i^2$$

where $\{Y_i\}$ are i.i.d. $N(0, 1)$.

Since $\|Au\|_2^2$ is a sum of independent random variables, it makes sense to use the method of Laplace transforms. So far we have only done this for bounded random variables, but since $N(0, 1)$ variables have a quickly diminishing tail, we can hope that the method will work here as well.

For some parameter $\lambda > 0$, we have

$$\begin{aligned} \mathbb{P}(\|Au\|_2^2 > 1 + \varepsilon) &= \mathbb{P}\left(\sum_{i=1}^k (Y_i^2 - 1) > \varepsilon k\right) \\ &= \mathbb{P}\left(e^{\lambda \sum_{i=1}^k (Y_i^2 - 1)} > e^{\varepsilon \lambda k}\right) \\ &\leq \frac{\mathbb{E}\left[e^{\lambda \sum_{i=1}^k (Y_i^2 - 1)}\right]}{e^{\varepsilon \lambda k}} \\ &= \frac{\prod_{i=1}^k \mathbb{E}\left[e^{\lambda(Y_i^2 - 1)}\right]}{e^{\varepsilon \lambda k}}. \end{aligned}$$

Thus our remaining goal is to bound $\mathbb{E}\left[e^{\lambda(Y^2 - 1)}\right]$ when Y is $N(0, 1)$. First, observe that

$$\mathbb{E}\left[e^{\lambda Y^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} e^{\lambda t^2} dt = \frac{1}{\sqrt{1 - 2\lambda}} \quad (1.2)$$

for $0 < \lambda < 1/2$. (Clearly the integral is divergent for $\lambda \geq 1/2$.) Thus in this range of λ ,

$$\mathbb{E}\left[e^{\lambda(Y^2 - 1)}\right] = \frac{e^{-\lambda}}{\sqrt{1 - 2\lambda}}.$$

Let us finally calculate (using the Taylor expansion $\log(1 - x) = -\sum_{k \geq 1} \frac{x^k}{k}$),

$$\begin{aligned} \left| \log \mathbb{E}\left[e^{\lambda(Y^2 - 1)}\right] \right| &= \left| -\lambda - \frac{1}{2} \log(1 - 2\lambda) \right| \\ &\leq 2\lambda^2(1 + (2\lambda) + (2\lambda)^2 + \dots) \\ &= \frac{2\lambda^2}{1 - 2\lambda}. \end{aligned}$$

Therefore:

$$\mathbb{P}(\|Au\|_2^2 > 1 + \varepsilon) \leq e^{2\lambda^2 k / (1 - 2\lambda) - \varepsilon \lambda k}.$$

Choosing $\lambda = \varepsilon/4$ and using $0 < \varepsilon < 1/2$, we conclude that $\mathbb{P}(\|Au\|_2^2 > 1 + \varepsilon) \leq e^{-\varepsilon k/8}$, completing the proof. A similar argument shows that $\mathbb{P}(\|Au\|_2^2 < 1 - \varepsilon) \leq e^{-\varepsilon k/8}$. \square

\square

Remark 1.3. A simple volume bound shows that that $\Theta(\log n)$ dependence in the dimension is necessary. A linear algebraic argument of Alon shows that the dimension must be at least $\Omega\left(\frac{\log n}{\varepsilon^2 \log(1/\varepsilon)}\right)$. It is unknown whether the upper or lower bound (which differ by a factor of $\log(1/\varepsilon)$) is the right answer.

Remark 1.4. Lemma 1.1 actually works for any independent family $X_i^{(j)}$ where each random variable satisfies a sub-Gaussian tail bound: $\mathbb{E}[e^{\alpha X}] \leq e^{C\alpha^2}$ for some constant $C \geq 1$. For instance, if X is a uniform ± 1 random variable, then $\mathbb{E}[e^{\alpha X}] = \frac{1}{2}(e^\alpha + e^{-\alpha}) \leq e^{\alpha^2/2}$ (recall that to prove this, you should Taylor expand both sides).

The proof uses a very clever trick. Note that if Z is a $N(0, 1)$ random variable (independent of X) then $\mathbb{E}[e^{\alpha Z}] = e^{\alpha^2/2}$ for any $\alpha > 0$. Now write

$$\mathbb{E}[e^{\lambda X^2}] = \mathbb{E}[e^{(\sqrt{2\lambda X})^2/2}] = \mathbb{E}[e^{\sqrt{2\lambda} Z X}] \leq \mathbb{E}[e^{2C\lambda Z^2}] = \frac{1}{\sqrt{1 - 2\lambda C}},$$

for $0 < \lambda < \frac{1}{2C}$, where the last equality is exactly the equality we proved in (1.2). Given this bound, we can finish the proof just as in [Claim 1.2](#).