Homework #1: Pinching CSE 599I, Spring 2021
Due: Apr 26, 2021

1 Problem 1: Spectral pinching

For a Hermitian matrix $A \in \mathbb{M}_n(\mathbb{C})$, recall that spec$(A)$ is the (finite) set of eigenvalues of A. For each $\lambda \in \text{spec}(A)$, denote the eigenspace $V_\lambda = \{v \in \mathbb{C}^n : Av = \lambda v\}$. Let $P_\lambda : \mathbb{C}^n \to V_\lambda$ be the orthogonal projection onto V_λ so that

$$\sum_{\lambda \in \text{spec}(A)} P_\lambda = I.$$

The *spectral pinching operator* (with respect to A) is the mapping

$$\mathcal{P}_A : X \mapsto \sum_{\lambda \in \text{spec}(A)} P_\lambda X P_\lambda.$$

(a) Prove that A commutes with $\mathcal{P}_A(X)$: $A\mathcal{P}_A(X) = \mathcal{P}_A(X)A$ for all Hermitian $A, X \in \mathbb{M}_n(\mathbb{C})$.

(b) Prove that $\text{Tr}\,(A\mathcal{P}_A(X)) = \text{Tr}\,(AX)$ for all Hermitian $A, X \in \mathbb{M}_n(\mathbb{C})$.

2 Problem 2: Pinching as averaging

Let $\omega_n := e^{2\pi i/n}$ denote the nth root of unity, and define $U := \text{diag}(1, \omega_n, \omega_n^2, \cdots, \omega_n^{n-1})$. For a matrix $X \in \mathbb{M}_n(\mathbb{C})$, let $\mathcal{D}(X) \in \mathbb{M}_n(\mathbb{C})$ denote the diagonal matrix with $\mathcal{D}(X)_{jj} = X_{jj}$.

(a) Suppose $A \in \mathbb{M}_n(\mathbb{C})$ is a diagonal matrix with $|\text{spec}(A)| = n$. Show that

$$\mathcal{P}_A(X) = \mathcal{D}(X) = \frac{1}{n}\sum_{j=1}^{n} U^j X U^{*j}.$$

(b) Consider now a Hermitian $A \in \mathbb{M}_n(\mathbb{C})$ with $\text{spec}(A) = \{\lambda_0, \lambda_1, \ldots, \lambda_{m-1}\}$. Define the matrices

$$U_k := \sum_{j=0}^{m-1} \omega_m^{kj} P_{\lambda_j},$$

where P_{λ_j} denotes the projection operator as in Problem 1. Prove that

$$\mathcal{P}_A(X) = \frac{1}{m}\sum_{k=0}^{m-1} U_k X U_k^*.$$

(c) Argue that for $X \geq 0$, we have

$$\mathcal{P}_A(X) \geq \frac{1}{|\text{spec}(A)|} X. \tag{2.1}$$

[Hint: Prove that $\mathcal{P}_A(X) \geq \frac{1}{m} U_0 X U_0^*$.]

3 Problem 3: Tensor products

For $A \in \mathbb{M}_n(\mathbb{C})$, denote the k-fold tensor product $A^{\otimes k} := A \otimes A \otimes \cdots \otimes A$. Note that if $u_1, u_2, \ldots, u_k$ are (not necessarily distinct) eigenvectors of A with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then

$$A^{\otimes k}(u_1 \otimes u_2 \otimes \cdots \otimes u_k) = \lambda_1 \lambda_2 \cdots \lambda_k (u_1 \otimes u_2 \otimes \cdots \otimes u_k).$$

(a) Show that such products $\prod_{j=1}^{k} \lambda_j$ are the *only* eigenvalues of $A^{\otimes m}$. (Hint: Find a complete basis of eigenvectors for $A^{\otimes k}$.)

(b) Prove that

$$|\mathrm{spec}(A^{\otimes k})| \leqslant (k+1)^n.$$

(c) Show that for all $X, Y \in \mathbb{M}_n(\mathbb{C})$, we have

$$\mathrm{Tr}(X \otimes Y) = \mathrm{Tr}(X)\mathrm{Tr}(Y).$$

(d) Show that for all Hermitian $X, Y \in \mathbb{M}_n(\mathbb{C})$,

$$e^{X \otimes I + I \otimes Y} = e^X \otimes e^Y,$$

and if additionally $X, Y \succ 0$, then

$$\log(X \otimes Y) = (\log(X) \otimes I) + (I \otimes \log(Y)).$$

4 Problem 4: Golden-Thompson

Consider now two positive definite matrices $A, B \in \mathbb{M}_n(\mathbb{C})$.

(a) Show that for all $k \geqslant 1$,

$$\log \mathrm{Tr}\left(\exp(\log A + \log B)\right) = \frac{1}{k} \log \mathrm{Tr}\left(\exp\left(\log(A^{\otimes k}) + \log(B^{\otimes k})\right)\right).$$

(b) Use Problem 2(c) and Problem 3(b) to show that

$$\log(A^{\otimes k}) \preceq \log\left(\mathcal{P}_{B^{\otimes k}}(A^{\otimes k})\right) + n \log(k+1) \cdot I.$$

(c) Combine parts (a) and (b) and take $k \to \infty$ to conclude that

$$\log \mathrm{Tr}\left(\exp(\log A + \log B)\right) \leqslant \log \mathrm{Tr}\left(AB\right).$$

You will need to use the properties of the spectral pinching established in Problem 1.

(d) Use this to prove the Golden-Thompson inequality: For all Hermitian marices $K, L \in \mathbb{M}_n(\mathbb{C})$, it holds that $\mathrm{Tr}(e^{K+L}) \leqslant \mathrm{Tr}(e^K e^L)$.