Problem 1: Strong subadditivity of the quantum entropy

We have established that the quantum relative entropy is monotone decreasing under partial trace: For bipartite states $\rho, \sigma \in \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, it holds that

$$\mathsf{S}(\mathrm{Tr}_{\mathcal{H}_1}(\rho) \,\|\, \mathrm{Tr}_{\mathcal{H}_1}(\sigma)) \leqslant \mathsf{S}(\rho \,\|\, \sigma). \tag{0.1}$$

Let $\rho^{ABC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ be a tripartite state, and denote the partial traces

$$\rho^{AB} = \mathrm{Tr}_C(\rho^{ABC}), \;\; \rho^{BC} = \mathrm{Tr}_A(\rho^{ABC}), \;\; \rho^{A} = \mathrm{Tr}_{BC}(\rho^{ABC}), \;\; \rho^{B} = \mathrm{Tr}_{AC}(\rho^{ABC}).$$

Using (0.1) with the states $\rho = \rho^{ABC}$ and $\sigma = \rho^A \otimes \rho^{BC}$, prove that strong subadditivity holds:

$$\mathcal{S}(\rho^{ABC}) + \mathcal{S}(\rho^{B}) \leqslant \mathcal{S}(\rho^{AB}) + \mathcal{S}(\rho^{BC}), \tag{0.2}$$

where $\mathcal{S}$ denotes the von Neumann entropy.

Remark 0.1. Recall from Lecture 9 that (0.2) can be interpreted as establishing nonnegativity of the quantum conditional mutual information:

$$\mathbf{I}(A, C \mid B)_\rho := \mathcal{S}(\rho^{AB}) + \mathcal{S}(\rho^{BC}) - \mathcal{S}(\rho^{ABC}) - \mathcal{S}(\rho^{B}).$$

Problem 2: Monotonicity of relative entropy under quantum channels

Suppose that $\mathcal{E} : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a quantum channel (i.e., a CPT map). Use Theorem 1.3 in Lecture 10, along with (0.1), to prove that for all $\rho, \sigma \in \mathcal{D}(\mathbb{M}_n(\mathbb{C}))$,

$$\mathsf{S}(\mathcal{E}(\rho) \,\|\, \mathcal{E}(\sigma)) \leqslant \mathsf{S}(\rho \,\|\, \sigma).$$

Problem 3: Quantum Pinsker inequality

Here you will prove the quantum Pinsker inequality, that for any density matrices ρ, σ, we have

$$\mathsf{S}(\rho \,\|\, \sigma) \geqslant \frac{1}{2}\|\rho - \sigma\|_1^2, \tag{0.3}$$

where $\|\cdot\|_1$ is the Schatten 1-norm:

$$\|A\|_1 = \mathrm{Tr}(|A|) = \mathrm{Tr}(\sqrt{A^*A}).$$

You can assume the classical Pinsker inequality for probability distributions p, q which asserts that

$$\mathsf{D}(p \,\|\, q) \geqslant \frac{1}{2}\|p - q\|_1^2,$$

and here $\|\cdot\|_1$ is the standard ℓ_1 norm.

1. Prove that for any density matrices ρ and σ, it holds that $\mathsf{S}(D_\rho \,\|\, D_\sigma) \leqslant \mathsf{S}(\rho \,\|\, \sigma)$, where D_ρ and D_σ are ρ and σ with the off-diagonal entries zeroed out. [Hint: It may help to recall Problem 2(a) on the first homework.]

2. Prove (0.3). [Hint: First reduce to the case when $\rho - \sigma$ is diagonal.]

Problem 4: Squashed entanglement

Denote a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_C$. For a given state $\rho^{AC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_C)$, consider all extensions $\rho^{ABC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ for some auxiliary Hilbert space $\mathcal{H}_B$ such that $\rho^{AC} = \mathrm{Tr}_B(\rho^{ABC})$. Define the *squashed entanglement of* ρ^{AC} by

$$\mathbf{E}^{\mathrm{sq}}(\rho^{AC}) := \frac{1}{2} \inf_{\rho^{ABC}} \left(\mathcal{S}(\rho^{AB}) + \mathcal{S}(\rho^{BC}) - \mathcal{S}(\rho^{ABC}) - \mathcal{S}(\rho^{B}) \right),$$

where the infimum is over extensions ρ^{ABC} of ρ^{AC} (in particular, over all finite-dimensional Hilbert spaces $\mathcal{H}_B$). Note that this quantity is nonnegative by Problem 1.

For intuition, one can use the quantum conditional mutual information (introduced in Lecture 9) to recast this as

$$\mathbf{E}^{\mathrm{sq}}(\rho^{AC}) := \frac{1}{2} \inf_{\rho} \mathbf{I}(A, C \mid B)_{\rho},$$

where the infimum is again over all extensions ρ of ρ^{AC}. Thus this captures the amount of "mutual quantum information" across the A-C partition that is not explained by some larger environment.

1. Show that if ρ^{AC} is pure, then $\mathbf{E}^{\mathrm{sq}}(\rho^{AC}) = \mathcal{S}(\rho^{A})$.

2. An entanglement measure $\mathbf{E}$ is called *faithful* if $\mathbf{E}(\rho^{AC}) = 0$ if and only if ρ^{AC} is separable (across the A-C partition). Show one direction (the easier one) for squashed entanglement: If ρ^{AC} is separable, then $\mathbf{E}^{\mathrm{sq}}(\rho^{AC}) = 0$.

   [Remark: It is also true that if ρ^{AC} is not separable, then $\mathbf{E}^{\mathrm{sq}}(\rho^{AC}) > 0$, but this is more difficult, and it was an open question for a while.]

3. Show that $\mathbf{E}^{\mathrm{sq}}$ is additive in the sense that

   $$\mathbf{E}^{\mathrm{sq}}(\rho^{AC} \otimes \rho^{A'C'}) = \mathbf{E}^{\mathrm{sq}}(\rho^{AC}) + \mathbf{E}^{\mathrm{sq}}(\rho^{A'C'}),$$

   where we think of $\rho^{AC} \otimes \rho^{A'C'}$ as a bipartite state in $\mathcal{D}\big((\mathcal{H}_A \otimes \mathcal{H}_{A'}) \otimes (\mathcal{H}_C \otimes \mathcal{H}_{C'})\big)$.

   For guidance, the proof can be broken into parts:

   (a) Prove that $\mathbf{E}^{\mathrm{sq}}(\rho^{AC} \otimes \rho^{A'C'}) \leqslant \mathbf{E}^{\mathrm{sq}}(\rho^{AC}) + \mathbf{E}^{\mathrm{sq}}(\rho^{A'C'})$.

   (b) Prove the following chain rule: For any $\sigma^{XYZU} \in \mathcal{D}(\mathcal{H}_X \otimes \mathcal{H}_Y \otimes \mathcal{H}_Z \otimes \mathcal{H}_U)$,

   $$\mathbf{I}(XY, Z \mid U)_{\sigma} = \mathbf{I}(X, Z \mid U)_{\sigma} + \mathbf{I}(Y, Z \mid UX)_{\sigma}.$$

   (c) Use the chain rule, along with nonnegativity of the quantum conditional mutual information to show that for *any* state $\rho^{AA'CC'} \in \mathcal{D}((\mathcal{H}_A \otimes \mathcal{H}_{A'}) \otimes (\mathcal{H}_C \otimes \mathcal{H}_{C'}))$, we have

   $$\mathbf{E}^{\mathrm{sq}}(\rho^{AA'CC'}) \geqslant \mathbf{E}^{\mathrm{sq}}(\rho^{AC}) + \mathbf{E}^{\mathrm{sq}}(\rho^{A'C'}).$$