- Some homework
- Format: Case study

Classical/scalar phenomenon $\Longleftrightarrow$ PSD analog

$n \times n$ real matrices $\quad M_n = M_n(\mathbb{R})$

$A \in M_n$ symmetric $\quad A = A^T$

A is positive definite if A is symmetric and

$$\langle x, Ax \rangle > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$$

$\langle u, v \rangle = u_1 v_1 + \dots + u_n v_n$

$\to$ PSD if $\langle x, Ax \rangle \geq 0 \quad \forall x \in \mathbb{R}^n$

Real symmetric matrices can be diagonalized

$$\boxed{A = UDU^T} \qquad UU^T = U^T U = I$$

$D = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \qquad D$ diagonal

$$Au_i = \lambda_i u_i \qquad U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}$$

A symmetric $\Longrightarrow \boxed{A \text{ is PSD} \Longleftrightarrow \lambda_1, \lambda_2, \dots, \lambda_n \geq 0}$

$$M_n^+(\mathbb{R}) := \{A \in M_n(\mathbb{R}) : A \text{ is PSD}\}$$

closed convex cone

$A, B \in M_n^+(\mathbb{R}) \Longrightarrow A + cB \in M_n^+(\mathbb{R}) \quad \forall c \geq 0.$

$$\langle x, (A + cB)x \rangle = \langle x, Ax \rangle + c\langle x, Bx \rangle$$

$A = \sum \lambda_i u_i u_i^T \qquad M_n^+(\mathbb{R}) = \mathrm{cone}(uu^T : u \in \mathbb{R}^n)$

rank-1 PSD matrix $\qquad a+bi \to a-bi$

$A \in M_n(\mathbb{C})$ is Hermitian if $A = A^*$

[$\langle x, Ax \rangle = \langle A^T x, x \rangle$ symmetric $\Leftrightarrow$ self-adjoint]

$\langle x, y \rangle_{\mathbb{C}^n} = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n$

$\langle x, Ay \rangle_{\mathbb{C}^n} = \langle A^* x, y \rangle$ Hermitian $\Leftrightarrow$ self-adjoint

$A \in M_n(\mathbb{C})$ is PSD if $\langle x, Ax \rangle_{\mathbb{C}^n} \geq 0 \quad \forall x \in \mathbb{C}^n$

A Hermitian:

$A = UDU^* \qquad D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

$\lambda_i \in \mathbb{R}$

$\underbrace{UU^* = U^*U = I}_{\text{unitary}}$

Quantum information theory:

classical $\longleftrightarrow$ quantum

probability distribution $\qquad$ density matrix $A \in M_n(\mathbb{C})$

A PSD, $\mathrm{Tr}(A) = 1$

$\{1, 2, \dots, n\} \quad \sum_{i=1}^{n} p_i = 1$

$p \in \mathbb{R}^n \longrightarrow \boxed{A} = \mathrm{diag}(p)$

POVM: $\rho_1 + \rho_2 + \cdots + \rho_m = I$

$\rho_1, \ldots, \rho_m$ PSD matrices

$\mathbb{P}[\text{outcome } i] = \mathrm{Tr}(\rho_i A)$

\succeq $A \succ 0 \iff A$ is pos. def.

Loewner order: $A \succeq 0 \iff A$ is PSD

$A \succeq B \iff A - B \succeq 0$

$\iff \langle x, Ax \rangle \geq \langle x, Bx \rangle \quad \forall x \in \mathbb{R}^n$

Gram matrix: $X \in M_{mn}(\mathbb{R})$ $m \times n$ real matrix

$A = X^T X \in M_n(\mathbb{R})$

$x_1, \ldots, x_n \in \mathbb{R}^m$: $A_{ij} = \langle x_i, x_j \rangle \quad X = \begin{pmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{pmatrix}$

✓ $\langle u, Au \rangle = \langle u, X^T X u \rangle = \langle Xu, Xu \rangle = \|Xu\|_2^2 \geq 0$

A PSD $\Rightarrow A = \sum_i \lambda_i u_i u_i^T \quad \lambda_i \geq 0 \quad \{u_i\}$ orthonormal

$\sqrt{A} := \sum_i \sqrt{\lambda_i}\, u_i u_i^T \qquad \sqrt{A} \cdot \sqrt{A} = \sum_i \lambda_i u_i u_i^T = A$

$A_{ij} = \langle \tilde{x}_i, \tilde{x}_j \rangle$

$A = \sqrt{A}^T \cdot \sqrt{A}$

$X^T \cdot X$

$m \gg n$

$\sqrt{A} = \begin{pmatrix} | & & | \\ \tilde{x}_1 & \cdots & \tilde{x}_n \\ | & & | \end{pmatrix}$

Given $x_1, \dots, x_n \in \mathbb{R}^m$, then $\exists\ \tilde{x}_1, \dots, \tilde{x}_n \in \mathbb{R}^n$

s.t. $\langle \tilde{x}_i, \tilde{x}_j \rangle = \langle x_i, x_j \rangle \quad \forall i,j$

<u>Ex:</u> If $\alpha_1, \alpha_2, \dots, \alpha_n > 0$ and $A \in M_n$ is def by

$$A_{ij} = \frac{1}{\alpha_i + \alpha_j}$$

Then A is PSD.

$$\frac{1}{\alpha_i + \alpha_j} = \int_0^\infty e^{-t(\alpha_i + \alpha_j)}\, dt = \langle f_i, f_j \rangle_{L^2([0,\infty))}$$

$$\langle f_i, f_j \rangle_{L^2([0,\infty))} = \int_0^\infty f_i(t) f_j(t)\, dt$$

$f_i(t) := e^{-\alpha_i t}$

$f_i : [0,\infty) \to \mathbb{R}$

$$\boxed{A_{S,T} = p^{|S \cup T|}}$$

$S, T \subseteq \{1, \dots, n\}$

$p \in [0,1]$

Claim: $A \succeq 0$.

$$A_{S,T} = \underbrace{\mathbb{E}[Y_S \cdot Y_T]}$$

$Y_S := \prod_{i \in S} X_i$

$$\mathbb{E}\Big[\prod_{i \in S \cup T} X_i\Big] = p^{|S \cup T|}$$

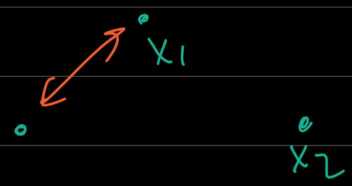
$\{X_i\}$ iid $\{0,1\}$ r.v.

$\mathbb{P}[X_i = 1] = p$

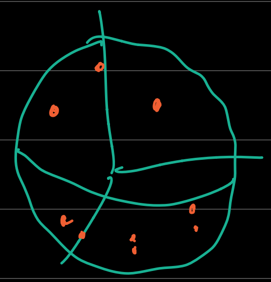
Consider $x_1, x_2, \dots, x_n \in \mathbb{R}^m$. There exist $y_1, y_2, \dots, y_n \in \mathbb{R}^n$ s.t. $\boxed{\|y_1\|_2 = \dots = \|y_n\|_2 = 1}$ $\leftarrow$

$$C^{-1}\min(\|x_i-x_j\|_2, 1) \le \|y_i - y_j\|_2 \le C \underbrace{\min(\|x_i-x_j\|_2, 1)} \quad \forall i,j$$

$$\left(C = \sqrt{\frac{e}{e-1}} \right)$$



$\longmapsto$



$f : \mathbb{R}^n \to \mathbb{C}$

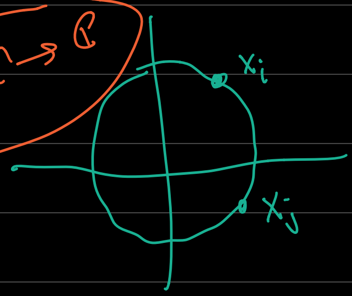
$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, d\gamma_n(x)$$

$Y_j \to y_j \in \mathbb{R}^n$

$$\boxed{Y_j} = \exp(i\langle x_j, g\rangle) \qquad g = (g_1, \dots, g_n)$$

Standard Gaussian (iid $N(0,1)$)

$f_j : \mathbb{R}^n \to \mathbb{C}$



$$\underbrace{(\mathbb{E}|Y_j|)^{1/2}} = 1 \qquad \forall j$$

$$\mathbb{E}|Y_i - Y_j|^2 = \mathbb{E}\left|\exp(i\langle x_i - x_j, g\rangle) - 1\right|^2$$

$$\langle x_i - x_j, g\rangle \overset{\text{law}}{\sim} \|x_i - x_j\|_2 \cdot g_1$$

$$= \mathbb{E}\left| \exp(i \|x_i - x_j\| g_1) - 1 \right|^2$$

$$= 2\,\mathbb{E}\,\underbrace{\left| \cos(\|x_i - x_j\|_2 \cdot g_1) - 1 \right|^2}_{\|x_i - x_j\|^2}$$

$$\cos(\varepsilon) = 1 - \varepsilon + O(\varepsilon^2)$$

$A, B \succeq 0$

AB symmetric $\iff AB = BA$

$\iff A, B$ simul. diag.

Hadamard product (aka Schur product) $\quad (A \circ B)_{ij} := A_{ij} B_{ij}$

$T : \mathbb{R}^n \to \mathbb{R}^n$ linear map $\iff A \in M_n(\mathbb{R})$

$$\boxed{T_A + T_B = T_{A+B}}$$

$$\langle e_j, T_A(e_i) \rangle = A_{ij}$$

$$\boxed{T_A \circ T_B = T_{AB}}$$

<u>Fact:</u> $A, B \succeq 0$, then $A \circ B \succeq 0$

$$A = \sum_i a_i u_i u_i^T$$

$$B = \sum_i b_i v_i v_i^T$$

$A, B \succeq 0$, then $\boxed{A \otimes B \succeq 0}$

$$(A \circ B)_{ij} = (A \otimes B)_{ij,\,ij}$$

$\left| \int_0 A^t B A^{1-t} \, dt \right|$ $\left| \right| \begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \left| \right|$ ab AB

$A, B \succeq 0$ $\boxed{\frac{1}{2}(AB + BA)}$ not necc. psd. $B^{1/2} A B^{1/2}$

Fact: If $A \succ 0$ and $\boxed{\frac{1}{2}(AB + BA) \succeq 0}$ then $B \succeq 0$.

$A = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$ $\alpha_i > 0 \;\; \forall i$

$$\frac{1}{2}(AB + BA) = B \circ \left[\left[\frac{\alpha_i + \alpha_j}{2} \right]\right]$$

$$B = \frac{1}{2}(AB + BA) \circ \left[\left[\frac{2}{\alpha_i + \alpha_j} \right]\right] \succeq 0$$

$A, B \succ 0$

$\rightarrow$ $\boxed{\text{Claim: } A \succeq B \implies A^{1/2} \succeq B^{1/2}}$ α

matrix sqrt is "operator monotone"

$$a^2 - b^2 = (a-b)(a+b)$$

$$X^2 - Y^2 = \frac{1}{2}\left[(X-Y)(X+Y) + (X+Y)(X-Y) \right]$$

If $(X^2 \succeq Y^2$ and $X + Y \succ 0 \implies X - Y \succeq 0$

$A \succeq B$ $\sqrt{A} + \sqrt{B} \succ 0 \implies X \succeq Y$

$X = \sqrt{A}$ $\sqrt{A} \succeq \sqrt{B}$

$Y = \sqrt{B}$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Not true that $A \succeq B \implies A^2 \succeq B^2$

$$\boxed{\begin{gathered} 0 \le p \le 1, \\ A, B \succeq 0 \quad A \succeq B \\ \implies A^p \succeq B^p \end{gathered}}$$

$\boxed{A_{S,T} = p^{|S \cup T|}}$ $\quad S, T \subseteq \{1, \dots, n\}$

$p \in [0,1]$

Claim: $A \succeq 0$.

$\sum_{\omega \in \{0,1\}^n} c_\omega \, y_S(\omega) \, y_T(\omega)$

$A_{S,T} = \mathbb{E}[Y_S \cdot Y_T]$ $\quad Y_S := \prod_{i \in S} X_i$

$\{X_i\}$ iid $\{0,1\}$ r.v.

$$\mathbb{E}\Big[\prod_{i \in S \cup T} X_i\Big] = p^{|S \cup T|} \qquad \mathbb{P}[X_i = 1] = p$$

$\Omega = \{0,1\}^n$ — n p-biased coins

Let (Ω, μ) be a prob space. $F : \Omega \to \mathbb{R}$

Let $\mathcal{J}$ the family of r.v.s on (Ω, μ)

s.t. $\mathbb{E}|X|^2 < \infty$

$\langle X, Y \rangle = \mathbb{E}[XY]$ $\quad (\mathcal{J}, \langle \cdot, \cdot \rangle)$

$$\int_\Omega f_S(\omega) f_T(\omega) \, d\mu(\omega)$$

$\omega \in \{0,1\}^n$

$y_S(\underline{\omega}) =$

$$\langle f, g \rangle_{L^2([0,1])} = \int_0^1 f(t) g(t) \, dt$$

$\int |f|^2 < \infty$

$(A \otimes B)_{ij,ij}$ $\cancel{X}$ BS

$A \otimes I$

$$(A \otimes B)_{ikjl} = A_{ij} B_{kl}$$

$$(A \otimes B)_{ij} = \boxed{(A \otimes B)_{(i,i),(j,j)} = A_{ij} B_{ij}}$$

$$\| $$

$$A_{ij} B_{ij}$$

$ii \; jj$

$\to A \in M_m \qquad A \otimes B \in \boxed{M_{mn}}$

$\to B \in M_n$

(i,j)

$$\langle e_i \otimes e_j, \; (A \otimes B)(e_k \otimes e_\ell) \rangle$$

$ijk\ell$

$\mathbb{R}^m \qquad \mathbb{R}^n \qquad \mathbb{R}^{mn}$

$e_1, \dots, e_m \qquad e_1, \dots, e_n \qquad e_i \otimes e_j$

$$v = \sum_{i,j} v_{ij} (e_i \otimes e_j) \qquad v_{ij} = \langle v, e_i \otimes e_j \rangle$$

$i \in [m]$

$j \in [n]$

$$(A \otimes B)_{(i,j),(i',j')} = A_{ii'} B_{jj'}$$