Noncommutative averaging: $\tilde{\Phi}$

$(p_1, p_2) \in \mathbb{R}^{n+n'}$ $\qquad \tilde{\Phi}(p) = (\Phi(p_1), \overline{p_2})$

Classical setting: $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ linear map that sends prob. distr on $[n] \to$ prob. on $[k]$.

→ (i) $p \in \mathbb{R}^n_+ \Rightarrow \Phi(p) \in \mathbb{R}^k_+$ ✓

(ii) $\sum_j p_j = \sum_j (\Phi(p))_j$ ✓

Random map $Y : [n] \to [k]$.

$X \xrightarrow{\Phi} Y(X)$

r.v. on $[n]$

operational def

"physical"

$\boxed{\mathcal{E}(AA^*) \succeq \mathcal{E}(A)\mathcal{E}(A^*)}$

$\mathcal{E}$ 2-pos

$\mathcal{E}(I_n) = I_k$

Quantum channel: $\mathcal{E} : M_n(\mathbb{C}) \to M_k(\mathbb{C})$ linear map

~~(i)~~ $A \succeq 0 \Rightarrow \mathcal{E}(A) \succeq 0$ $\quad \forall A \in M_n(\mathbb{C})$ positive map

(ii) $\mathrm{Tr}(\mathcal{E}(A)) = \mathrm{Tr}(A)$ trace preserving

CPT

$\mathcal{E}$ is ℓ-positive if $\mathcal{E} \otimes I_\ell : M_{n\ell}(\mathbb{C}) \to M_{k\ell}(\mathbb{C})$ is a positive map.

$\mathcal{E}$ is CP (completely pos.) if $\mathcal{E}$ is ℓ-pos. for all $\ell \geq 1$

(i) $\mathcal{E}$ is a CP map

Thm [Kraus]: $\mathcal{E}$ ($M_n \to M_k$) is CP $\Leftrightarrow$

(*) $\boxed{\mathcal{E}(\rho) = A_1 \rho A_1^* + \dots + A_m \rho A_m^*}$

for some $A_1, \dots, A_m : \mathbb{C}^n \to \mathbb{C}^k$,

and $\mathcal{E}$ is CPT iff (*) holds and $\boxed{A_1^* A_1 + \dots + A_m^* A_m = I}$

Ex: $\Phi : A \mapsto A^T$ is a pos. map but $A \otimes I_2$ is not pos.

$(\Phi \otimes I_2)\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^T & B^T \\ C^T & D^T \end{bmatrix}$

This is a quantum ch. and

Thm [Stinespring Dilation Thm]:

$\rho \in \mathcal{D}(\mathbb{C}^n)$, $\sigma \in \mathcal{D}(\mathbb{C}^m)$

I_m/m

$\mathcal{E}(\rho) = \mathrm{T}\,(U(\rho \otimes \sigma)U^*)$

every quantum ch. is of this form.

$\mathcal{E}(\rho) = \mathrm{Tr}_B(U(\rho \otimes \sigma)U^*)$ for some $\mathbb{C}^n \otimes \mathbb{C}^m = \mathcal{H}_A \otimes \mathcal{H}_B$

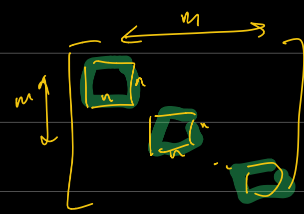
Classical: If Φ is a stoch. channel, then

$$D(\Phi(p) \,\|\, \Phi(q)) \le D(p \,\|\, q) \quad \forall p, q$$

Quantum: If $\mathcal{E}$ is a quantum channel, then

$$S(\mathcal{E}(\rho) \,\|\, \mathcal{E}(\sigma)) \le S(\rho \,\|\, \sigma) \quad \forall \rho, \sigma \text{ densities}$$

$\mathcal{E}(\rho) = \mathrm{Tr}_A(\rho)$ is CPT



Thm: $\rho, \sigma \in \mathcal{D}(\underset{\mathbb{C}^m}{\mathcal{H}_A} \otimes \underset{\mathbb{C}^n}{\mathcal{H}_B})$

Then $\underbrace{S(\mathrm{Tr}_A(\rho) \,\|\, \mathrm{Tr}_A(\sigma)) \le S(\rho \,\|\, \sigma)}$

follow from joint convexity of $S(\cdot \| \cdot)$

$$E = \begin{bmatrix} \omega_m & & & & & \\ & \omega_m^2 & & & 0 & \\ & & \omega_m^3 & & & \\ & 0 & & \ddots & & \\ & & & & \omega_m^{m-1} & \\ & & & & & 1 \end{bmatrix} \qquad \omega_m = \exp(2\pi i/m)$$

$$D_Y = \frac{1}{m}\sum_{j=1}^{m} E^{-j} Y E^{j} \quad \text{where } D_Y \text{ is the diagonal } Y$$

$$\frac{\mathrm{Tr}(Y)}{m} I_m = \frac{1}{m}\sum_{j=1}^{m} V_j D_Y V_j^*$$

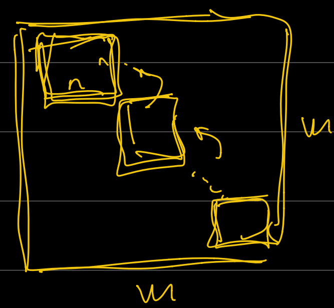
$$\begin{bmatrix} y_{11} & & & \\ & y_{22} & & \\ & & \ddots & \\ & & & y_{mm} \end{bmatrix} \to \begin{bmatrix} y_{22} & & & \\ & y_{33} & & \\ & & \ddots & \\ & & & y_{mm} \\ & & & & y_{11} \end{bmatrix}$$

$$= \frac{1}{N}\sum_{j=1}^{N} U_j Y U_j^*$$

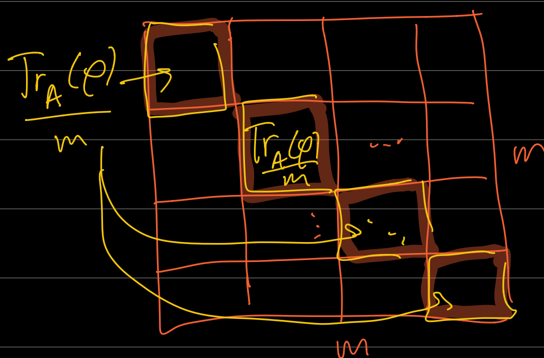
$e_i \to e_{(i+1) \bmod m}$

$Y \in M_{mn}(\mathbb{C})$

$$E = \begin{bmatrix} \omega_m I_n & & & & 0 \\ & \omega_m^2 I_n & & & \\ & & \omega_m^3 I_n & & \\ & & & \ddots & \\ & & & & \omega_m^{m-1} I_n & \\ 0 & & & & & 1\cdot I_n \end{bmatrix}$$



$$D_Y = \frac{1}{m}\sum_{j=1}^{m} E^{-j} Y E^{j}$$



$A \quad B$

$\rho \in D(\mathbb{C}^m \otimes \mathbb{C}^n)$

$$\frac{Tr_A(\rho)}{m} \otimes I_m = \frac{1}{N}\sum_{j=1}^{N} U_j \rho U_j^*$$

$$S(Tr_A(\rho) \,\|\, Tr_A(\sigma)) = S\left(\frac{Tr_A(\rho)}{m} \otimes I_m \,\Big\|\, \frac{Tr_A(\rho)}{m} \otimes I_m\right)$$

$$= S\left(\frac{1}{N}\sum_{j=1}^{N} U_j \rho U_j^* \,\Big\|\, \frac{1}{N}\sum_{j=1}^{N} U_j \sigma U_j^*\right)$$

$$\le \frac{1}{N}\sum_{j=1}^{N} S(U_j \rho U_j^* \,\|\, U_j \sigma U_j^*)$$

$$= S(\rho \| \sigma).$$

$$\sum_{i=1}^{m} \lambda_i = 1$$

noncommutative average

Thm: If $f: \mathbb{R}_+ \to \mathbb{R}$ is operator convex, then for any $A_1, A_2, \dots, A_m \in M_n(\mathbb{C})$ s.t. $A_1^* A_1 + \cdots + A_m^* A_m = I$, and any Hermitian $X_1, X_2, \dots, X_m \in M_n(\mathbb{C})$, it holds that

$$f\left(A_1^* X_1 A_1 + \cdots + A_m^* X_m A_m\right) \le A_1^* f(X_1) A_1 + \cdots + A_m^* f(X_m) A_m$$

$$\boxed{\begin{gathered} S^*S + T^*T = I, \quad A, B \succeq 0 \\ (S^*AS + T^*BT)^2 \le S^*A^2S + T^*B^2T \end{gathered}}$$

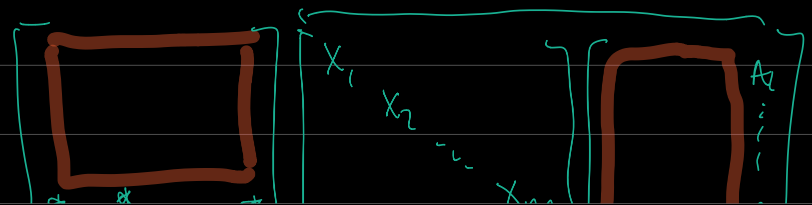
Pf:

$$\overbrace{A_1^* X_1 A_1 + \cdots + A_m^* X_m A_m}^{n \times n} = \overbrace{\left(U^* X U\right)_{mm}}^{}$$

$m \times m$ block matrix, blocks $n \times n$

$$X = \begin{bmatrix} X_1 & & & \\ & X_2 & & \\ & & \ddots & \\ & & & X_m \end{bmatrix} \qquad U = \begin{array}{c} \underbrace{\qquad\qquad\qquad\qquad\qquad}_{mn} \qquad \overbrace{\quad}^{n \text{ columns}} \\ \left[\begin{array}{cccc|c} | & | & & | & A_1 \\ u_1 & u_2 & \cdots & u_{n(m-1)} & A_2 \\ | & | & & | & \vdots \\ & & & & A_m \end{array}\right] \end{array} \; mn$$

$$f(X) = \begin{bmatrix} f(X_1) & & \\ & \ddots & \\ & & f(X_m) \end{bmatrix} \begin{bmatrix} \cdots & X_1 A_1 \\ & \vdots \\ & X_m A_m \end{bmatrix}$$



$$mn \updownarrow \overbrace{\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}}^{n} : \mathbb{C}^n \to \mathbb{C}^{mn}$$

$$K \oplus \boxed{K'}$$

$\to [A_1^* \; A_2^* \; \cdots \; A_m^*]$ $\mathbb{C}^n$ $\mathbb{C}^{n(m-1)}$

$(U^* \quad X \quad U)_{mm}$

$V^* V = \sum A_i^* A_i = I$

$$A_1^* X_1 A_1 + \cdots + A_m^* X_m A_m = (U^* X U)_{mm}$$

$$\boxed{f(A_1^* X_1 A_1 + \cdots + A_m^* X_m A_m)} = f((U^* X U)_{mm})$$

$$f\left(\left(\frac{1}{m}\sum_{j=1}^{m} E^{-j} U^* X U E^{j}\right)_{mm}\right)$$

$$f\left(\begin{bmatrix} \boxed{M_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{M_m} \end{bmatrix}\right) = \begin{bmatrix} f(M_1) & & \\ & \ddots & \\ & & \boxed{f(M_m)} \end{bmatrix}$$

$$= \left(f\left(\frac{1}{m}\sum_{j=1}^{m} E^{-j} U^* X U E^{j}\right)\right)_{mm}$$

$$\preceq \left(\frac{1}{m}\sum_{j=1}^{m} f(E^{-j} U^* X U E^{j})\right)_{mm}$$

$$\begin{bmatrix} A_{11} & & \\ & \ddots & \\ & & A_{mm} \end{bmatrix} \succeq \begin{bmatrix} B_{11} & & \\ & \ddots & \\ & & B_{mm} \end{bmatrix} \iff A_{ii} \succeq B_{ii} \;\forall i$$

$$= \left(\frac{1}{m}\sum_{j=1}^{m} E^{-j} U^* f(X) U E^{j}\right)_{mm}$$

$$f(U^* A U) = U^* f(A) U$$

$$= (U^* f(X) U)_{mm}$$

$$= \boxed{A_1^* f(X_1) A_1 + \cdots + A_m^* f(X_m) A_m}$$