CUT_n = convex hull of indicators of cuts in the complete graph $\}$

$$\mathrm{QML}_+^n = \left\{ f : \{0,1\}^n \to \mathbb{R}_+ : f(x) = \sum_{i,j} a_{ij} x_i x_j \right\}$$

$\mathrm{CUT}_{n+1} \overset{\text{aff.}}{\longleftrightarrow} \mathrm{CORR}_n$

$\mathcal{M}_n : \mathrm{QML}_+^n \times \{0,1\}^n \to \mathbb{R}_+ \qquad \mathcal{M}_n(f,x) := f(x)$

is a slack matrix for CORR_n

$$\mathrm{rk}_+(\mathcal{M}_n) = \min\left\{ r : \mathcal{M}_n(f,x) = \langle u_f, v_x \rangle, \; u_f, v_x \in \mathbb{R}_+^r \;\; \forall f, x \right\}$$

$$\mathrm{rk}_{psd}(\mathcal{M}_n) = \min\left\{ r : \mathcal{M}_n(f,x) = \langle U_f, V_x \rangle, \; U_f, V_x \in \mathcal{S}_r^+ \right\}$$

Lemma: $\mathrm{rk}_+(\mathcal{M}_n) = \min\left\{ r : \mathrm{QML}_+^n \subseteq \mathrm{cone}(f_1, \ldots, f_r), \; \underbrace{f_1, \ldots, f_r : \{0,1\}^n \to \mathbb{R}_+}_{\text{"r axioms"}} \right\}$

suppose that $\mathrm{QML}_+^n \subseteq \mathrm{cone}(f_1, \ldots, f_r)$

$f \in \mathrm{QML}_+^n \qquad f = \lambda_1(f) f_1 + \cdots + \lambda_r(f) f_r \qquad \lambda_i(f) \geq 0$

$\mathcal{M}_n(f,x) = \langle \lambda(f), g(x) \rangle \qquad \mathrm{rk}_+(\mathcal{M}_n) \leq r$

$g(x) = (f_1(x), f_2(x), \ldots, f_r(x))$.

$\square$

$$L^2(\{0,1\}^n) := \{ f : \{0,1\}^n \to \mathbb{R} \} \qquad (\mathbb{R}^{\{0,1\}^n})$$

$U \subseteq L^2(\{0,1\}^n)$, define $\overbrace{\mathrm{sos}(U)}^{\text{"sos cone over } U\text{"}} := \mathrm{cone}(q^2 : q \in U)$

i.e., $f \in \mathrm{sos}(U) \iff f(x) = \sum^{m} q_i(x)^2 \qquad q_1, \ldots, q_r \in U$.

Lemma: $\mathrm{rk}_{psd}(\mathcal{M}_n) \leq \boxed{\min\{\dim(U) : QML_+^n \subseteq sos(U)\}}$
$\leq \mathrm{rk}_{psd}(\mathcal{M}_n)^2$

Pf: Suppose $QML_+^n \subseteq sos(U)$ and let $q_1, q_2, \ldots, q_r : \{0,1\}^n \to \mathbb{R}$ be a basis for $U = \mathrm{span}\{q_1, \ldots, q_r\}$.

$Q : \{0,1\}^n \to \mathcal{S}_r^+$ $\qquad Q(x)_{ij} = q_i(x) q_j(x)$

(i.e. $Q(x) = (q_1(x), \ldots, q_r(x))(q_1(x), \ldots, q_r(x))^T$)

Given $p \in U$, write $p = \sum_{i=1}^r c_i q_i$, $\Lambda(p^2)_{ij} := c_i c_j$

$$p(x)^2 = \mathrm{Tr}(\Lambda(p^2)\, Q(x)) = \sum_{i,j} c_i c_j\, q_i(x)\, q_j(x)$$
$$= \left(\sum_i c_i q_i(x)\right)^2$$

$f \in QML_+^n$, then $f = \sum_{i=1}^m \lambda_i p_i^2$ $\qquad p_1, \ldots, p_m \in sos(U)$

$$\Lambda(f) := \sum_{i=1}^m \lambda_i \Lambda(p_i^2)$$

$$\mathrm{Tr}(\underbrace{\Lambda(f)}_{\mathcal{S}_r^+} Q(x)) = \sum_{i=1}^m \lambda_i \mathrm{Tr}(\Lambda(p_i^2)\, Q(x))$$
$$= \sum_{i=1}^m \lambda_i\, p_i(x)^2 = f(x) = \mathcal{M}_n(f, x)$$

$\Rightarrow \mathrm{rk}_{psd}(\mathcal{M}_n) \leq r.$

$f \in QML_+^n$ $\quad f(x) = \mathrm{Tr}(P(f)\, \underline{Q(x)})$ $\qquad \Lambda(f), Q(x) \in \mathcal{S}_r^+$

$$P(f) = \sum^r a_i(f)\, a_i(f)^T, \quad Q(x) = \sum^r \overbrace{q_i(x)}^{\mathbb{R}^r} q_i(x)^T$$

$$f(x) = \mathrm{Tr}(P(f)Q(x)) = \sum_{i,j} \langle q_i(f), q_j(x)\rangle^2$$

Ex

$$\sum_{j=1}^{r}\left(\sum_{k} a_{ik}(f) q_{jk}(x)\right)^2 = \sum_{i,j}\left(\underline{\sum_{k=1}^{r} a_{ik}(f)\, q_{jk}(x)}\right)^2$$

$\dim(U) \le r^2$ $\qquad U = \mathrm{span}\underbrace{(q_{jk} : 1 \le j, k \le r)}_{r^2}$

$L^2(\{0,1\}^n; \ell_2)$ = vector space of functions $f : \{0,1\}^n \to \ell_2$

$$\ell_2 = \{(x_j) : \sum x_j^2 < \infty\}$$

$$\boxed{\begin{array}{l} U \subseteq L^2(\{0,1\}^n; \ell_2) \text{ a subspace} \\ \mathrm{sos}(U) := \mathrm{cone}(\|q(x)\|_{\ell_2}^2 : q \in U) \end{array}}$$

$$\|x\|_{\ell_2} = \left(\sum_{j\ge1} x_j^2\right)^{1/2}$$

Lemma: $\mathrm{rk}_{psd}(M_n) = \min\{\dim(U) : QML_+^n \subseteq \mathrm{sos}(U)\}$

Lemma: $\mathrm{rk}_+(M_n) = \min\underbrace{\{r : QML_+^n \subseteq \mathrm{cone}(f_1, \dots, f_r)\ f_1, \dots, f_r : \{0,1\}^n \to \mathbb{R}_+\}}$

S_n = group of permutations on n elements $\quad S_n \curvearrowright \{0,1\}^n$

$\sigma \in S_n \qquad \sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$

$f : \{0,1\}^n \to \mathbb{R} : \qquad \sigma f(\underline{x}) := f(\sigma x).$

Fact: QML_+^n is invariant under the coord. action of S_n

$\mathcal{F} = \{f_1, \dots, f_r : \{0,1\}^n \to \mathbb{R}_+\}$

Suppose $\mathcal{F}$ invariant under the action of S_n.

$f : \{0,1\}^n \to \mathbb{R}$ is a k-junta if f depends on $\le k$ of the input variables.

$\boxed{\mathcal{J}_k := \text{core}(\text{non-neg } k\text{-juntas})}$ $\quad \mathcal{J}_k = \text{cone}(\binom{n}{k} 2^k \text{ non-neg fns})$

Theorem: If $QML_+^{2n} \subseteq \boxed{\text{cone}(f_1, \dots, f_r)}$ and $\mathcal{F} = \{f_1, \dots, f_r\}$ is S_{2n}-invariant and $r < \binom{2n}{k}$ and $k < n/2$. Then $\boxed{QML_+^n \subseteq \text{core}(\mathcal{J}_k)}$

$r \simeq n^{O(k)} \qquad n^{O(k)}$

Sherali-Adams

$\text{Orb}(f_1) = \{\pi f_1 : \pi \in S_n\}$

$\text{Stab}(f_1) = \{\pi \in S_n : \pi f_1 = f_1\}$

Fact: $|\text{Orb}(f_1)| \cdot |\text{Stab}(f_1)| = |S_n| = n!$

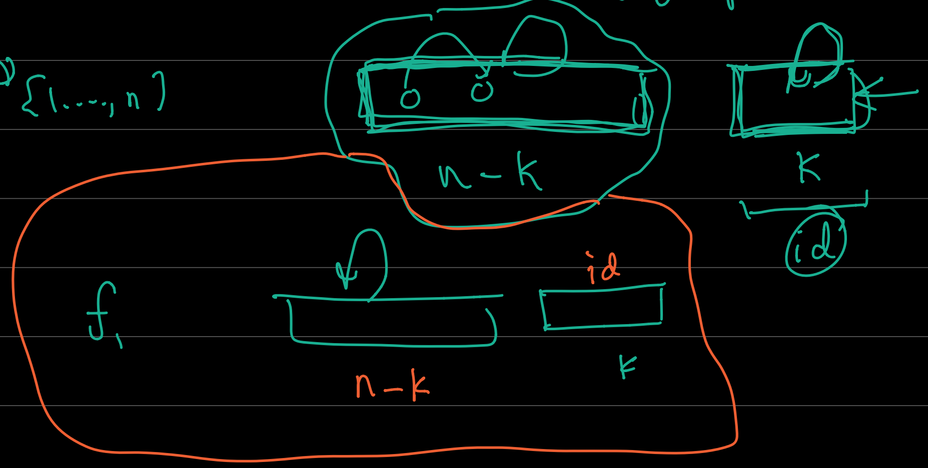
$\boxed{r < \binom{n}{k}}$

$|\text{Orb}(f_1)| \le |\mathcal{F}| \le r \Rightarrow \underbrace{|\text{Stab}(f_1)|}_{H} \ge \frac{n!}{r} > \boxed{k!(n-k)!}$

$\mathrm{Stab}(f_1) \leq S_n$ $\qquad$ $A_n \subseteq S_n$

alternating group

$S_n \curvearrowright \{1,\dots,n\}$



$f(\underbrace{x_1,\dots,x_{n-k}}, \overbrace{x_{n-k+1},\dots,x_n})$

$\boxed{f(x) = x_1 + x_2 + \dots + x_n}$

$f(x) = g(\underbrace{x_{n-k+1},\dots,x_n}_{k \text{ words}}, \boxed{|x|})$ $\qquad$ $|x| = x_1 + \dots + x_n$

$\underbrace{\tilde{f} : \{0,1\}^{2n} \to \mathbb{R}}$ $\qquad$ $f(x) := \boxed{\tilde{f}(x, \bar{x})}$ $\qquad$ $|x| + |\bar{x}| = n$

$\bar{x}_i = 1 - x_i$

$\boxed{QML_+^n \not\subseteq \mathcal{J}_{n-1}}$ $\qquad$ $\boxed{QML_+^n \subseteq \mathcal{J}_n}$

$\boxed{f(e_1) = \dots = f(e_n) = 0}$ $\qquad$ $f(0) = 1$

$f(x) = (x_1 + x_2 + \dots + x_n - 1)^2$ $\qquad$ $\{0,1\}^n$

$g : \{0,1\}^n \to \mathbb{R}_+$ is an $\underbrace{(n-1)\text{-junta}}$

and $g(e_1) = \cdots = g(e_n) = 0$

$\Longrightarrow g(0,\dots,0) = 0$

$g(e_j) = 0$
$\shortparallel$
$g(0)$

$|B_1| \geq \cdots \geq |B_m|$

$H \leq S_n \qquad \{1,\dots,n\} = B_1 \cup B_2 \cup \cdots \cup B_m$

$$k!(n-k)! \leq |H| \leq \prod_{i=1}^{m} (|B_i|!)$$

e_2

$\underline{|B_1| \geq n-k}$ $\qquad H_1 \leq S_{n-k} \qquad H_2 \leq S_k$

$n-k \qquad k$

$H = H_1 \rtimes H_2 \qquad H_1 = S_{n-k} \qquad \underset{=}{|H|} = \underbrace{|H_1|}_{\geq} \cdot |H_2|$