Pattern matrices and smooth factorizations

$A \in \mathbb{R}_+^{a\times r},\ B \in \mathbb{R}_+^{r\times b}$

$\mathrm{rank}_+(M)$

$M \in \mathbb{R}^{a\times b}$ $\quad \mathrm{rank}(M) = \min\{ r : M = AB \quad M_{ij} = \langle a_i, b_j \rangle$

$A \in \mathbb{R}^{a\times r}, B \in \mathbb{R}^{r\times b}\}$

$M : \mathbb{R}^b \to \mathbb{R}^a$

$\mathbb{R}^b \to \mathbb{R}^r \to \mathbb{R}^a \qquad M \in \mathbb{R}_+^{a\times b}$

$$\boxed{\mathrm{rank}_{psd}(M) := \min\{ r : M_{ij} = \langle A_i, B_j \rangle,\ \{A_i\}, \{B_j\} \subseteq \mathcal{S}_r^+ \}}$$

$M_n(f,x) := \boxed{f(x)}$ $\qquad$ psd rank $\swarrow cn^d$ $\qquad x \in \{0,1\}^n$

$$M_n(f,x) = \mathrm{Tr}(A(f)\,B(x))\}$$

$f(x) = \sum_{i\le j} a_{ij} x_i x_j$

s.t. $f(x) \ge 0 \ \forall x \in \{0,1\}^n$

$\to \left\{ \begin{array}{l} B(x)_{S,T} := x_{S\cup T} = \prod_{i\in S\cup T} x_i = x_S x_T \quad (x \in \{0,1\}^n) \\ |S|, |T| \le d \end{array} \right.$

$$= \sum_{j=0}^{d} \binom{n}{j} \approx \boxed{cn^d}$$

$g_k(x) = (x_1 + \dots + x_n - k)(x_1 + \dots + x_n - (k+1))$

k a pos. integer

$\shortparallel$

$(x_1 + \dots + x_n - \frac{n}{2})^2 - \frac{1}{4}$

n odd, $k = \frac{n}{2}$

Thm [Gr.]:

$\to \boxed{\deg_{sos}(g_k) \ge k/2}$ for every $1 \le k \le n/2$.

$\quad h$

and [KLW]: $\deg_{sos}(g_1) \ge \Omega(\sqrt{n})$

$\{\deg_{sos}(g_k) \gtrsim \sqrt{k(n-k)}$

$$\mathrm{rank}_{psd}(M_n) \le \underline{\left[\min\{ r : QML_+^n \subseteq sos(U),\ \dim(U) = r \}\right]}$$

$\sqrt{g} : \{0,1\}^n \to \mathbb{R}_+$

$U = \mathrm{span}(\sqrt{g})$

$g \in sos(U)$, $g = (\sqrt{g})^2$

$g : \{0,1\}^m \to \mathbb{R}_+$ $\qquad$ $\deg_{sos}(g) \ge d$

rank_{psd} is large?

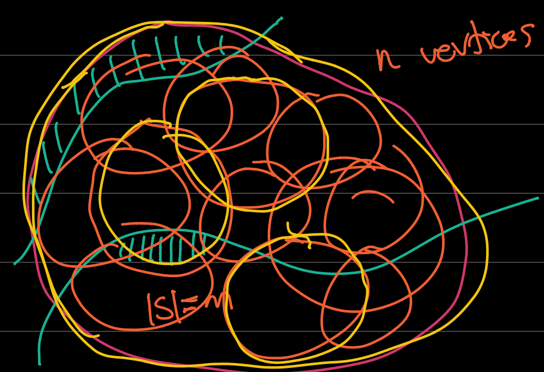
$\mathcal{L} := \mathcal{L}^{n,g}$ $\qquad$ $\boxed{\mathcal{L}(S, x) := g(x|_S)}$ $\qquad$ $S \subseteq [n]$, $|S| = m$

$x \in \{0,1\}^n$

$S = \{i_1, i_2, \dots, i_k\}$

$x|_S = (x_{i_1}, x_{i_2}, \dots, x_{i_k})$

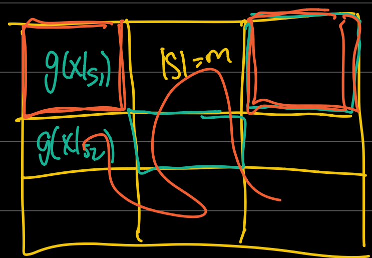
if $i_1 < i_2 < \dots < i_k$



If $g \in QML_+^m$, then

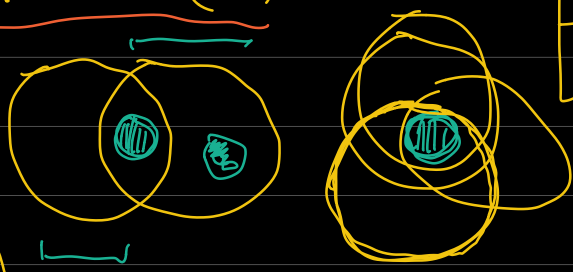
$\mathcal{L}(S,x)$ is a submatrix

of the slack matrix of $CORR_n$.



$\underline{\mathrm{rank}_{psd}(\mathcal{L}) \le \binom{n}{m}}$ $\qquad$ $n^{c \cdot m}$

$g(x|S)$

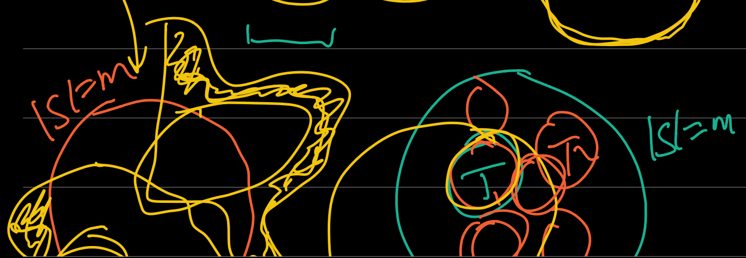


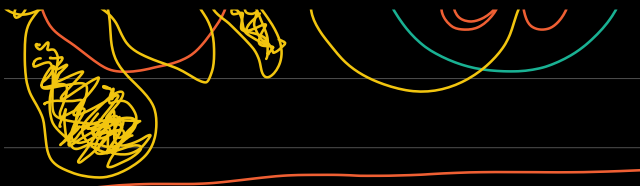
$\dim(U) \ge n/m$

$\uparrow$

$U = sos(\underline{\text{degree} \le d})$

$X_T = \prod_{i \in T} x_i$ $\qquad$ $|T| \le d$



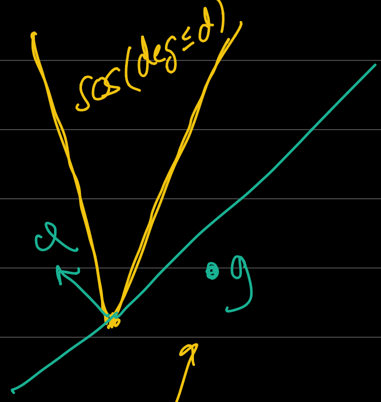


$d = d(n)$

Thm: For any $g : \{0,1\}^m \to \mathbb{R}_+$ with $\deg_{sos}(g) = d$,

if $f := f^{n,g}$ $\quad f(S,x) = g(x|_S)$ $\quad S \in \binom{[n]}{m}$, $\quad x \in \{0,1\}^n$

$$n^d + 1 \geq \mathrm{rank}_{psd}(f^{n,g}) \gtrsim \left(\frac{n}{\log n}\right)^d \quad \text{as } n \to \infty.$$



$\varphi : \{0,1\}^m \to \mathbb{R}$

degree-d pseudo density if

(i) $\langle \varphi, \mathbb{1} \rangle = 1$

(ii) $\langle \varphi, p^2 \rangle = \mathbb{E}_x \left[\varphi(x) p(x)^2 \right] \geq 0$

$\forall \ \deg(p) \leq d$

$\langle \varphi, g \rangle < 0$

$\mathbb{E}[\varphi(x) g(x)] < 0$

$\boxed{M : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}_+}$

$$\mathbb{E}_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} \left[\psi(x,y) M(x,y) \right] < 0$$

$\|A(x)\|_{S_\infty} \cdot \|B(y)\|_{S_1} \leq \gamma$

$\boxed{\psi(S,x) = \varphi(x|_S)}$

$$\mathbb{E}_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} \left[\psi(x,y) \, \mathrm{Tr}(A(x) B(y)) \right] \geq 0$$

$$\mathbb{E}_{|S| = m} \left[\varphi(x|_S) \, g(x|_S) \right]$$

whenever $A : \mathcal{X} \to \mathbb{S}_r^+$

$x \in \{0,1\}^n$ $\quad$ ‖

$B : Y \to S_r^+$

$$\mathbb{E}_{z \in \{0,1\}^m} \left[q(z)\, g(z) \right] < 0$$

$\mathrm{rank}_{psd}(M') > r$

(Ex) <u>Fact</u>: $\left\{ \mathrm{rank}_{psd}(M) \le r \right\}$ is not convex

<u>John's theorem</u>: $K \subseteq \mathbb{R}^n$ centrally-sym. convex body (convex and $x \in K \Leftrightarrow -x \in K$)

Then $\exists$ a linear map $J : \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$\underbrace{B_{\ell_2}}_{=\{x \in \mathbb{R}^n : \|x\|_2 \le 1\}} \subseteq J(K) \subseteq B_{\ell_2} \cdot \sqrt{n}$$

$M : X \times Y \to \mathbb{R}_+$

"smooth psd rank"

$$\gamma_{psd}(M) := \min \left\{ \gamma : M_{ij} = \mathrm{Tr}(A_i B_j) \quad A_i \succeq 0,\ B_j \succeq 0 \right.$$
$$\left. \text{s.t. } \boxed{\underbrace{\|A_i\|_{S_\infty} \cdot \|B_j\|_{S_1}}_{\|A_i\| \cdot \mathrm{Tr}(B_j)} \le \gamma} \right\}$$

(1) $\{ \gamma_{psd}(M) \le \gamma \}$ convex set for every γ ✓

$\rightarrow$ $M_{ij} = \mathrm{Tr}(A_i B_j)$ $\quad$ $N_{ij} = \mathrm{Tr}(P_i Q_j)$

$$\underline{\frac{1}{2}(M_{ij} + N_{ij})} = \mathrm{Tr}\left(\begin{pmatrix} \boxed{A_i} & \\ & \boxed{P_i} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\boxed{B_j} & \\ & \frac{1}{2}\boxed{Q_j} \end{pmatrix} \right)$$

$\frac{1}{2}\left(\mathrm{Tr}(R_j) + \mathrm{Tr}(G_j)\right)$

(ii) If $\underline{\mathrm{rank}_{psd}(M) \le r}$, then

$$\underline{\gamma_{psd}(M) \le r^2 \|M\|_{\ell_\infty}}$$