$\mathbb{E}\,\mathrm{Tr}\left(e^{\beta(A_1+A_2+\cdots+A_n)}\right)$ $\qquad \mathrm{Tr}\left(e^{A+B}\right) \neq \mathrm{Tr}\left(e^A e^B\right)$

$e^{A+B} \neq e^A e^B$

Golden-Thompson: $A, B \in M_n(\mathbb{R})$ symmetric, then

$$\boxed{\mathrm{Tr}\left(e^{A+B}\right) \leq \mathrm{Tr}\left(e^A e^B\right)}$$

$A \succeq 0 \Rightarrow$

$U^* A U \succeq 0$

$$\mathrm{Tr}\left(e^{A+B}\right) \leq \mathrm{Tr}\left(e^{B/2} e^A e^{B/2}\right)$$

False: $\mathrm{Tr}\left(e^{A+B+C}\right) \leq \mathrm{Tr}\left(e^{A/2} e^{C/2} e^B e^{C/2} e^{A/2}\right)$

$$e^X = \sum_{n \geq 0} \frac{X^n}{n!}$$

$$e^{A+B} = \sum_{n \geq 0} \frac{(A+B)^n}{n!} \qquad e^A e^B = \sum_{n \geq 0} \sum_{j=0}^{n} \frac{A^j B^{n-j}}{n!} \binom{n}{j}$$

$$(A+B)^n = (A \mid B)(A \mid B) \cdots (A \mid B)$$

$$ABA^2B^3A \cdots$$

std inner product on $\mathbb{R}^{d^2}$

Frobenius inner product: $A, B \in M_d(\mathbb{R})$

$$(A, B) \mapsto \mathrm{Tr}\left(A^T B\right) = \sum_{i,j} A_{ij} B_{ij}$$

$$\left((A, B) \mapsto \mathrm{Tr}\left(A^* B\right) \right)$$

(Frob. norm, HS norm, Schatten 2-norm)

$A = UDV^T$

$$\|A\|_2 := \left(\mathrm{Tr}\left(A^T A\right)\right)^{1/2}$$

Singular-value decomp.

$$A = \sum_i \sigma_i u_i v_i^T$$

$$\mathrm{Tr}(A^TA) = \sum_{i,j} \sigma_i\sigma_j v_i \underbrace{u_i^T u_j}\, v_j^T$$

$\sigma_1, \ldots, \sigma_d \geq 0$

$\{u_i\}, \{v_i\}$ orthonormal bases

$$= \sum_i \sigma_i^2 v_i v_i^T$$

$\|A\|_2 = \|(\sigma_1, \sigma_2, \ldots, \sigma_d)\|_2$

$$\boxed{|\mathrm{Tr}(A^TB)| \leq \|A\|_2 \cdot \|B\|_2 \qquad \forall A, B \in \mathbb{M}_d(\mathbb{R})}$$

$|\mathrm{Tr}(A^2)| \leq \|A\|_2^2$

$\|A\|_2^2 = \mathrm{Tr}(A^TA)$

$\mathrm{Tr}(A^2) = \sum_{i,j} A_{ij} A_{ji}$

$\mathrm{Tr}(A^TA) = \sum_{i,j} A_{ij}^2$

A, B PSD

$\mathrm{Tr}((AB)^2) \leq \mathrm{Tr}(A^2B^2)$ ✓

$$\underbrace{\mathrm{Tr}((AB)^2)} \leq \|AB\|_2^2 = \mathrm{Tr}((AB)^TAB) = \mathrm{Tr}(B^TA^TAB)$$
$$= \mathrm{Tr}(BA^2B)$$
$$= \mathrm{Tr}(A^2B^2)$$

$$\mathrm{Tr}((AB)^4) = \mathrm{Tr}((AB)^2(AB)^2)$$
$$\leq \|(AB)^2\|_2^2 = \mathrm{Tr}((ABAB)^T ABAB)$$
$$= \mathrm{Tr}(\underline{BA}\,\underline{BA}\;\underline{AB}\,\underline{AB})$$
$$= \mathrm{Tr}(AB(AB)^T(AB)^TAB)$$
$$\leq \|AB(AB)^T\|_2 \cdot \|(AB)^TAB\|_2$$
$$= \|AB(AB)^T\|_2^2$$
$$= \mathrm{Tr}(AB(AB)^TAB(AB)^T)$$
$$= \mathrm{Tr}((AB^2A)^2)$$
$$= \mathrm{Tr}((A^2B^2)^2)$$

$A \leftarrow A^2,\ B \leftarrow B^2$

$A+B \quad A, B$

Goal: $\mathrm{Tr}(e^{A+B}) \le \mathrm{Tr}(e^A e^B) \le \mathrm{Tr}(A^4 B^4)$ $\quad e \,=e\, e$

* Lemma: $U, V \in M_d(\mathbb{C})$ and $k \ge 1$ integer, then

$$\to \mathrm{Tr}((UV)^{2^k}) \le \mathrm{Tr}(U^{2^k} V^{2^k})$$

$$\boxed{e^{A+B} \overset{?}{=} e^A e^B}$$

$$|\mathrm{Tr}(A^* B)| \le \left(\mathrm{Tr}(|A||B|) \cdot \mathrm{Tr}(|A^*||B^*|)\right)^{1/2}$$

$$|A| = (A^* A)^{1/2} \qquad A = \sum_i \sigma_i u_i v_i^*$$

$$\boxed{|A| = \sum_i \sigma_i v_i v_i^*}$$

$$U = e^{A/p}, \quad V = e^{B/p}, \quad p = 2^k$$

$$\mathrm{Tr}((e^{A/p} e^{B/p})^p) \le \mathrm{Tr}(e^A e^B)$$

Lie-Trotter: $\boxed{\lim_{p\to\infty} (e^{A/p} \cdot e^{B/p})^p = e^{A+B}}$ $\quad \sum_n \frac{\|X\|^n}{n!}$

$$\boxed{e^X = \sum_{n\ge 0} \frac{X^n}{n!}}$$

Pf: $S := e^{(A+B)/p}$, $T := e^{A/p} e^{B/p}$

$$S = I + \frac{A+B}{p} + \boxed{\frac{(A+B)^2}{2p^2}} + \cdots \frac{(A+B)^n}{p^n n!} \qquad T = \left(I + \frac{A}{p} + \cdots\right)\left(I + \frac{B}{p} + \cdots\right)$$

$$T = I + \frac{A+B}{p} + \boxed{\frac{A^2 + B^2 + 2AB}{2p^2}} + \cdots$$

implicit constant depends on A, B

$\to \|S - T\| \le O(1/p^2)$ as $p \to \infty$

$$\|S\|, \|T\| \le e^{(\|A\| + \|B\|)/p}$$

$$\|T\| = \|e^{A/p} e^{B/p}\| \le \|e^{A/p}\| \cdot \|e^{B/p}\| = e^{\|A\|/p} e^{\|B\|/p}$$

$$\|S^p - T^p\| \le \sum_{k=0}^{p} \|S^{p-k} T^k - S^{p-k-1} T^{k+1}\|$$

$$= \sum_{k=0}^{p} \|S^{p-k-1} (S - T) T^k\|$$

p

$$\le \|S-T\| \cdot \underbrace{\sum_{k=0} \|S\|^{p-k-1}\|T\|^{k}}$$

$$\le \|S-T\| \cdot p \cdot \underbrace{e^{\|A\|+\|B\|}} \le O(1/p) \quad , p \to \infty$$

$$T^p = (e^{A/p}e^{B/p})^p \to S^p = e^{A+B} \qquad \square$$

$$(e^{A/p}e^{B/p})^p = (e^{\frac{A+B}{p}+O(1/p^2)})^p = e^{A+B+O(1/p)}$$

$$e^{A/p} = I + \frac{A}{p} + O(1/p^2)$$

Lemma: $U, V \in M_d(\mathbb{C})$ and $k \ge 1$ integer, then

$$\mathrm{Tr}((UV)^{2^k}) \le \mathrm{Tr}(U^{2^k}V^{2^k})$$

$$\mathrm{Tr}[(A^*A)(A^*A)(A^*A)(A^*A)]$$

$$\vee|$$

$$\mathrm{Tr}(AA^*AAA^*A^*AA) \quad 4$$

Consider $A \in M_d(\mathbb{C})$ and

$$P = A_1 A_2 \cdots A_{2n}, \qquad A_i \in \{A, A^*\}$$

$$\boxed{\text{Claim: } |\mathrm{Tr}(P)| \le \mathrm{Tr}((A^*A)^n)}$$

Take P that maximizes $|\mathrm{Tr}(P)|$. If $P = (A^*A)^n$ or $P = (AA^*)^n$, done.

Otherwise, $\exists$ adjacent AA or A^*A^*

$$P = \underbrace{\overset{n}{\longrightarrow} \cdots A}\,\underbrace{A \cdots \overset{n}{\longleftarrow}}$$

$$AA^* \qquad A^*A$$

$$Q \curvearrowright R$$

$$|\mathrm{Tr}(P)|^2 \leq \|Q\|_2^2 \, \|R\|_2^2 = \mathrm{Tr}(Q^* Q)\, \mathrm{Tr}(R^* R)$$

$$P' = Q^* Q, \quad \underline{P'' = R^* R} \qquad |\mathrm{Tr}(P)| = |\mathrm{Tr}(P')| = |\mathrm{Tr}(P'')|$$

Count the # of $A \to A^*$ or $A^* \to A$ transitions

$$k_P = \text{\# trans for } P = \tilde{K}_Q + \tilde{K}_R + \underbrace{K_{RQ}}_{\leq 1}$$

$$K_{P'} = 2\tilde{K}_Q + 2$$

$$K_{P''} = 2\tilde{K}_R + 2$$

$$P' = \underbrace{A^* \quad X}_{Q^*} \,|\, \underbrace{X^* \quad A}_{Q}$$

$$\tfrac{1}{2}(K_{P'} + K_{P''}) > K_P \Rightarrow \text{One of } P' \text{ or } P'' \text{ has } \underline{\underline{\text{more}}}\ A \leftrightarrow A^* \text{ trans}$$

<u>Lemma:</u> $U, V \in M_d(\mathbb{C})$ (Hermitian) $(U = U^*,\ V = V^*)$ and $k \geq 1$ integer, then

$$\mathrm{Tr}((UV)^{2^k}) \leq \mathrm{Tr}(U^{2^k} V^{2^k})$$

$$A = UV \qquad \mathrm{Tr}(A^{2^k}) \leq \mathrm{Tr}((A^* A)^{2^{k-1}})$$

$$\begin{aligned}
\mathrm{Tr}((UV)^{2^k}) &\leq \mathrm{Tr}((V^* U^* U V)^{2^{k-1}}) \\
&= \mathrm{Tr}((V U^2 V)^{2^{k-1}}) \\
&= \mathrm{Tr}((U^2 V^2)^{2^{k-1}}) \\
&\leq \mathrm{Tr}((U^4 V^4)^{2^{k-2}}) \leq \cdots \leq \mathrm{Tr}(U^{2^k} V^{2^k}) \quad \square
\end{aligned}$$

Lemma: If $A_1, \dots, A_{2^k} \in M_d(\mathbb{C})$, then

$$|\mathrm{Tr}(A_1 A_2 \cdots A_{2^k})| \le \|A_1\|_{2^k} \cdots \|A_{2^k}\|_{2^k}$$

$$|\mathrm{Tr}((UV)^{2^k})| \le \|UV\|_{2^k}^{2^k} = \mathrm{Tr}(((UV)^* UV)^{2^{k-1}})$$

$$\boxed{|\mathrm{Tr}(A_1 A_2 \cdots A_n)| \le \|A_1\|_n \cdots \|A_n\|_n}$$

$A \prec_w B$ if $\sigma_1(A) \ge \dots \ge \sigma_d(A)$

$\sigma_1(B) \ge \dots \ge \sigma_d(B)$

$$\Longleftrightarrow \begin{cases} \sigma_1(A) \le \sigma_1(B) \\ \sigma_1(A) + \sigma_2(A) \le \sigma_1(B) + \sigma_2(B) \\ \vdots \\ \sigma_1(A) + \dots + \sigma_d(A) \le \sigma_1(B) + \dots + \sigma_d(B) \end{cases}$$

$p = 1$

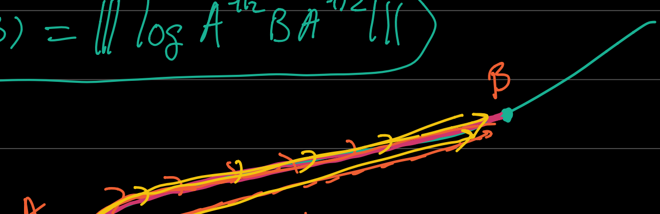
$$\boxed{e^{A+B} \prec_w e^{A/2} e^{B} e^{A/2}} \qquad \|e^{A+B}\|_p \le \|e^A e^B\|_p$$

$$\Updownarrow$$

$$|||UAV||| = |||A|||$$

$$\boxed{|||e^{A+B}||| \le |||e^{A/2} e^{B} e^{A/2}|||} \quad U, V \text{ unitary}$$

$$\delta_{|||\cdot|||}(A,B) = |||\log A^{-1/2} B A^{-1/2}|||$$



$A \cdots\to$

$$\underbrace{e^{\varepsilon H/2} A e^{\varepsilon H/2}}_{\varepsilon |||H|||}$$

$$e^{\varepsilon H/2} = \left(I + \varepsilon \frac{H}{2}\right) \qquad \boxed{a'(t) = a(t)\, v}$$

$$\partial_t(\log a) = v$$

$$\frac{a'(t)}{a(t)} = v$$

$$e^{\log a(t) + v\,dt}$$

$$= a(t)\, v\, dt$$

$$\delta_{|||}(e^H, e^K) = |||\log e^{-H/2} e^K e^{-H/2}||| \geq |||H-K|||$$

$$|||e^{-H/2} e^K e^{-H/2}||| \geq |||e^{H-K}|||$$

$$\mathrm{Tr}(e^{A+B}) \leq \mathrm{Tr}(e^A e^B) \quad \forall A, B$$

Symmeh.