Homework #1 is posted. Due: Apr 26 (Mon)

$A, B \in M_n(\mathbb{C})$

$$(A, B) \mapsto \mathrm{Tr}(A^* B) = \sum_{i,j} \overline{A_{ij}} B_{ij}$$

$$|\mathrm{Tr}(A^* B)| \le \|A\|_2 \|B\|_2 \qquad \underbrace{|A|}_{} = (A^* A)^{1/2}$$

PSD and its eigenvalues are the singular vals of A

$$\|A\|_p := \left(\mathrm{Tr}(|A|^p)\right)^{1/p}$$

$$= \underline{\|(\sigma_1, \sigma_2, \dots, \sigma_n)\|_p}$$ where $\sigma_1, \sigma_2, \dots, \sigma_n$ are the sig. vals of A

$\|UAV\|_p = \|A\|_p \quad \forall\, U, V$ unitaries

Von Neumann's trace inequality: If $A, B \in M_n(\mathbb{C})$ w/ singular values $\sigma_1(A) \ge \dots \ge \sigma_n(A)$, $\sigma_1(B) \ge \dots \ge \sigma_n(B)$, then

$$|\mathrm{Tr}(A^* B)| \le \sum_{i=1}^{n} \sigma_i(A)\, \sigma_i(B)$$

Lemma: $|\mathrm{Tr}(A^* B)| \le \left(\mathrm{Tr}(|A|\,|B|) \cdot \mathrm{Tr}(|A^*|\,|B^*|) \right)^{1/2}$

$$A = \sum_i \sigma_i u_i v_i^*, \quad B = \sum_i \tilde{\sigma}_i \tilde{u}_i \tilde{v}_i^*$$

$\sigma_1, \dots, \sigma_n \ge 0$

$\tilde{\sigma}_1, \dots, \tilde{\sigma}_n \ge 0$

$$A^* = \sum_i \sigma_i v_i u_i^*$$

$\{u_i\}, \{v_i\}, \{\tilde{u}_i\}, \{\tilde{v}_i\}$ orthonormal bases

$|\mathrm{Tr}(A^* B)|$

$$\mathrm{Tr}(\sigma_i \tilde{\sigma}_j v_i u_i^* \tilde{u}_j \tilde{v}_j^*)$$

$$= \sigma_i \tilde{\sigma}_j \langle u_i, \tilde{u}_j \rangle \mathrm{Tr}(v_i \tilde{v}_j^*)$$

$$= \left| \sum_{i,j} \sigma_i \tilde{\sigma}_j \langle u_i, \tilde{u}_j \rangle \langle v_i, \tilde{v}_j \rangle \right|$$

$$\left| \sum_i c_i a_i b_i \right| \le \left(\sum_i c_i |a_i|^2 \right)^{1/2} \left(\sum_i c_i |b_i|^2 \right)^{1/2}$$

$$\le \left(\sum_{i,j} \sigma_i \tilde{\sigma}_j |\langle u_i, \tilde{u}_j \rangle|^2 \right)^{1/2} \left(\sum_{i,j} \sigma_i \tilde{\sigma}_j |\langle v_i, \tilde{v}_j \rangle|^2 \right)^{1/2}$$

$$= \left(\mathrm{Tr}(|A| \, |B|) \right)^{1/2} \left(\mathrm{Tr}(|A^*| \, |B^*|) \right)^{1/2}$$

$$|A| = (A^* A)^{1/2} = \sum_{i,j} \sigma_i \sigma_j \, v_i u_i^* u_j v_j^*$$

$$= \left(\sum_i \sigma_i^2 v_i v_i^* \right)^{1/2}$$

$$= \sum \sigma_i v_i v_i^*$$

$$|B| = \sum_i \tilde{\sigma}_i \tilde{v}_i \tilde{v}_i^*$$

$$\mathrm{Tr}(v_i v_i^* \tilde{v}_j \tilde{v}_j^*)$$

$$\mathrm{Tr}(|A| \, |B|) = \sum_{i,j} \sigma_i \tilde{\sigma}_j |\langle v_i, \tilde{v}_j \rangle|^2$$

Von Neumann's trace inequality: If $A, B \in M_n(\mathbb{C})$ w/ singular values $\sigma_1(A) \ge \cdots \ge \sigma_n(A)$, $\sigma_1(B) \ge \cdots \ge \sigma_n(B)$, then

$$|\mathrm{Tr}(A^* B)| \le \sum_{i=1}^n \sigma_i(A) \sigma_i(B)$$

✓ Lemma: $|\mathrm{Tr}(A^T B)| \le \left(\mathrm{Tr}(|A|\,|B|) \cdot \mathrm{Tr}(|A^*|\,|B^*|)\right)$

Suffices to consider $A, B \succeq 0$. $\quad A = \sum_i a_i u_i u_i^*$

$B = \sum_i b_i v_i v_i^*$

$$\mathrm{Tr}(AB) = \sum_{i,j} a_i b_j \underbrace{|\langle u_i, v_j\rangle|^2}$$

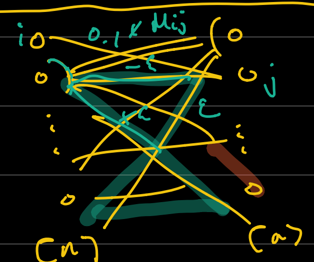
M is doubly-stochastic

$$= \sum_{i,j} a_i b_j M_{ij}$$

i.e. $\sum_i M_{ij} = \sum_j M_{ij} = 1$

$$= \sum_{\pi} w_\pi \sum_{i=1}^{n} a_i b_{\pi(i)}$$

$w_\pi \ge 0$ and $\sum_\pi w_\pi = 1$

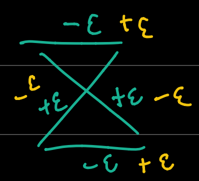


$$\le \max_\pi \sum_i a_i b_{\pi(i)} = \sum_i a_i b_i$$

M min # of fractional entries

$\varepsilon \to 0$

(?) $\quad \underline{\underline{M}} = \frac{1}{2}(\underline{\underline{M_1}} + \underline{\underline{M_2}})$



$-\varepsilon$
ε

Von Neumann's trace inequality: If $A, B \in M_n(\mathbb{C})$ w/ singular values $\sigma_1(A) \ge \dots \ge \sigma_n(A)$
$\sigma_1(B) \ge \dots \ge \sigma_n(B)$, then

$$|\mathrm{Tr}(A^* B)| \le \sum_{i=1}^{n} \sigma_i(A)\,\sigma_i(B)$$

$= \|\sigma(A)\|_p \, \|\sigma(B)\|_q$

Hölder ineq: $|\mathrm{Tr}(A^*B)| \le \|A\|_p \cdot \|B\|_q$

when $\frac{1}{p} + \frac{1}{q} = 1$.

$\Phi : \mathbb{R}^n \to \mathbb{R}_+$ is a symmetric gauge function

if Φ is a norm on $\mathbb{R}^n$ that's invariant wrt

(i) permute coords [symm.]

(ii) flip the signs coord. [gauge]

(Φ is sym. gauge fn $\Rightarrow$ same for Φ^*)

$$\Phi^*(u) = \max\{\langle u, v\rangle : \Phi(v) \le 1\} \leftarrow$$

$$\boxed{|\mathrm{Tr}(A^*B)| \le \Phi(\sigma(A))\,\Phi^*(\sigma(B))}$$

$$|\langle u, v\rangle| \le \Phi(u)\,\Phi^*(v)$$

$$\Phi(u) = \|u\|_p$$

$$\Phi^*(u) = \|u\|_q$$

Say that a norm $|||\cdot|||$ is unitarily invariant

if $|||UAV||| = |||A|||$ $\forall\, U, V$ unitary, $A \in M_n(\mathbb{C})$

If $|||A||| = \Phi(\sigma(A))$ and Φ is a sym. gauge fn, then $|||\cdot|||$ is unitarily inv. norm.

And this is a characterization of u.i.n.s.

A, B PSD $\qquad \boxed{A \preceq B}$

$\langle x, Ax \rangle \leq \langle x, Bx \rangle \quad \forall x \in \mathbb{C}^n$

$\lambda_1(A) \geq \cdots \geq \lambda_n(A)$

$\lambda_1(B) \geq \cdots \geq \lambda_n(B) \implies \boxed{\lambda_i(A) \leq \lambda_i(B)}$

$\implies \Phi(\lambda(A)) \leq \Phi(\lambda(B))$

$\forall$ sym gauge fn

$\mathrm{Tr}(e^{A+B}) \leq \mathrm{Tr}(e^A e^B)$

$\{ |||e^{A+B}||| \leq |||e^{B/2} e^A e^{B/2}||| \implies |||A||| \leq |||B||| \quad \forall$ u.i.n.

Suppose $|||A||| \leq |||B|||$ for every u.i.n. $|||\cdot|||$

Majorization $\quad x \in \mathbb{R}^n \quad x_1^\downarrow \geq x_2^\downarrow \geq \cdots \geq x_n^\downarrow$

$x, y \in \mathbb{R}^n$: $\quad x \prec_w y$ if $x_1^\downarrow + \cdots + x_k^\downarrow \leq y_1^\downarrow + \cdots + y_k^\downarrow \quad \forall\ 1 \leq k \leq n$

$x \prec y$: if $x \prec_w y$ and $x_1^\downarrow + \cdots + x_n^\downarrow = y_1^\downarrow + \cdots + y_n^\downarrow$

Thm: For all $A, B \in M_n(\mathbb{C})$,

$|||A||| \leq |||B|||$ for every u.i. $|||\cdot||| \iff \sigma(A) \prec_w \sigma(B)$

Schur-convex φ

$x \prec y \implies \varphi(x) \prec \varphi(y)$

$A \preceq B \;\not\Rightarrow\; e^A \preceq e^B$

$\implies \log(A) \leq \log(B)$

$[\, |||A||| \leq |||B||| \quad \forall$ u.i. norm $]$

$\implies$

If $x \prec_w y$, then $e^x \prec_w e^y$

$f: \mathbb{R} \to \mathbb{R}$

mon. incr, convex,

then

$\left[\{ |||e^A||| \leq |||e^B||| \} \ \forall \text{ u.i. norm} \right]$

$x \prec_w y \Rightarrow f(x) \prec_w f(y)$

$A \prec_w B \Leftrightarrow \sigma(A) \prec_w \sigma(B)$

Lemma: $x, y \in \mathbb{R}^n$, then

$\boxed{x \prec y \Leftrightarrow x = Py}$, P doubly stochastic

$\Leftrightarrow x \in$ convex hull (perms of y)

$x \prec_w y \Leftrightarrow \exists u$ s.t. $x \leq u$ (pointwise) and $u \prec y$

$\Leftrightarrow x = By$ where B is doubly substochastic

$x, y \in \mathbb{R}^n_+$

$x \prec_w y \Leftrightarrow \Phi(x) \leq \Phi(y)$ $\forall$ sym. gauge fn

$\left(\Phi_k(x) := |x_1^\downarrow| + \dots + |x_k^\downarrow| \right)$

$\Downarrow$

$x \leq u$ and $u = Py$

$\Rightarrow \Phi(x) \leq \Phi(u) = \Phi(\underline{Py}) \leq \underline{\Phi(y)}$

$A \preceq B \Rightarrow \log(A) \preceq \log(B)$ $\quad A = \varphi(\lambda_1 \ldots$

Thm: A PSD, then $(\lambda_1(A), \ldots, \lambda_n(A)) \succ \boxed{(A_{11}, \ldots, A_{nn})}$

$$\det(A) \le \prod_{i=1}^{n} A_{ii} \qquad A \succeq 0$$