$A_1, A_2, \dots, A_n$ random symmetric matrices

$$\mathrm{Tr}\left(\mathbb{E}\left[e^{\beta(A_1+\dots+A_n)}\right]\right) \le \mathrm{Tr}(I) \cdot \prod_{i=1}^{n} \underbrace{\|\mathbb{E}\, e^{\beta A_i}\|}$$

$\|A_i\| \le L$

$$\left\{ \mathbb{E}\left[e^{\beta(X_1+\dots+X_n)}\right] \le \underbrace{\prod_{i=1}^{n} \mathbb{E}\left[e^{\beta X_i}\right]} \right.$$

$\sum \mathbb{E}(X_i^2) \le$ small

$$\mathrm{Tr}(e^{A+B+C}) \le \mathrm{Tr}(e^A e^B e^C)$$

(doesn't hold)

$$\Lambda_A(\beta) := \log \mathbb{E}\left[e^{\beta A}\right]$$

$A = \underbrace{A_1 + \dots + A_n}$

indep. random sym. matrices

$$\rightarrow \mathrm{Tr}\left(\exp(\Lambda_A(\beta))\right) \le \mathrm{Tr}\left(\exp\left(\Lambda_{A_1}(\beta) + \dots + \Lambda_{A_n}(\beta)\right)\right)$$

$$\mathrm{Tr}\left(\mathbb{E}[e^{\beta A}]\right) \le \mathrm{Tr}\left(\exp\left(\log \mathbb{E}[e^{\beta A_1}] + \dots + \log \mathbb{E}[e^{\beta A_n}]\right)\right)$$

Theorem (Lieb's concavity thm): For every Hermitian H, the function

$$A \mapsto \mathrm{Tr}\left(e^{\log A + H}\right)$$

is concave on positive def. matrices.

$X = \beta A_{n-1}$

$H = \beta(A_1 + \dots + A_{n-2}) + \log(\mathbb{E}\, e^{A_n})$

$H = \beta(A_1 + \dots + A_{n-1})$

$X = \beta A_n$

X - random Herm. matrix

$$\mathrm{Tr}\left(\mathbb{E}\, e^{H+X}\right) \le \mathrm{Tr}\left(\mathbb{E}\left[e^{H + \log(\mathbb{E}\, e^X)}\right]\right)$$

$Y = e^X$

$$\boxed{\mathrm{Tr}\left(\mathbb{E}\, e^{H + \log Y}\right) \le \mathrm{Tr}\left(\mathbb{E}\, e^{H + \log \mathbb{E} Y}\right)}$$

Operator monotonicity and convexity

$I \subseteq \mathbb{R}$ interval

$(0,\infty)$ $\quad f: I \to \mathbb{R}$ $\qquad f(A) = f(UDU^*) := U\begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n)\end{pmatrix}U^*$

$(\mathrm{spec}(A) \subseteq I)$

f is operator monotone if $\quad f_n(A) \geq f_n(B) \quad f_n \to f$

$$A \geq B \implies f(A) \geq f(B) \qquad \forall\ \mathrm{spec}(A), \mathrm{spec}(B) \subseteq I$$

Lec #1: $f(x) = \sqrt{x}$ is operator monotone.

$f(x) = x^2$ not op. monotone

$$(A+\varepsilon B)^2 = A^2 + \varepsilon\underbrace{(AB+BA)}_{\text{not nec. psd}} + \varepsilon^2 B^2 \qquad \varepsilon \to 0$$

$$(A+\varepsilon B)^2 \not\geq A^2$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AB + BA = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \qquad -1$$

not nec. psd

f is operator convex if $\forall\ \mathrm{spec}(A), \mathrm{spec}(B) \subseteq I$

$$f(tA + (1-t)B) \leq t f(A) + (1-t) f(B) \qquad \forall t \in [0,1]$$

$t = \frac{1}{2}$ $\qquad f\left(\frac{A+B}{2}\right) \leq \frac{1}{2}\left(f(A) + f(B)\right)$

"midpoint convex"

Fact: f cont, then midpoint op convex $\implies$ op convex

$$\left(\frac{A+B}{2}\right)^2 \leq \left(\frac{A+B}{2}\right)^2 + \left(\frac{A-B}{2}\right)^2 = \frac{A^2+B^2}{2}$$

$f(x) = x^2$ midpoint op. convex $\implies$ op. convex

$f(x) = x^3$ is not operator convex.

Def: f is operator concave if $-f$ is op. convex

f is operator mono decr. if $-f$ is op. monotone

$p=0$
$p=1$

Thm (Loewner–Heinz): $f:(0,\infty)\to\mathbb{R}$

$f(t)=t^p$ is
- op monotone decr. and op convex $\quad -1\le p\le 0$
- op. monotone and op. concave $\quad 0\le p\le 1$
- op. convex, not monotone for $p>1$ $\quad 1\le p\le 2$

$f(t)=\log t$ is op. concave and op. monotone

$f(t)=t\log t$ is op. convex (but not monotone)

$$\log A=\lim_{p\to 0}\frac{A^p-I}{p} \qquad \log t=\lim_{p\to 0}\frac{t^p-1}{p}$$

$p\in[-1,1]$ $\frac{A^p-I}{p}$ is op concave and monotone

$$\lim_{p\to 0}\frac{e^{p\log t}-1}{p}=\lim_{p\to 0}\frac{\log t\, e^{p\log t}}{1}=\log t$$

$$A\log A=\lim_{p\to 1}\frac{A^p-A}{p-1} \qquad t\log t=\lim_{p\to 1}\frac{t^p-t}{p-1}$$

$\frac{A^p-A}{p-1}$ is convex for $p\in[0,2]$

$(AB)^{-1}=B^{-1}A^{-1}$

Lemma: On positive matrices, $A\mapsto A^{-1}$ is op. monotone decr. and op. convex

≥ 0

$A>0,\ B>0$ $\quad \left(C := A^{-1/2} B A^{-1/2}\right) \succeq 0$

$$\boxed{(A+B)^{-1} - A^{-1} = A^{-1/2}\left[(I+C)^{-1} - I\right]A^{-1/2}}$$

$$D_{\underline{\underline{B}}}(A^{-1}) := \lim_{\varepsilon\to 0} \frac{(A+\varepsilon B)^{-1} - A^{-1}}{\varepsilon} = A^{-1} B A^{-1}$$

Fact: $A \succeq B \implies UAU^* \succeq UBU^* \quad \forall U$

$\langle x, Ax\rangle \geq \langle x, Bx\rangle$

$\forall x$

$\langle x, UAU^* x\rangle$

$\geq \langle x, UBU^* x\rangle$

$\langle U^* x, AU^* x\rangle$

$\geq \langle U^* x, BU^* x\rangle$

$C \geq 0$

$$\boxed{(I+C)^{-1} \preceq I} \Leftarrow \frac{1}{1+c} \leq 1$$

$$\boxed{\left(\frac{A+B}{2}\right)^{-1} \preceq \frac{A^{-1}}{2} + \frac{B^{-1}}{2}}$$

$$\left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1} \leq \frac{a+b}{2}$$

$$\left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1} = \frac{2ab}{a+b} = \sqrt{ab} \cdot \frac{\sqrt{ab}}{(a+b)/2}$$

$$\leq \sqrt{ab} \leq \frac{a+b}{2}$$

$$\underbrace{\frac{A^{-1}}{2} + \frac{B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1}}_{\geq\ 0} = A^{-1/2}\underbrace{\boxed{\frac{I}{2} + \frac{C^{-1}}{2} + \left(\frac{I+C}{2}\right)^{-1}}}\, A^{-1/2}$$

$$C = A^{-1/2} B A^{-1/2}$$

$\frac{1}{2} + \frac{c^{-1}}{2} \geq \left(\frac{1+c}{2}\right)^{-1}$ $\quad$ AM-HM

$f(t) = t^p$ is
$$\begin{cases} \text{op monotone decr. and op convex} & -1 \le p \le 0 \\ \text{op. monotone and op. concave} & 0 \le p \le 1 \\ \text{op. convex, not monotone for } p>1 & 1 \le p \le 2 \end{cases}$$

Integral representations: Claim that for $0<p<1$, $\exists$ constant C_p s.t.

$$C_p\int_0^\infty t^p\left(\frac{1}{t}-\frac{1}{t+a}\right)dt = c_p\, a^p \quad \forall a>0$$

$$\boxed{\frac{a}{t(t+a)}}$$

$p \to \infty$ integrand $\le O(t^{p-2})$

$p \to 0$ integrand $\le O(t^{p-1})$

$t = as \qquad \frac{dt}{ds} = a$

$$a^p\int_0^\infty s^p\left(\frac{1}{\cancel{a}s}-\frac{1}{\cancel{a}(s+1)}\right)\cancel{a}\,ds = a^p\underbrace{\int_0^\infty s^p\left(\frac{1}{s}-\frac{1}{s+1}\right)ds}_{1/C_p}$$

$$C_p\int_0^\infty t^p\left(\frac{1}{t}-\frac{1}{t+a}\right)dt = a^p$$

$0<p<1$

$A > 0 \Rightarrow A^p = \int_0 t^p \left(\frac{1}{t} \cdot I - (tI + A)^{-1} \right) dt$

$A \mapsto A^{-1}$

$A \mapsto (tI+A)^{-1}$

is convex

and op. mono. decr

$A \mapsto \frac{1}{t} \cdot I - (tI + A)^{-1}$

is op. concave and

op. monotone

$$A \gtrsim B \Rightarrow \underbrace{A + C \gtrsim B + C \Rightarrow f(A+C) \gtrsim f(B+C)}$$

$$f((1-t)(A+C) + t(B+C)) \precsim (1-t) f(A+C) + t f(B+C)$$

$$\shortparallel$$

$$f((1-t)A + tB + \underline{\underline{C}}) \leq (1-t) f(A+C) + t f(B+C)$$

$A \mapsto f(A)$ is convex $\Rightarrow$ $A \mapsto f(A+C)$ is convex

$$C_p \int_0^\infty t^p \left(\frac{1}{t} - \frac{1}{t+a} \right) dt = a^p$$

$0 < p < 1$

$$a^{p+1} = \int_0^\infty t^p \left(\frac{a}{t} - \frac{a}{t+a} \right) dt = \int_0^\infty t^p \left(\frac{a}{t} + \frac{t}{t+a} - 1 \right) dt$$

$$\underbrace{A^{p+1}} = \underbrace{\int_0^\infty} t^p \underbrace{\left(\underbrace{\frac{A}{t}} + \frac{t}{\underline{tI + A}} - I \right)} dt \qquad 0 < p < 1$$

A^q, $q \in [1, 2]$

convex

$$\frac{1}{A + t \cdot I}$$

$$\underline{\underline{a^{p-1}}} = c_p \int_0^\infty t^p \left(\frac{1}{at} - \frac{1}{a(t+a)} \right) dt$$

$$= c_p \int_0^\infty t^{p-1} \left(\frac{1}{a} - \frac{t}{a(t+a)} \right) = \int_0^\infty t^{p-1} \left(\frac{1}{t+a} \right) dt$$

$$\frac{t+c-t}{a(t+c)} = \frac{a}{a(t+a)} = \frac{1}{t+a}$$