$\tilde p_i = \frac{p|_{S_i}}{p(S_i)}$

Quantum probability

$\tilde p_i = \frac{p_i}{\mathbb{P}(S_\alpha)}$

$\{1,2,\dots,n\} = \boxed{S_1 \cup S_2 \cup \dots \cup S_m}$

Space of outcomes $\{1,2,\dots,n\}$

$p \in \mathbb{R}^n_+$, $\sum_{i=1}^n p_i = 1$

$\frac{1}{2}p_1 + \frac{1}{2}p_1 + p_2 = 1$

Density matrix:

State $\rho \in M_n(\mathbb{C})$ s.t.

$\rho \geq 0$, $\mathrm{Tr}(\rho) = 1$

$$\rho = \mathrm{diag}(p) = \begin{pmatrix} p_1 & & 0 \\ & p_2 & \\ 0 & \ddots & p_n \end{pmatrix}$$

$\rho_1 + \dots + \rho_m = I$

$|0\rangle$, $\frac{1}{\sqrt{2}}(|01\rangle + |00\rangle)$

Measurements

$P_1 + P_2 = I$

$U_1, U_2, \dots, U_m \in M_n(\mathbb{C})$

s.t. $U_1U_1^* + U_2U_2^* + \dots + U_mU_m^* = I$

$\langle x, Ix\rangle = |x|^2 = 1$

m outcomes

$\mathbb{P}($outcome i when measuring $\rho) = \mathrm{Tr}(U_i \rho U_i^*)$

State after measuring outcome i is $\dfrac{U_i \rho U_i^*}{\mathrm{Tr}(U_i \rho U_i^*)}$

$\sum_i \mathrm{Tr}(U_i \rho U_i^*)$

$= \sum_i \mathrm{Tr}(U_i^* U_i \rho)$

$= \mathrm{Tr}(\sum_i U_i^* U_i \rho)$

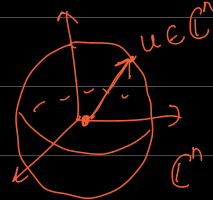
$= \mathrm{Tr}(I\rho) = \mathrm{Tr}(\rho) = 1$

Pure states:

$\mathcal{D} \subseteq M_n(\mathbb{C})$ convex set

extreme points are the pure states

$\equiv \{uu^* : u \in \mathbb{C}^n, \|u\| = 1\}$



$\rho = \sum_i \lambda_i u_i u_i^*$

$\rho = vv^* \longrightarrow \sum_{i=1}^m U_i \rho U_i^*$ $\mathrm{Tr}(\dots)$ $\mathrm{Tr}(\dots)$

Composite systems

S_1 S_2 $\cancel{(p_1, p_2)}$

$x \in \{1,\dots,m\}$, $y \in \{1,\dots,n\}$ $\{1,2,\dots,m\}$ $\{1,\dots,n\}$

$p_1 \in \mathbb{R}^m$ $p_2 \in \mathbb{R}^n$

independent: $q = p_1 \otimes p_2$ $q \in \mathbb{R}^m \otimes \mathbb{R}^n$

$p(x,y)$

$q(x,y) = p_1(x)\,p_2(y)$ $\qquad \cong \mathbb{R}^{mn}$ $\qquad \rho^A \otimes \rho^B \in D(\mathcal{H}_S)$

$\mathcal{H}_A = \mathbb{C}^m, \quad \mathcal{H}_B = \mathbb{C}^n$ $\qquad \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$

$\mathcal{D}(\mathbb{C}^n) := \{\rho \in M_n(\mathbb{C}) : \rho \geq 0, \operatorname{Tr}\rho = 1\}$

independent: $\rho^{AB} = \rho^A \otimes \rho^B$

$S_1 \;\vdots\; S_2$

If q distr. on $[m] \times [n]$

Marginal: $q_1(x) := \sum_{y \in [n]} q(x,y)$

$S_1 \cup S_2 \cup \dots \cup S_k = [m]$

$S_1 \times [n] \cup \dots \cup S_k \times [n] = [m] \times [n]$

Want: $\rho^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, $\quad \operatorname{Tr}((U \otimes I)\rho^{AB}(U^* \otimes I)) = \operatorname{Tr}(U \rho^A U^*)$

ρ^A = "marginal" of ρ^{AB} on A-part

$$\begin{cases} \operatorname{Tr}((U \otimes I)\rho^{AB}) = \operatorname{Tr}(U\rho^A) & \forall U \text{ hermitian} \\ \operatorname{Tr}((I \otimes V)\rho^{AB}) = \operatorname{Tr}(V\rho^B) & \forall V \text{ hermitian} \end{cases}$$

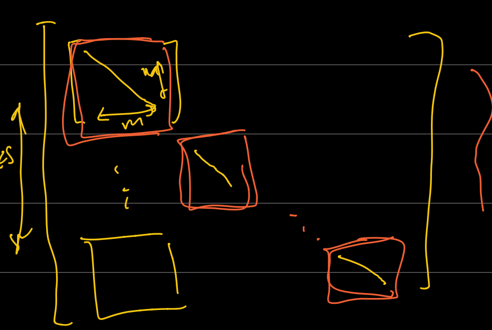
$\rho^A := \operatorname{Tr}_B(\rho^{AB})$ $\qquad \operatorname{Tr}_B(\overset{A}{U} \otimes \overset{B}{V}) = U \cdot \operatorname{Tr}(V)$

$\operatorname{Tr}_A(U \otimes V) = V \operatorname{Tr}(U)$

$\rho \in D(\overset{A}{\mathbb{C}^m} \otimes \overset{B}{\mathbb{C}^n})$

$mn \times mn$ matrix

$\rho =$ n blocks



$\operatorname{Tr}_B(\rho) = \boxed{\sum_{i=1}^{n} \rho_{ii}} \in D(\mathbb{C}^m)$

$\operatorname{Tr}(\operatorname{Tr}_B(\rho)) = \sum_{i=1}^{n} \operatorname{Tr}(\rho_{ii})$

$= \operatorname{Tr}(\rho)$

$\rho =$ m blocks

$\mathrm{Tr}_A(\rho) = \boxed{\sum_{j=1}^{m} \rho^{jj}} \in D(\mathbb{C}^n)$

$\boxed{\rho^{AB} = \rho^A \otimes \rho^B}$

$\mathrm{Tr}_B(\rho^{AB}) = \rho^A$

$\rho^A \cdot \sum_{i=1}^{n} \rho^B_{ii} = \rho^A$

<u>Entanglement</u> $\quad S \subseteq D(\mathcal{H}_A \otimes \mathcal{H}_B)$

"separable states" $\quad = \left\{ \sum_i c_i (A_i \otimes B_i) : \begin{array}{l} c_i \geq 0, \ \sum_i c_i = 1 \\ A_i \in D(\mathcal{H}_A), \ B_i \in D(\mathcal{H}_B) \end{array} \right\}$

The rest of the states are "<u>entangled</u>."

<u>Examples:</u> $\quad v = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1) \qquad \rho = vv^*$

$\rho \in \underbrace{D(\mathbb{C}^n \otimes \mathbb{C}^n)}_{n^2 \times n^2 \text{ matrices}}$

$\rho = \frac{1}{n^2} \sum_{i,j} e_{ij} \otimes e_{ij}$ maximally entangled states

$I_{ij,ij} = 1 \qquad I_{ij,i'j'} = 0$ for $ij \neq i'j'$

$(e_{ij} \otimes e_{ij})_{a,b,c,d} = (e_{ij})_{ab} (e_{ij})_{cd}$

$(e_i e_j^T \otimes e_i e_j^T)$

$\boxed{(A \otimes B)_{abcd} = A_{ac} B_{bd}}$

$(e_{ij} \otimes e_{ij})_{abcd} = 1_{\{a=i,\, c=j,\, b=i,\, d=j\}}$

$(e_{ij})_{ac} (e_{ij})_{bd}$

$\|\Psi\|_{acc} = 4$ $\boxed{U \rho U^*}$

von Neumann entropy $\rho \in \mathcal{D}(\mathbb{C}^n)$ $\rho = \sum_i \lambda_i v_i v_i^*$

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho)$$ concave

$$= H(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_i \lambda_i \log \frac{1}{\lambda_i}$$

$1 \cdot \log 1 = 0$.

If q prob. on $[m] \times [n]$. (x, y)

Motonicity of entropy: $\boxed{H(q_1) \le H(q)}$ q_1 = marginal of q on $[n]$

$$H(X,Y) = H(X) + \underbrace{H(Y|X)}_{\text{non-neg}}$$

If $\rho = uu^*$ is a pure state, then $S(\rho) = 0$

(Purification) I/n

For every $\rho \in \mathcal{D}(\mathbb{C}^n)$, there is density $\tilde{\rho} \in D(\mathbb{C}^n_A \otimes \mathbb{C}^n_B)$

s.t. (i) $\tilde{\rho}$ is pure state $S(\tilde{\rho}) = 0$

(ii) $\mathrm{Tr}_B(\tilde{\rho}) = \rho$

Montonicity of relative entropy: $\rho, \sigma \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$

$$S(\mathrm{Tr}_B(\rho) \,\|\, \mathrm{Tr}_B(\sigma)) \le S(\rho \| \sigma)$$

$$S(\rho_1 \| \rho_2) = \mathrm{Tr}(\rho_1(\log \rho_1 - \log \rho_2))$$

$$S(\Phi(\rho) \,\|\, \Phi(\sigma)) \le S(\rho \| \sigma)$$ $\forall$ quantum

L

channel Φ

$\mathbb{R}_+$

$\boxed{\sum_i w_i S_i}$

$\{1, 2, \ldots, n\} \times [m]$

$\forall j \in [n] \quad \sum_i w_i \mathbb{1}_{S_i}(j) = 1$

(1, 0)

(1, 1)

(1, 2)

(1, 3)