$\rho \in \underbrace{\mathcal{D}(\mathbb{C}^n)}$ $\quad \rho \in M_n(\mathbb{C})$, $\rho \geq 0$, $\mathrm{Tr}(\rho) = 1$

concave

$$\boxed{S(\rho) = -\mathrm{Tr}(\rho \log \rho)} \qquad \rho = \sum_i \lambda_i v_i v_i^*$$

$$= H(\lambda_1, \lambda_2, \dots, \lambda_n) = \underbrace{\sum_{i=1}^{n} \lambda_i \log \frac{1}{\lambda_i}}.$$

$$\boxed{\frac{\rho + \varepsilon I/n}{1+\varepsilon} \quad \varepsilon \to 0}$$

Fact: $S(u^* \rho u) = S(\rho)$

$\forall$ unitaries u

$$\left[\lim_{x \to 0} x \log x = 0\right] \qquad \rho > 0$$

X, Y random variables: $\boxed{H(X,Y) \leq H(X) + H(Y)}$

$$H(X,Y) = H(X) + \underbrace{H(Y|X)}_{} \leq H(X) + \underbrace{H(Y)}_{}$$

$\to$ $\boxed{H(Y|X) = H(X,Y) - H(X)}$

$\leq$ fails in the quantum world

$\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ $\qquad \rho^A = \mathrm{Tr}_B(\rho)$, $\rho^B = \mathrm{Tr}_A(\rho)$

$$S(A|B)_\rho := S(\rho) - S(\rho^B) \qquad \left[\begin{array}{l} S(\rho) = 0 \text{ but} \\ S(\rho^B) > 0 \end{array}\right]$$

Given $\sigma \in \mathcal{D}(\overbrace{\mathbb{C}^n}^{\mathcal{H}_A})$ $\qquad \{e_1, e_2, \dots, e_n\}$ orthonormal basis for $\mathcal{H}_B \cong \mathbb{C}^n$

$\sigma = \sum_{i=1}^{n} \lambda_i v_i v_j^*$ $\qquad$ [$\sigma^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$

Goal: $S(\sigma^{AB}) = 0$, and $\sigma = \mathrm{Tr}_B(\sigma^{AB})$

$S(\sigma^{AB}) = 0$

$$u^{AB} = \sum_{i=1}^{n} \sqrt{\lambda_i}\,(v_i \otimes e_i) \in \mathcal{H}_A \otimes \mathcal{H}_B \qquad \boxed{\sigma^{AB}} = (u^{AB})(u^{AB})^*$$

$$= \left[\sum_{i,j} \sqrt{\lambda_i \lambda_j}\,(v_i \otimes e_i)(v_j \otimes e_j)^*\right.$$

$\sum (\sqrt{\lambda_i})^2 = \sum \lambda_i = 1$

$\to (\sigma^{AB})_{ij} = \sqrt{\lambda_i \lambda_j}\, v_i v_j^*$ $\qquad \sum a_{ij} e_i e_j^*$

$A = (a_{ij})$

$\mathrm{T}\ (\ {}^{AB})\ \ \angle\ \ \ \ {}^{*}$

$$\mathrm{Tr}_B(\sigma^{\cdots}) = \sum_i \lambda_i v_i v_i^* = \sigma$$

Lemma: If $\rho^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, then
$$S(\rho^{AB}) \leq S(\rho^A) + S(\rho^B),$$
where $\rho^A = \mathrm{Tr}_B(\rho^{AB})$, $\rho^B = \mathrm{Tr}_A(\rho^{AB})$,
with equality iff $\rho^{AB} = \rho^A \otimes \rho^B$.

$S(\rho\|\sigma) \geq 0$ and $S(\rho\|\sigma) = 0 \iff \rho = \sigma$

$$S(\rho\|\sigma) = \mathrm{Tr}(\rho(\log\rho - \log\sigma))$$

$$S(\rho^{AB}\|\rho^A\otimes\rho^B) = \mathrm{Tr}(\rho^{AB}(\log\rho^{AB} - \log(\rho^A\otimes\rho^B)))$$

$$\log(\rho^A\otimes\rho^B) = \log(\rho^A)\otimes I_B + I_A\otimes\log(\rho^B)$$

$$= -S(\rho^{AB}) - \mathrm{Tr}(\rho^{AB}(\log(\rho^A)\otimes I_B)) - \cdots$$

$\sigma \in \mathcal{D}(\mathcal{H}_A\otimes\mathcal{H}_B)$

$\mathrm{Tr}(\sigma(U\otimes I_B))$

$= \mathrm{Tr}(\mathrm{Tr}_B(\sigma)U) \quad \forall U$

$$-\mathrm{Tr}(\mathrm{Tr}_B(\rho^{AB})\log(\rho^A))$$
$$= -\mathrm{Tr}(\rho^A\log\rho^A)$$
$$= S(\rho^A)$$

$$0 \leq S(\rho^{AB}\|\rho^A\otimes\rho^B) = \mathrm{Tr}(\rho^{AB}(\log\rho^{AB} - \log(\rho^A\otimes\rho^B)))$$

$$= -S(\rho^{AB}) + S(\rho^A) + S(\rho^B)$$

$$S(\rho^{AB}) \leq S(\rho^A) + S(\rho^B)$$

Classically: $\cancel{H(X,Y) \geq \min(H(X),H(Y))}$ $\quad >0 \quad >0$

fails for density matrices

σ^{AB} pure state and $\mathrm{Tr}_B(\sigma^{AB}) = \sigma$

equal when σ^{AB} is pure

(?) $\underbrace{S(\sigma^{AB})}_{0} \geq \min\left(\underbrace{S(\mathrm{Tr}_B(\sigma^{AB}))}_{0}, \underbrace{S(\mathrm{Tr}_A(\sigma^{AB}))}\right)$

Lemma: If $\rho^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is pure, then

$$S(\rho^A) = S(\rho^B) \quad \text{for} \quad \rho^A = \mathrm{Tr}_B(\rho^{AB}), \quad \rho^B = \mathrm{Tr}_A(\rho^{AB})$$

$u \in \underbrace{\overset{\mathbb{C}^m}{\mathcal{H}_A} \otimes \overset{\mathbb{C}^n}{\mathcal{H}_B}}$

Pf: ρ^A and ρ^B have the same non-zero eigenvalues

$\mathcal{H}_A$ $\mathcal{H}_B$

$\{u_i\}, \{v_j\}$ orthonormal bases

$$\rho^{AB} = uu^* \qquad u = \sum_{i,j} a_{ij}(u_i \otimes v_j)$$

$a_{ij}\bar{a}_{kj}$

$$= \sum_{i,j,k,l} a_{ij}\bar{a}_{kl}\, \underbrace{(u_i \otimes v_j)(u_k \otimes v_l)^*}_{j=l}$$

$A = (a_{ij})$

$$\rho^A = \sum_{i,k} (AA^*)_{ik}\, u_i u_k^* \qquad \rho^B = \sum_{j,l} (A^*A)_{jl}\, v_j v_l^*$$

$\boxed{AA^*}$ $\sum_j a_{ij}\bar{a}_{kj}$ $\boxed{A^*A}$

spec(AA^*) = spec(A^*A) w/ mult. except 0

Lemma: $\underline{S(\rho^{AB}) \geq |S(\rho^A) - S(\rho^B)|}$ $\forall\, \rho^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$

$\rho^A = \mathrm{Tr}_B(\rho^{AB}), \ \rho^B = \mathrm{Tr}_A(\rho^{AB})$

Pf:

$$\rho^{AB} \xrightarrow{\text{purify}} \underbrace{\rho^{ABC}}_{\text{pure}} \qquad \rho^{AB} = \mathrm{Tr}_C(\rho^{ABC})$$

ρ^{ABC}

$S(\rho^C) = S(\rho^{AB}) \leftarrow$

$S(\rho^B) = S(\rho^{AC}) \leftarrow$

$\rho^C = \mathrm{Tr}_{AB}(\rho^{ABC})$ $\quad \rho^{AB} = \mathrm{Tr}_C(\rho^{ABC})$

$$S(\rho^B) = S(\rho^{AC}) \le S(\rho^A) + S(\rho^C)$$
$$= S(\rho^A) + S(\rho^{AB})$$

$S(\rho^{AB}) \ge S(\rho^B) - S(\rho^A)$ + symmetric sign.

$$|S(\rho^A) - S(\rho^B)| \le S(\rho^{AB}) \le S(\rho^A) + S(\rho^B)$$

$X = Y$ $\qquad = \iff \rho^{AB} = \rho^A \otimes \rho^B$

$= H(X) + H(Y) - H(X,Y)$ $\qquad H(X|Z) = H(X,Z) - H(X)$

$I(X,Y) := D(p_{XY} \| p_X \otimes p_Y) \ge 0$ (subadd of entropy)

$I(A,B)_\rho := S(\rho^{AB} \| \rho^A \otimes \rho^B) \ge 0$

$$I(X,Y|Z) := \mathbb{E}_Z \left[D(p_{XY|Z} \| p_{X|Z} \otimes p_{Y|Z}) \right] \ge 0$$

(true) $\qquad = H(X,Z) + H(Y,Z) - H(X,Y,Z) - H(Z)$

$$\boxed{I(A,B|C)_\rho := S(\rho^{AC}) + S(\rho^{BC}) - S(\rho^{ABC}) - S(\rho^C)} \ge 0$$

$\rho^{ABC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ $\qquad$ (e.g. $\rho^{AC} = \mathrm{Tr}_B(\rho^{ABC})$)

Thm: For all $\rho, \sigma \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$,

$$S(\mathrm{Tr}_B(\rho) \| \mathrm{Tr}_B(\sigma)) \le S(\rho \| \sigma)$$

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$S(\mathcal{E}(\rho) \| \mathcal{E}(\sigma)) \le S(\rho \| \sigma)$

$$S\left(\frac{1}{m} \sum_{j=1}^{m} U_j \rho U_j^* \,\Big\|\, \frac{1}{m} \sum_{j=1}^{m} U_j \sigma U_j^*\right)$$

$U_1, \dots, U_m$ unitaries $\qquad \le \frac{1}{m} \sum_{j=1}^{m} S(U_j^* \rho U_j \| U_j^* \sigma U_j)$

$$= S(\rho \| \sigma)$$

$f: \mathbb{R} \to \mathbb{R}$ $\quad f(U^* A U) = U^* f(A) U$

$\log(U^* \rho U) =$

$U^* \log(\rho)\, U$

$$S(\rho \| \sigma) = \mathrm{Tr}(\rho(\log \rho - \log \sigma))$$

$$S(U\rho U^* \| U\sigma U^*) = \mathrm{Tr}(U\rho U^*(\log(U\rho U^*) - \log(U\sigma U^*)))$$

$$= \mathrm{Tr}(U\rho \underline{U^*}(\underline{U} \log(\rho)\, \underline{U^*} - U \log(\sigma)\, \underline{U^*})$$

$$= \mathrm{Tr}(\rho(\log \rho - \log \sigma)).$$