1 The cut polytope

Consider the set \([n] = \{1, 2, \ldots, n\}\). A partition \(S \cup \bar{S} = [n]\) is called a cut, and we associate to it the vector \(v^S \in \mathbb{R}^\binom{n}{2}\) given by

\[
v^S_{\{i,j\}} = |1_S(i) - 1_S(j)|, \quad \{i,j\} \in \binom{[n]}{2},
\]

which indicates whether \(i, j\) are on the same side of \((S, \bar{S})\).

The cut polytope \(\text{CUT}_n \subseteq \mathbb{R}^\binom{n}{2}\) is defined as the convex hull of these vectors:

\[
\text{CUT}_n \equiv \text{conv}\left\{v^S_{\{i,j\}} : \{i,j\} \in \binom{[n]}{2}\right\}.
\]

It is NP-hard to optimize linear functions over \(\text{CUT}_n\). For instance, the MAX-CUT value of a weighted graph \(G = (V, E, w)\) can be written

\[
\max \{\langle w, v \rangle : v \in \text{CUT}_n\}.
\]

Thus if \(P \neq NP\), we don’t expect that \(\text{CUT}_n\) that the SDP extension complexity of \(\text{CUT}_n\) should be bounded by a polynomial in \(n\). (Formally speaking, this would only show that \(NP \subseteq P/\text{poly}\).)

1.1 Slack matrices of \(\text{CUT}_n\)

To understand the structure of \(\text{CUT}_n\), we examine its slack matrices. Instead of studying \(\text{CUT}_n\), we will look at the correlation polytope \(\text{CORR}_n\) defined by

\[
\text{CORR}_n = \text{conv}\left(\{xx^T : x \in \{0,1\}^n\}\right).
\]

Lemma 1.1. For every \(n \geq 1\), the polytopes \(\text{CUT}_{n+1}\) and \(\text{CORR}_n\) are affine-equivalent.

Proof. Note that for every \(X := xx^T \in \text{CORR}_n\), we have \(\text{diag}(X) = x\), and we can associate the cut \(S(x) = \{i \in [n] : x_i = 1\} \subseteq [n+1]\). The corresponding cut vector is a linear function of the entries of \(X\):

\[
v^S_{\{i\}}(x) := x_i(1 - x_j) + x_j(1 - x_i) = X_{ii} + X_{jj} - 2X_{ij}, \quad 1 \leq i < j \leq n,
\]

\[
v^S_{\{i,n+1\}}(x) := x_i = X_{ii}.
\]

It is straightforward to check that this map is invertible. \(\square\)

In light of this equivalence, we focus on the correlation polytopes \(\text{CORR}_n\). Define the set of quadratic multilinear functions \(f : \{0,1\}^n \to \mathbb{R}\):

\[
\text{QML}^n := \left\{ f : \{0,1\}^n \to \mathbb{R} \mid f(x) = a_0 + \sum a_ix_i + \sum a_{ij}x_ix_j \right\},
\]

\[
\text{QML}^n_+ := \left\{ f \in \text{QML}^n : f(x) \geq 0 \quad \forall x \in \{0,1\}^n \right\}.
\]

Finally, let us define the (infinite) nonnegative matrix \(M_n : \text{QML}^n_+ \times \{0,1\}^n \to \mathbb{R}_+\) by

\[
M_n(f, x) := f(x).
\]
Lemma 1.2. For every $n \geq 1$, $M_n$ is a slack matrix for CORR_n.

Proof. By definition, $\{0, 1\}^n$ represents a set of vertices for CORR_n. Now consider $f \in \text{QML}_n^+$ and note that since $x_i = x_i^2$ on $\{0, 1\}^n$, we can write

$$f(x) = b - \sum_i A_{ii}x_i^2 - \sum_{i \neq j} A_{ij}x_ix_j,$$

for some real symmetric matrix $A \in \mathbb{S}_n$ and $b \in \mathbb{R}$. Observe that $f(x) = b - \langle A, xx^T \rangle$.

Since $f(x) \geq 0$ for $x \in \{0, 1\}^n$, it holds that $\langle A, xx^T \rangle \leq b$ and thus, by convexity, $\langle A, Y \rangle \leq b$ holds for all $Y \in \text{CORR}_n$. Furthermore, it is clear that every valid inequality on CORR_n can be represented by some $f \in \text{QML}_n^+$. Since $f(x)$ is precisely the slack of the $\langle A, Y \rangle \leq b$ constraint on the vertex $x$, the proof is complete. \qed