1 The cut polytope

Consider the set $[n] = \{1, 2, \ldots, n\}$. A partition $S \cup \bar{S} = [n]$ is called a *cut*, and we associate to it the vector $v^S \in \mathbb{R}^{\binom{n}{2}}$ given by

$$v^S_{\{i,j\}} = |\mathbf{1}_S(i) - \mathbf{1}_S(j)|, \quad \{i,j\} \in \binom{[n]}{2},$$

which indicates whether i, j are on the same side of $(S, \bar{S})$.

The *cut polytope* $\mathrm{CUT}_n \subseteq \mathbb{R}^{\binom{n}{2}}$ is defined as the convex hull of these vectors:

$$\mathrm{CUT}_n := \mathrm{conv}\left(\left\{v^S_{i,j} : \{i,j\} \in \binom{[n]}{2}\right\}\right).$$

It is NP-hard to optimize linear functions over CUT_n. For instance, the MAX-CUT value of a weighted graph $G = (V, E, w)$ can be written

$$\max \{\langle w, v \rangle : v \in \mathrm{CUT}_n\}.$$

Thus if $P \neq NP$, we don't expect that CUT_n that the SDP extension complexity of CUT_n should be bounded by a polynomial in n. (Formally speaking, this would only show that $NP \subseteq P/\mathrm{poly}$.)

1.1 Slack matrices of CUT_n

To understand the structure of CUT_n, we examine its slack matrices. Instead of studying CUT_n, we will look at the correlation polytope CORR_n defined by

$$\mathrm{CORR}_n = \mathrm{conv}\left(\left\{xx^\top : x \in \{0,1\}^n\right\}\right).$$

Lemma 1.1. *For every $n \geqslant 1$, the polytopes CUT_{n+1} and CORR_n are affine-equivalent.*

Proof. Note that for every $X := xx^\top \in \mathrm{CORR}_n$, we have $\mathrm{diag}(X) = x$, and we can associate the cut $S(x) = \{i \in [n] : x_i = 1\} \subseteq [n+1]$. The corresponding cut vector is a linear function of the entries of X:

$$v^{S(x)}_{\{ij\}} := x_i(1 - x_j) + x_j(1 - x_i) = X_{ii} + X_{jj} - 2X_{ij}, \quad 1 \leqslant i < j \leqslant n,$$
$$v^{S(x)}_{\{i,n+1\}} := x_i = X_{ii}.$$

It is straightforward to check that this map is invertible. □

In light of this equivalence, we focus on the correlation polytopes CORR_n. Define the set of *quadratic multilinear functions* $f : \{0,1\}^n \to \mathbb{R}$:

$$\mathsf{QML}^n := \left\{ f : \{0,1\}^n \to \mathbb{R} \;\middle|\; f(x) = a_0 + \sum_i a_i x_i + \sum_{i<j} a_{ij} x_i x_j \right\},$$
$$\mathsf{QML}^n_+ := \left\{ f \in \mathsf{QML}^n : f(x) \geqslant 0 \quad \forall x \in \{0,1\}^n \right\}.$$

Finally, let us define the (infinite) nonnegative matrix $\mathcal{M}_n : \mathsf{QML}^n_+ \times \{0,1\}^n \to \mathbb{R}_+$ by

$$\mathcal{M}_n(f, x) := f(x).$$

Lemma 1.2. *For every $n \geqslant 1$, $\mathcal{M}_n$ is a slack matrix for* CORR_n.

Proof. By definition, $\{0,1\}^n$ represents a set of vertices for CORR_n. Now consider $f \in \mathsf{QML}_+^n$ and note that since $x_i = x_i^2$ on $\{0,1\}^n$, we can write

$$f(x) = b - \sum_i A_{ii} x_i^2 - \sum_{i \neq j} A_{ij} x_i x_j,$$

for some real symmetric matrix $A \in \mathbb{S}_n$ and $b \in \mathbb{R}$. Observe that $f(x) = b - \langle A, xx^T \rangle$.

Since $f(x) \geqslant 0$ for $x \in \{0,1\}^n$, it holds that $\langle A, xx^T \rangle \leqslant b$ and thus, by convexity, $\langle A, Y \rangle \leqslant b$ holds for all $Y \in \mathrm{CORR}_n$. Furthermore, it is clear that every valid inequality on CORR_n can be represented by some $f \in \mathsf{QML}_+^n$. Since $f(x)$ is precisely the slack of the $\langle A, Y \rangle \leqslant b$ constraint on the vertex x, the proof is complete. □