1 Low-degree SOS lower bounds

Recall that $\mathcal{M}_n(f,x) = f(x)$ is a slack matrix of CORR_n, where $f \in \mathsf{QML}_+^n$ and $x \in \{0,1\}^n$. Define the *equivariant PSD rank of* $\mathcal{M}_n$ to be the smallest r such that there is a PSD factorization

$$\mathcal{M}_n(f,x) = \mathrm{Tr}(P(f)Q(x)),$$

where $P : \mathsf{QML}_+^n \to \mathbb{S}_r^+, Q : \{0,1\}^n \to \mathbb{S}_r^+$, and such that for every permutation $\sigma \in \mathcal{S}_n$, there is a permutation matrix Π_σ such that

$$\mathcal{M}_n(f,\sigma x) = \mathrm{Tr}(P(f)\Pi_\sigma Q(x)\Pi_\sigma^\dagger).$$

Let us denote this quantity by $\mathrm{rk}_{\mathsf{psd}}^{\mathcal{S}_n}(\mathcal{M}_n)$.

Recall that Q_d is the subspace of functions $f : \{0,1\}^n \to \mathbb{R}$ spanned by $\{\chi_S : |S| \leqslant d\}$, where $\chi_S(x) = \prod_{i\in S} x_i$. In the previous lecture, we asserted the following.

Lemma 1.1. *If* $\mathrm{rk}_{\mathsf{psd}}^{\mathcal{S}_n}(\mathcal{M}_{2n}) < \binom{2n}{d}$ *and* $d < n/2$, *then* $\mathsf{QML}_+^n \subseteq \mathsf{sos}(Q_d)$.

For a nonnegative function $f : \{0,1\}^n \to \mathbb{R}_+$, define the *sum-of-squares degree*

$$\deg_{\mathsf{sos}}(f) := \min\{d : f \in \mathsf{sos}(Q_d)\}.$$

In light of Lemma 1.1, we can rule out the existence of symmetric PSD lifts for the cut polytope by finding a function $f \in \mathsf{QML}_+^n$ with $\deg_{\mathsf{sos}}(f)$ large.

Theorem 1.2 (Grigoriev)**.** *Define the functions* $g_n : \{0,1\}^n \to \mathbb{R}_+$ *by*

$$g_n(x) = \left(x_1 + x_2 + \cdots + x_n - \frac{n}{2}\right)^2 - \frac{1}{4}.$$

For n *odd, it holds that* $g_n \in \mathsf{QML}_+^n$ *and* $\deg_{\mathsf{sos}}(g_n) \geqslant \Omega(n)$.

We sketch the proof of this theorem in Section 2 below, but let's first establish the following simpler (but weaker) lower bound.

Theorem 1.3 (Kaniewski-T. Lee-de Wolf)**.** *Define the functions* $h_n : \{0,1\}^n \to \mathbb{R}_+$ *by*

$$h_n(x) = (|x|-1)(|x|-2),$$

where $|x| = x_1 + \cdots + x_n$. *Then* $h_n \in \mathsf{QML}_+^n$ *and* $\deg_{\mathsf{sos}}(h_n) \geqslant \Omega(\sqrt{n})$.

Proof. Suppose that

$$h_n(x) = \sum_{i=1}^m q_i(x)^2,$$

where $\deg(q_i) \leqslant d$ for each $i = 1,\ldots,m$. Define functions $Q_i : [n] \to \mathbb{R}$ by

$$Q_i(k) := \mathop{\mathbb{E}}_{\substack{x\in\{0,1\}^n:\\ |x|=k}}[q_i(x)].$$

Let us first argue that we can extend Q_i to a (univariate) polynomial $\tilde{Q}_i : \mathbb{R} \to \mathbb{R}$ with $\deg(\tilde{Q}_i) \leqslant d$. By linearity, it suffices to do this for every monomial $\prod_{i \in S} x_i$ with $|S| \leqslant d$. Consider

$$\mathop{\mathbb{E}}_{x \in \{0,1\}^n : |x| = k} \left[\prod_{i \in S} x_i \right] = \mathbb{P}[S \subseteq \boldsymbol{S}_k],$$

where $\boldsymbol{S}_k \subseteq [n]$ is a uniformly random subset of size k. This probability equals 0 if $|S| > k$, and otherwise

$$\mathbb{P}\left[S \subseteq \boldsymbol{S}_k\right] = \frac{\binom{n-|S|}{k-|S|}}{\binom{n}{k}} = \frac{(n-|S|)!k!(n-k)!}{(k-|S|)!(n-k)!n!} = \frac{(n-|S|)!}{n!}\left[k \cdot (k-1) \cdots (k-|S|+1)\right].$$

Clearly the latter expression is a polynomial of degree $|S|$ in k.

Define now the univariate polynomial $P : \mathbb{R} \to \mathbb{R}$ by

$$P(t) := \sum_{i=1}^{m} \tilde{Q}_i(t)^2.$$

We have $\deg(P) \leqslant 2d$ and $P(1) = P(2) = 0$ since $h_n(x) = 0$ for $|x| = 1$ or $|x| = 2$ (which implies $q_i(x) = 0$ for $|x| \in \{1, 2\}$ and all i, which implies $Q_i(1) = Q_i(2) = 0$ for all i).

Now, every nonnegative real polynomial must contain every zero with multiplicity at least 2, hence

$$P(t) = (t-1)^2(t-2)^2 Q(t)$$

for a polynomial Q with $\deg(Q) \leqslant 2d - 4$.

We will argue now that $\deg(Q)$ should be large using the following lemma of A. A. Markov.

Lemma 1.4. *For any univariate polynomial Q and interval $I \subseteq \mathbb{R}$, we have*

$$\deg(Q) \geqslant \sqrt{\frac{|I|}{2} \cdot \frac{\max_{t \in I} |Q'(t)|}{\max_{t \in I} |Q(t)|}},$$

where $|I|$ denotes the length of I.

Back to our setting, for $k \in \{0, 1, \ldots, n\}$, it holds that

$$0 \leqslant P(k) = \sum_{i=1}^{m} \left(\mathop{\mathbb{E}}_{|x|=k} [q_i(x)] \right)^2 \leqslant \sum_{i=1}^{m} \mathop{\mathbb{E}}_{|x|=k} \left[q_i(x)^2 \right] = (k-1)(k-2).$$

This implies

$$0 \leqslant Q(k) \leqslant \frac{1}{(k-1)(k-2)}, \quad k \in \{3, 4, \ldots, n\}. \tag{1.1}$$

Moreover, note that $Q(0) = P(0)/2 = 1/4$ since $Q_i(0) = q_i(0, \ldots, 0)$ for each i.

Define $D := \max_{t \in [0,n]} |Q'(t)|$. Then since $Q(0) = 1/4$ and $Q(4) \leqslant 1/6$, we have

$$D \geqslant \frac{1/4 - 1/6}{4} = 1/48.$$

And since $Q(k) \leqslant 1/2$ for $k \in \{0, 3, 4, \ldots, n\}$, this implies

$$\max_{t \in [0,n]} |Q(t)| \leqslant 1/2 + 3D/2 \leqslant (51/2)D.$$

Hence Lemma 1.4 gives

$$\deg(q) \geqslant \sqrt{\frac{n}{2}\frac{D}{(51/2)D}} = \sqrt{\frac{n}{51}}.$$

which implies that

$$\deg_{\mathsf{sos}}(h_n) \geqslant \Omega(\sqrt{n}).$$

□

2 Pseudodensities

Recall that $L^2(\{0,1\}^n)$ is the vector space of functions $f : \{0,1\}^n \to \mathbb{R}$. Let us now equip this space with an inner product

$$\langle f, g \rangle = \mathop{\mathbb{E}}_{x}[f(x)g(x)],$$

where $x \in \{0,1\}^n$ is chosen uniformly at random.

Recall that $Q_d \subseteq \{0,1\}^n$ is the space of degree at most d multilinear polynomials on $\{0,1\}^n$, i.e., $Q_d = \operatorname{span}(\chi_S : |S| \leqslant d)$, where $\chi_S(x) = \prod_{i \in S} x_i$, and $\mathsf{sos}(Q_d)$ is the sum-of-squares cone over Q_d. Consider $g : \{0,1\}^n \to \mathbb{R}_+$. If it holds that $g \notin \mathsf{sos}(Q_d)$, then there is some separating functional $\varphi : \{0,1\}^n \to \mathbb{R}$ such that $\langle \varphi, g \rangle < 0$ and

$$\langle \varphi, q^2 \rangle \geqslant 1, \quad \forall q \in Q_d. \tag{2.1}$$

Since the constant function $\mathbf{1} \in Q_d$, we may normalize so that

$$\langle \varphi, \mathbf{1} \rangle = 1. \tag{2.2}$$

Let us call a functional φ satisfying (2.1) and (2.2) a *degree-d pseudodensity.* Often in the literature, one sees expressions like

$$\tilde{\mathbb{E}}_\varphi[f] := \langle \varphi, f \rangle,$$

emphasizing the fact that $\tilde{\mathbb{E}}_\varphi$ behaves "like an expectation" from the perspective of low-degree squares: Indeed, (2.1) immediately implies that for any $f \in \mathsf{sos}(Q_d)$, we have $\tilde{\mathbb{E}}_\varphi[f] \geqslant 0$.

2.1 Grigoriev's lower bound

For a number $w \in \{0,1,\ldots,n\}$, let $c_w(t)$ denote the unique degree-n univariate polynomial such that

$$c_w(t) = \begin{cases} 1 & \text{if } t = w \\ 0 & \text{if } t \in \{0,1,\ldots,n\} \setminus \{w\}. \end{cases}$$

We can alternately specify c_w via polynomial interpolation:

$$c_w(t) = \frac{\prod_{j=0, j\neq w}^{n}(t-j)}{\prod_{j=0, j\neq w}^{n}(w-j)}.$$

Define $\varphi_k : \{0,1\}^n \to \mathbb{R}$ by

$$\varphi_k(x) := 2^n \frac{c_{|x|}(k/2)}{\binom{n}{|x|}}.$$

Intuitively, $c_{|x|}(k/2)$ checks whether $|x| = k/2$. But if k is odd, this is impossible, and $c_{|x|}(k/2)$ will sometimes take negative values.

One can think of φ_k as a "pseudodensity" that tries to pretend it is a real probability density supported on strings with $|x| = k/2$. Observe that

$$\begin{aligned}
\mathop{\mathbb{E}}_x \left[\varphi_k(x) \prod_{i \in S} x_i\right] &= \mathop{\mathbb{E}}_x \left[\varphi_k(x) \frac{1}{\binom{n}{|S|}} \binom{|x|}{|S|}\right] \\
&= \sum_{w=0}^{n} \frac{\binom{n}{w}}{2^n} \mathop{\mathbb{E}}_x \left[\varphi_k(x) \frac{1}{\binom{n}{|S|}} \binom{|x|}{|S|} \mid |x| = w\right] \\
&= \sum_{w=0}^{n} c_w(k/2) \frac{\binom{w}{|S|}}{\binom{n}{|S|}}.
\end{aligned}$$

Define the fractional binomial coefficient

$$\binom{t}{k} := \frac{t \cdot (t-1) \cdots (t-k+1)}{k \cdot (k-1) \cdots 1}, \tag{2.3}$$

and define the univariate polynomial p_S with $\deg(p_S) \leqslant n$ by

$$p_S(t) := \frac{\binom{t}{|S|}}{\binom{n}{|S|}}.$$

We have shown that

$$\mathop{\mathbb{E}}_x \left[\varphi_k(x) \prod_{i \in S} x_i\right] = \sum_{w=0}^{n} c_w(k/2) p_S(w) = p_S(k/2), \tag{2.4}$$

where in the last equality we have used the fact that for any real polynomial p with $\deg(p) \leqslant n$, we have

$$\sum_{w=0}^{n} p(w) c_w(t) = p(t), \qquad \forall t \in \mathbb{R},$$

since both sides are polynomials of degree n that agree at the $n+1$ points $t = 0, 1, \ldots, n$.

Let us use this to evaluate

$$\begin{aligned}
\mathbb{E}\left[\varphi_k(x)\,(x_1 + \cdots + x_n - k/2)^2\right] &= n(1-k)\,\mathbb{E}[\varphi_k(x) x_1^2] + n(n-1)\,\mathbb{E}[\varphi_k(x) x_1 x_2] + \frac{k^2}{4} \\
&= (1-k)\frac{k}{2} + \frac{k}{2}\left(\frac{k}{2} - 1\right) + \frac{k^2}{4} = 0.
\end{aligned}$$

In this sense, $\varphi_k(x)$ is indeed successful at "pretending" that $|x| = k/2$. We conclude that if $g_{n,k}(x) := (x_1 + \cdots + x_n - k/2)^2 - 1/4$, then $\langle \varphi_k, g_{n,k} \rangle < 0$. So if we can establish that φ_k is a degree-$k/2$ pseudodensity, we conclude that $\deg_{\mathsf{sos}}(g_{n,k}) \geqslant k/2$.

To this end, suppose $\deg(q) = d \leqslant k/2$, write $q = \sum_{|S| \leqslant d} c_S x_S$ where $x_S = \prod_{i \in S} x_i$. Then we have:

$$\mathop{\mathbb{E}}_x[\varphi_k(x) q(x)^2] = \sum_{|S|,|T| \leqslant d} c_S c_T \mathop{\mathbb{E}}_x[\varphi_k(x) x_S(x) x_T(x)] = \sum_{|S|,|T| \leqslant d} c_S c_T \mathop{\mathbb{E}}_x[\varphi_k(x) x_{S \cup T}(x)] = \mathop{\mathbb{E}}_x \operatorname{Tr}\left(M^{n,d} Q\right),$$

where $Q_{S,T} = c_S c_T$, and $M^{n,k}$ is the matrix defined by

$$M^{n,k}_{S,T} := p_{S \cup T}(k/2) = \frac{\binom{k/2}{|S \cup T|}}{\binom{n}{|S \cup T|}}, \quad |S|, |T| \leqslant k/2.$$

Thus the following result of Grigoriev completes the argument.

Lemma 2.1. *For every $n \geqslant k \geqslant 1$, it holds that $M^{n,k} \succeq 0$.*

As far as I know, there is no slick proof of this lemma known. Instead, one uses the theory of association schemes to account for all the eigenvalues. First, it holds that

$$\dim(\ker(M^{n,k})) \geqslant \sum_{i=0}^{r-1} \binom{n}{i},$$

where $r := \lfloor k/2 \rfloor$. Secondly, if $\tilde{M}^{n,k}$ denotes the principal submatrix corresponding to indicies with $|S| = |T| = r$, then using the theory of the Johnson scheme, one establishes that $\tilde{M}^{(n)}$ has at least $\binom{n}{r}$ positive eigenvalues. Since this accounts for the entire dimension of $M^{n,k}$, we conclude that $M^{n,k} \succeq 0$.

Say that a matrix $M_{S,T}$ indexed by subsets $S, T \subseteq [n]$ with $|S|, |T| = r$ is *set-symmetric* if it holds that $M_{S,T}$ depends only on $|S \cap T|$. Let $\mathscr{J}_r \subseteq \mathbb{M}_n(\mathbb{R})$ denote the subspace spanned by such matrices. The members of $\mathscr{J}_r$ are simultaneously diagonalizable and the Johnson scheme givese a canonical eigenbasis for $\mathscr{J}_r$.

Remark 2.2 (Open problem). Prove Theorem 1.3 by giving an explicit degree-d pseudodensity φ with $\langle \varphi, h_n \rangle < 0$ and $d \geqslant \Omega(\sqrt{n})$.