Lecture 5: Monotonicity and convexity CSE 599I, Spring 2021
Instructor: James R. Lee

1 Monotonicity and convexity

Recall that our initial motivation for the Golden-Thompson inequality was in attempting to bound the moment generating function for a sum of independent random matrices:

$$\mathrm{Tr}\left(\mathbb{E}\left[e^{\beta(\boldsymbol{A}_1+\cdots+\boldsymbol{A}_n)}\right]\right) \leqslant \mathrm{Tr}(I)\prod_{i=1}^{n}\left\|\mathbb{E}\, e^{\beta \boldsymbol{A}_i}\right\|$$

While effective in the case when the operator norms $\|\boldsymbol{A}_i\|$ are uniformly bounded, this falls short of the corresponding scalar bound

$$\mathbb{E}\left[e^{\beta(\boldsymbol{X}_1+\cdots+\boldsymbol{X}_n)}\right] \leqslant \prod_{i=1}^{n}\mathbb{E}[e^{\beta \boldsymbol{X}_i}]. \tag{1.1}$$

This is problematic when attempting to extend more nuanced tail bounds to the matrix setting. On the other hand, a straightforward adaptation fails because, as we noted, the 3-matrix version of Golden-Thompson: $\mathrm{Tr}(e^{A+B+C}) \leqslant \mathrm{Tr}(e^A e^B e^C)$ does not hold.

For a random symmetric matrix $\boldsymbol{A} \in \mathbb{M}_n(\mathbb{C})$, let us denote

$$\Lambda_{\boldsymbol{A}}(\beta) := \log \mathbb{E}[e^{\beta \boldsymbol{A}}].$$

As discovered by Tropp, it turns out that the following analog of (1.1) *does hold* when $\boldsymbol{A} = \boldsymbol{A}_1 + \cdots + \boldsymbol{A}_n$ is a sum of independent random symmetric matrices: For all $\beta > 0$,

$$\mathrm{Tr}\exp\left(\Lambda_{\boldsymbol{A}}(\beta)\right) \leqslant \mathrm{Tr}\exp\left(\Lambda_{\boldsymbol{A}_1}(\beta) + \cdots + \Lambda_{\boldsymbol{A}_n}(\beta)\right). \tag{1.2}$$

This is a straightforward consequence of the following fundamental theorem of Lieb.

Theorem 1.1 (Lieb's concavity theorem)**.** *For every H Hermitian and $A \succeq 0$, the map*

$$A \mapsto \mathrm{Tr}\left(e^{H+\log(A)}\right)$$

is concave.

There are actually many results called "Lieb's concavity theorem" that are all mutual consequences of each other. Let us see how Theorem 1.1 gives (1.2) immediately. It suffices to prove that we can peel off one factor, i.e., if H is Hermitian and $\boldsymbol{X}$ is a random Hermitian matrix, then $\mathbb{E}\,\mathrm{Tr}(e^{H+\boldsymbol{X}}) \leqslant \mathrm{Tr}\exp\left(H + \Lambda_{\boldsymbol{X}}(\beta)\right)$, and we can obtain this directly by setting $\boldsymbol{Y} := e^{\boldsymbol{X}}$ and applying Theorem 1.1:

$$\mathbb{E}\,\mathrm{Tr}(e^{H+\log \boldsymbol{Y}}) \leqslant \mathrm{Tr}(\exp(H + \log(\mathbb{E}\,\boldsymbol{Y})) = \mathrm{Tr}(\exp(H + \log(\mathbb{E}\, e^{\boldsymbol{X}}))).$$

In a certain sense, Theorem 1.1 is deeper than the Golden-Thompson inequality, and understanding it will require us to study monotonicity and convexity for spectral functions.

1.1 Operator monotonicity and convexity

We will use $\mathbf{H}_n \subseteq \mathbb{M}_n(C)$ to denote the set of Hermitian matrices. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be *operator monotone* if for all $A, B \in \mathbf{H}_n$ with $\mathrm{spec}(A), \mathrm{spec}(B) \subseteq I$, it holds that

$$A \succeq B \implies f(A) \succeq f(B).$$

It is *not true* that if f is monotone then it is also operator monotone.

Indeed, we have already mentioned that this fails for $f(x) = e^x$, and we saw in Lecture 1 an example where it fails for $f(x) = x^2$. Indeed, consider any pair of matrices where $AB + BA$ has a negative eigenvalue. Then,

$$(A + \varepsilon B)^2 = A^2 + \varepsilon(AB + BA) + \varepsilon^2 B^2.$$

Observe that $(A + \varepsilon B)^2 \succeq A^2$ will fail for $\varepsilon > 0$ sufficiently small. For example,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B^2 = B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad AB + BA = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.3}$$

On the other hand, we also saw that $f(x) = x^{1/2}$ *is* operator monotone.

A function $f : I \to \mathbb{R}$ is said to be *operator convex* if it holds that for all $A, B \in \mathbf{H}_n$ with $\mathrm{spec}(A), \mathrm{spec}(B) \subseteq I$,

$$f((1-t)A + tB) \preceq (1-t)f(A) + tf(B), \qquad \forall t \in [0,1]. \tag{1.4}$$

It is elementary to show that $f(x) = x^2$ is operator convex: Expanding the square gives

$$\left(\frac{A+B}{2}\right)^2 \preceq \left(\frac{A+B}{2}\right)^2 + \left(\frac{A-B}{2}\right)^2 = \frac{A^2 + B^2}{2},$$

verifying (1.4) for $t = 1/2$. It is a convenient fact that this weaker property (midpoint convexity) is equivalent to convexity for f continuous.

On the other hand, the cube $f(x) = x^3$ is *not* operator convex. We can see this by taking the derivative of A^3 in the direction of a matrix B:

$$\mathsf{D}_B(A^3) := \lim_{\varepsilon \to 0} \frac{(A + \varepsilon B)^3 - A^3}{\varepsilon} = A^2 B + ABA + BA^2$$

But operator convexity would entail that

$$(A + \varepsilon B)^3 = ((1-\varepsilon)A + \varepsilon(A+B))^3 \preceq (1-\varepsilon)A^3 + \varepsilon(A+B)^3,$$

which gives

$$\mathsf{D}_B(A^3) \preceq (A+B)^3 - A^3,$$

and simplifying yields

$$0 \preceq (B^3 + BAB) + (B^2 A + AB^2).$$

If we evalute the RHS for the choice (1.3) and observe that $B^3 = B^2 = B = BAB$, then

$$0 \preceq 2B + (BA + AB) = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix},$$

which is false.

1.2 The Loewner-Heinz Theorem

Fortunatley, the Loewner-Heinz Theorem gives us many positive examples. We will say that $f : I \to \mathbb{R}$ is *operator concave* if $-f$ is operator convex, and f is *operator monotone decreasing* if $-f$ is operator monotone.

Theorem 1.2 (Loewner-Heinz)**.** *It holds that the function $f : (0, \infty) \to \mathbb{R}$ given by*

$$f(t) = t^p \quad \textit{is} \quad \begin{cases} \textit{operator monotone decreasing and operator convex} & -1 \leqslant p \leqslant 0, \\ \textit{operator monotone and operator concave} & 0 \leqslant p \leqslant 1, \\ \textit{operator convex but not operator monotone} & 1 \leqslant p \leqslant 2. \end{cases}$$

Furthermore, the function $f(t) = \log t$ is operator concave and operator monotone, and $f(t) = t \log t$ is operator convex.

Since operator convexity (and concavity) and operator monotonicity are preserved under taking limits, the second set of results follows from the first:

$$\log A = \lim_{p \to 0} \frac{A^p - I}{p}$$
$$A \log A = \lim_{p \to 1} \frac{A^p - A}{p - 1}.$$

Note that all matrices involved are simultaneously diagonalizable, so these two equalities reduce to their scalar versions. Since Theorem 1.2 gives that $(A^p - I)/p$ is operator monotone and operator concave for all $p \in [-1, 1]$, and that $(A^p - A)/(p - 1)$ is operator convex for all $p \in [0, 2] \setminus \{1\}$, the result follows.

Lemma 1.3. *The function $f : (0, \infty) \to \mathbb{R}$ given by $f(t) = 1/t$ is operator convex and operator monotone decreasing.*

Proof. Consider $f(A) := A^{-1}$ for $A > 0$. Then for $B \geq 0$, we have

$$\mathsf{D}_B \left(A^{-1}\right) := \lim_{\varepsilon \to 0} \frac{(A + \varepsilon B)^{-1} - A^{-1}}{\varepsilon} = -A^{-1} B A^{-1} \preceq 0,$$

hence A^{-1} is monotone decreasing.

Indeed, one can evaluate the derivative using the Woodbury formula for updating the inverse matrix:

$$(A + B)^{-1} - A^{-1} = A^{-1/2} \left[(I + C)^{-1} - I \right] A^{-1/2}, \tag{1.5}$$

where $C := A^{-1/2} B A^{-1/2}$, and then noting that as $\|C\| \to 0$, we have

$$(I + C)^{-1} = 1 - C + O(\|C\|^2).$$

Of course, if $B \succeq 0$, then $C \succeq 0$, hence $I \succeq (I + C)^{-1}$ is also true, and (1.5) gives monotonicity directly.

Similarly, for $B > 0$, we have

$$\frac{A^{-1}}{2} + \frac{B^{-1}}{2} - \left(\frac{A + B}{2}\right)^{-1} = A^{-1/2} \left[\frac{I}{2} + \frac{C^{-1}}{2} - \left(\frac{I + C}{2}\right)^{-1} \right] A^{-1/2},$$

and now

$$\frac{I}{2} + \frac{C^{-1}}{2} \geq \left(\frac{I+C}{2}\right)^{-1}$$

follows from the arithmetic-harmonic mean inequality *for scalars:* For all $a, b > 0$,

$$\frac{a}{2} + \frac{b}{2} \geqslant \left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1}.$$

(Note that the RHS is $\frac{2ab}{a+b} = \sqrt{ab}\frac{\sqrt{ab}}{(a+b)/2}$, hence this is an immediate conequence of the AM-GM inequality.) Since f is midpoint operator convex, operator convexity follows from continuity. □

Remark 1.4 (Matrix AM-HM inequality). Indeed, one can see from the proof that midpoint convexity of $f(A) = A^{-1}$ is equivalent to a matrix arithmetic-harmonic mean inequality: For $A, B > 0$,

$$\frac{A+B}{2} \geq \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}.$$

Unlike in the scalar case, it is a bit difficult to conceive of a corresponding AM-GM inequality, with a major difficulty being that it's not even clear how to *define* the geometric mean of two matrices in a satisfactory way.

1.3 Integral representations

We have now proved Theorem 1.2 for precisely one value of $p = -1$. It's also clearly true for $p = 1$, where $f(t) = t^p$ is linear. It turns out these cases are sufficient for deducing the others.

Lemma 1.5. *For $0 < p < 1$, there is a constant $C_p > 0$ such that for all $a > 0$,*

$$a^p = C_p \int_0^\infty t^p \left(\frac{1}{t} - \frac{1}{t+a}\right) dt, \qquad 0 < p < 1$$
$$a^p = C_{p+1} \int_0^\infty t^p \frac{1}{t+a}\, dt, \qquad -1 < p < 0$$
$$a^p = C_{p-1} \int_0^\infty t^{p-1} \left(\frac{a}{t} + \frac{t}{t+a} - 1\right) dt \qquad 1 < p < 2.$$

Proof. Consider the integral $\int_0^\infty t^p \left(\frac{1}{t} - \frac{1}{t+a}\right) dt$. Note that it converges because the argument grows like $O(t^{p-2})$ as $t \to \infty$, and like $O(t^{p-1})$ as $t \to 0$. And making the change of variables $t = as$ gives

$$\int_0^\infty t^p \left(\frac{1}{t} - \frac{1}{t+a}\right) dt = a^{p+1} \int_0^\infty s^p \left(\frac{1}{as} - \frac{1}{a(s+1)}\right) ds = a^p \underbrace{\int_0^\infty s^p \left(\frac{1}{s} - \frac{1}{s+1}\right) ds}_{1/C_p}.$$

Multiplying the integral by a gives

$$a^{p+1} = \int_0^\infty t^p \left(\frac{a}{t} - \frac{a}{t+a}\right) dt = \int_0^\infty t^p \left(\frac{a}{t} + \frac{t}{t+a} - 1\right) dt,$$

and dividing by a gives

$$a^{p-1} = \int_0^\infty t^p \left(\frac{1}{at} - \frac{1}{a(t+a)}\right) dt = \int_0^\infty t^{p-1} \left(\frac{1}{t+a}\right) dt. \qquad \square$$

As a consequence of the lemma, for any $A > 0$ and $-1 < p < 0$, we can write

$$A^p = C_p \int_0^\infty t^p (tI + A)^{-1}\, dt.$$

Since the map $A \mapsto (tI + A)^{-1}$ is operator convex and operator monotone decreasing by Lemma 1.3, this holds also for A^p (which is a nonnegative sum of such operators). Similarly, for $0 < p < 1$, we can write

$$A^p = C_p \int_0^\infty t^p \left(\frac{1}{t} - (tI + A)^{-1}\right) dt, .$$

Since the map $A \mapsto t^{-1} - (tI + A)^{-1}$ is operator concave and operator monotone by Lemma 1.3, this holds also for $A \mapsto A^p$. Finally, for $1 < p < 2$, we write

$$A^p = C_{p-1} \int_0^\infty t^{p-1} \left(\frac{A}{t} + t(tI + A)^{-1} - I\right) dt.$$

Since $A \mapsto A/t + (tI + A)^{-1}$ is operator convex by Lemma 1.3, the same holds for A^p. This finishes our proof of Theorem 1.2.

1.4 Non-monotonicity of the matrix exponential

The map $f(x) = e^x$ is not operator monotone or operator convex. Observe that if f is differentiable, then f is operator monotone if and only if the derivative is nonnegative in every nonnegative direction:

$$\mathsf{D}_B f(A) = \lim_{\varepsilon \to 0} \frac{f(A + \varepsilon B) - f(A)}{\varepsilon} \geq 0, \quad \forall B \geq 0.$$

The *Duhamel formula* asserts that

$$\mathsf{D}_B\, e^A = \int_0^1 e^{tA} B e^{(1-t)A}\, dt.$$

From the formula for the logarithmic mean, for all $a, b \geqslant 0$, we have

$$\int_0^1 e^{at} e^{b(1-t)}\, dt = \frac{e^a - e^b}{a - b}.$$

where we take the latter quantity equal to e^a when $a = b$.

Threfore if $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, it holds that

$$\mathsf{D}_B e^A = B \circ \left[\!\left[\frac{e^{\lambda_i} - e^{\lambda_j}}{\lambda_i - \lambda_j} \right]\!\right].$$

Take now $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $A = \mathrm{diag}(0, y)$ so that

$$\mathsf{D}_B e^A = \begin{bmatrix} 1 & \frac{e^y - 1}{y} \\ \frac{e^y - 1}{y} & e^y. \end{bmatrix}$$

The determinant of this matrix is

$$e^y - \left(\frac{e^y - 1}{y}\right)^2,$$

which clearly becomes negative as $y \to \infty$.