Unitary matrices

0.a Non-SVD proof

Proof. Recall that for $v \in \mathbb{C}^d$ we have that $\|v\|^2 = v^* v$. Thus,

$$\|Av\|_2^2 = (Av)^*(Av) = v^* A^* A v = \|v\|_2^2 = v^* v \tag{1}$$

Then, for all $v \in \mathbb{C}^d$,

$$v^*(A^* A - \mathbb{1})v = 0 \tag{2}$$

Observe that $A^* A - \mathbb{1}$ is Hermitian and has all zero eigenvalues. Thus, by diagonalizability, it is the all zeros matrix. Therefore, $A^* A = \mathbb{1}$, as desired.

(Alternatively, can use the provided proof here to show that a matrix with quadratic form equalling zero implies all zero elements over the complex field).

□

0.b SVD

Proof. Note that $A = U \Sigma V^*$ by SVD. Furthermore, consider v, an eigenvector of $A^* A$ with eigenvalue λ:

$$\|Av\|_2^2 = v^* A^* A v = v^* \lambda v = \lambda \|v\|_2^2 \implies \lambda = 1 \tag{3}$$

Thus, all of $A^* A$'s eigenvalues are 1, so all of the singular values for A are also 1. This implies:

$$A^* A = V \Sigma U^* U \Sigma V^* = V \Sigma^2 V^* = V V^* = \mathbb{1} \tag{4}$$

As desired. □

Bras and kets

0.c Q

Recall that $|\psi\rangle = |0\rangle = (1,0)^*$ and $|\phi\rangle = |+\rangle = \frac{1}{\sqrt{2}}(1,1)^*$, so:

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \tag{5}$$

0.d P

Projection matrix onto $|\psi\rangle$; only keep the component in the $|\psi\rangle$ direction.

0.e $I - 2P$

Using $|\psi\rangle$, we can create a full orthonormal basis using $|\psi\rangle |e_i\rangle$ and decompose an arbitrary state as $|v\rangle = a|\psi\rangle + \sum c_i |e_i\rangle$. Then,

$$(I - 2P)|v\rangle = |v\rangle - 2a|\psi\rangle = -a|\psi\rangle + \sum c_i |e_i\rangle \tag{6}$$

Thus, this matrix simply negates the coefficient of the component in $|\psi\rangle$.

To demonstrate unitarity, note that:

$$(I - 2P)^\dagger (I - 2P) = (I - 2P)(I - 2P) = I - 2P - 2P + 4P^2 = I - 4P + 4P = I \tag{7}$$

Because projection matrices observe $P^2 = P$. So $I - 2P$ unitary, as desired.

0.f Abracadbra, you're a new basis

Call the transformation T. We use:

$$T = |\psi_1\rangle \langle\phi_1| + |\psi_2\rangle \langle\phi_2| + ... + |\psi_n\rangle \langle\phi_n| \tag{8}$$

This works because $\langle\phi_i|\phi_j\rangle = \delta_{ij}$, so the T on any basis state will precisely transform it into its counterpart.

To demonstrate unitarity, note:

$$T^*T = \left(\sum |\phi_i\rangle \langle\psi_i|\right) \left(\sum |\psi_i\rangle \langle\phi_i|\right) = \sum_i \sum_j |\phi_i\rangle \langle\psi_i|\psi_j\rangle \langle\phi_j| = \sum_i |\phi_i\rangle \langle\phi_i| = \mathbb{1} \tag{9}$$