Unitary matrices

0.a Non-SVD proof

Proof. Recall that for $v \in \mathbb{C}^d$ we have that $\|v\|^2 = v^*v$. Thus,
\[ ||Av||^2 = (Av)^*(Av) = v^*A^*Av = ||v||^2 = v^*v \]  \hspace{1cm} (1)

Then, for all $v \in \mathbb{C}^d$,
\[ v^*(A^*A - I)v = 0 \]  \hspace{1cm} (2)

Observe that $A^*A - I$ is Hermitian and has all zero eigenvalues. Thus, by diagonalizability, it is the all zeros matrix. Therefore, $A^*A = I$, as desired.

(Alternatively, can use the provided proof here to show that a matrix with quadratic form equalling zero implies all zero elements over the complex field).

0.b SVD

Proof. Note that $A = U\Sigma V^*$ by SVD. Furthermore, consider $v$, an eigenvector of $A^*A$ with eigenvalue $\lambda$:
\[ ||Av||^2 = v^*A^*Av = v^*\lambda v = \lambda ||v||^2 \Rightarrow \lambda = 1 \]  \hspace{1cm} (3)

Thus, all of $A^*A$’s eigenvalues are 1, so all of the singular values for $A$ are also 1. This implies:
\[ A^*A = V\Sigma U^*U\Sigma V^* = V\Sigma^2V^* =VV^* = I \]  \hspace{1cm} (4)

As desired.

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0.c $Q$

Recall that $|\psi\rangle = |0\rangle = (1,0)^*$ and $|\phi\rangle = |+\rangle = \frac{1}{\sqrt{2}}(1,1)^*$, so:
\[ Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \]  \hspace{1cm} (5)

0.d $P$

Projection matrix onto $|\psi\rangle$; only keep the component in the $|\psi\rangle$ direction.

0.e $I - 2P$

Using $|\psi\rangle$, we can create a full orthonormal basis using $|\psi\rangle |e_i\rangle$ and decompose an arbitrary state as $|v\rangle = a |\psi\rangle + \sum c_i |e_i\rangle$. Then,
\[ (I - 2P) |v\rangle = |v\rangle - 2a |\psi\rangle = -a |\psi\rangle + \sum c_i |e_i\rangle \]  \hspace{1cm} (6)

Thus, this matrix simply negates the coefficient of the component in $|\psi\rangle$.

To demonstrate unitarity, note that:
\[ (I - 2P)^\dagger (I - 2P) = (I - 2P)(I - 2P) = I - 2P - 2P + 4P^2 = I - 4P + 4P = I \]  \hspace{1cm} (7)

Because projection matrices observe $P^2 = P$. So $I - 2P$ unitary, as desired.
0.f Abracadbra, you’re a new basis

Call the transformation $T$. We use:

$$T = |\psi_1\rangle \langle \phi_1| + |\psi_2\rangle \langle \phi_2| + \ldots + |\psi_n\rangle \langle \phi_n|$$  \hspace{1cm} (8)

This works because $\langle \phi_i|\phi_j\rangle = \delta_{ij}$, so the $T$ on any basis state will precisely transform it into its counterpart.

To demonstrate unitarity, note:

$$T^*T = \left( \sum_i |\phi_i\rangle \langle \psi_i| \right) \left( \sum_i |\psi_i\rangle \langle \phi_i| \right) = \sum_i \sum_j |\phi_i\rangle \langle \psi_i| \langle \psi_j| \langle \phi_j| = \sum_i |\phi_i\rangle \langle \phi_i| = 1$$  \hspace{1cm} (9)