MONOTONICITY OF THE VOLUME OF INTERSECTION OF BALLS

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1. Consider points $x_i \in I\!R^n$, for $i = 1, \ldots, k$ and let $B(x_i, r_i) \subset I\!R^n$ be the balls around these points of given radii $r_i \geq 0$. Let

$$V(x_i, r_i) = \mathrm{Vol} \bigcap_i B(x_i, r_i) \quad .$$

1.A THEOREM. *if $k \leq n+1$ then the function V is monotone decreasing in $d_{ij} = \|x_i - x_j\|$, that is if k-tuples x_i and x_i' have $d_{ij} \geq d_{ij}'$ then*

$$V(x_i, r_i) \leq V(x_i', r_i) \quad .$$

1.B Take the sphere $S_r^{n-1} \subset I\!R_n$ of a fixed radius R around the origin and consider the spherical volume of the intersections of the balls B_i with S_r^{n-1},

$$V_{n-1}(x_i, r_i, r) = \mathrm{Vol}\ S_r^{n-1} \bigcap_i B_i(x_i, r_i)$$

1.B$'$ LEMMA. *If $k \leq n$ and some k-tuples of points x_i and x_i' in $I\!R^n$ have $\|x_i\| = \|x_i'\|$ for $i = 1, \ldots, k$ and $d_{ij} \geq d_{ij}'$, then*

$$V_{n-1}(x_i, r_i, r) \leq V(x_i', r_i, r)$$

for all $r \geq 0$ and $r_i \geq 0$.

1.B$''$ REMARK: The most important case of 1.B$'$ is that where $\|x_i\| = \|x_i'\| = r$ which gives a version of 1.A for the sphere S_r^{n-1}. Notice that 1.B$'$ follows from this special case applied to the radial projection $\bar{x}_i$ of x_i to S_r^n and to $\bar{r}_i$, such that

$$B(\bar{x}_i, \bar{r}_i) \cap S_i^{n-1} = B(x_i, r_i) \cap S_r^{n-1} \quad .$$

Furthermore, since the geometry of S_r^{n-1} converges to that of $I\!R^{n-1}$ the spherical version of 1.A implies the Euclidean version.

1.C THE PROOF OF 1.B$'$: Assume, that 1.B$'$ by induction holds true for given k and $n-1$ and let us prove it for $(k+1)$-tuples in $S_r^n \subset I\!R^{n-1}$. By the above remark we may assume that the $(k+1)$-tuples in question, say $(x_0, \dots, x_k)$ have $\|x_i\| = r$ for $i = 0, \dots, k$. We also may assume for two $(k+1)$-tuples (x_i) and (x_i') under comparison that $x_0 = x_0'$. Now we prove 1.B$'$ for S_r^n under an additional

(a) *Technical Assumption*

$$\|x_i - x_0\| = \|x_i' - x_0\| = r_i' \quad \text{for } i = 1, \dots, k \quad .$$

The intersection in question is

$$S_r^n \cap B(x_0, r_0) \bigcap_{i=1}^{k} B(x_i, r_i) = \bigcup_{t=0}^{r} S^{n-1}(r_t) \bigcap_{i=1}^{k} B(x_i, r_i) \quad ,$$

where $S^{n-1}(r_t)$ is the sphere

$$S^{n-1}(r_t) = S_r^n \bigcap B(x_0, t) \quad .$$

By the induction assumption

$$\mathrm{Vol}_{n-1} S^{n-1}(r_t) \bigcap_{i=1}^{k} B(x_i, r_i) \le \mathrm{Vol}_{n-1} S^{n-1}(r_t) \bigcap_{i=1}^{k} B(x_i', r_i)$$

and the proof under assumption (a) follows by integration in t.

2. REDUCING THE GENERAL CASE TO THAT SATISFYING (A):

2.A. LEMMA. *Let x_i and x_i' in $I\!R^n$ for $i = 1, \dots, k \le n$ have $d_{ij} \ge d_{ij}'$ then there exists a continuous family of k-tuples x_i^t, such that $x_i^0 = x_i$ and $x_i^1 = x_i'$ and d_{ij}^t is decreasing in t.*

PROOF: The mutual distances d_{ij} can be replaced by scalar products $\langle x_i, x_j \rangle$ and the pertinent deformation is the linear homotopy between the matrices $\langle x_i, x_j \rangle$ and $\langle x_i', x_j' \rangle$. Since the matrix $(1-t)\langle x_i, x_j \rangle + t\langle x_i', x_j' \rangle$ is positive semidefinite for all $t \in [0,1]$ (as well as $\langle x_i, x_j \rangle$ and $\langle x_i', x_j' \rangle$) it represents some points x_i^t with the mutual distances d_{ij}^t. Q.E.D.

CONCLUSION OF THE PROOF:

Any monotone (for d_{ij}^t) homotopy can be approximated by another homotopy which also is (nonstrictly) monotone and which, on every subsegment in $[0,1]$ of the form $[\frac{\nu}{N}, \frac{\nu+1}{N}]$ (for N depending on desired the precision of approximation) has all but one among d_{ij}^t constant in t.

Now the special case applies to the tuples x_i^t and $x_i^{t'}$ for $t = \frac{\nu}{N}$ and $t' = \frac{\nu+1}{N}$ for $\nu = 1, \dots, N$ and the proof in the general case is concluded.

REMARK: It is likely that 1.A extends to the hyperbolic space with curvature $\equiv$ −const. (as well as to the spheres S_r^n with curvature $\equiv$ +const $= r^{-2}$). The above proof breaks down at the only 'non-trivial'point that is Lemma 2.A.

3. COROLLARIES.

3.A *(Kirszbraun Intersection Property; see [W.W]) if x_i and x_i' for $i = 1, \dots, k$, in $\mathbb{R}^n$, now for any k, have $d_{ij} \geq d'_{ij}$, then*

$$\bigcap_i B(x_i, r_i) \neq \emptyset \Rightarrow \bigcap_i B(x_i', r_i) \neq \emptyset \quad .$$

PROOF: Apply 1.A. to the balls around x_i and x_i' in the ambient space $\mathbb{R}^m \supset \mathbb{R}^n$ for $m = \max(n, k-1)$.

3.B COROLLARY. *(Kirszbraun; see [W.W]) Let X and Y be (finite or infinite dimensional) Hilbert spaces and set $f : X_0 \to Y$ be a distance decreasing map of a subset $X_0 \subset X$. Then f extends to a distance decreasing map $X \to Y$. Furthermore, a similar result holds for maps of subsets in the hemisphere of the Hilbert sphere.*

3.C. *Let $e(r)$ be a monotone decreasing function in $r \in [0, \infty)$ such that*

$$\int_0^\infty |e(r)| < \infty \quad .$$

Assign to each $x \in \mathbb{R}^n$ the function $e_x = e_x(y) = e(|x - y|)$.

3.D SLEPIAN INEQUALITY. *[S] For the above x_i and x_i'*

$$\int_{\mathbb{R}^n} \prod_i e_{x_i} \leq \int_{\mathbb{R}^n} \prod_i e_{x_i'} \quad .$$

The proof obviously follows from 1.A.

QUESTION: Who is the author of 1.A.? My guess is this was known to Archimedes. Undoubtedly the theorem can be located (in the form 3.D as well as 1.A) somewhere in the 17th century.

REFERENCES

[S] Slepian D. The one-sided barrier problem for Gaussian noise, Bell System Tech. J. 41 (1962), 463-501.

[W.W] Wells J.H. and Williams L.R. Embeddings and Extensions in Analysis, Ergebuisse n.84, Springer Verlag 1975.