1. **Trees and $L_1$.**

Recall that $\ell^k_p$ is the metric space where $\mathbb{R}^k$ is equipped with the $\ell_p$ norm.

(a) Show that the 3-pointed star metric $(S, d)$ does not embed isometrically (i.e. with distortion 1) into $\ell^2_2$. The star metric has four points $S = \{c, v_1, v_2, v_3\}$ (a center and 3 leaves) with $d(c, v_i) = 1$ for $i = 1, 2, 3$ and $d(v_i, v_j) = 2$ for $i \neq j$.

(b) Show that $(S, d)$ does embed isometrically into $\ell^1_2$.

(c) Let $T = (V, E)$ be a graph-theoretic tree, and let $d_T$ be the (unweighted) shortest path metric on $T$. Show that $(V, d_T)$ embeds isometrically into $\ell^{|E|}_1$.

2. **Group norms and metrics.**

Let $G$ be a multiplicative group. A **group norm on $G$** is a non-negative function $N : G \to \mathbb{R}$ satisfying the following three axioms.

- For all $g \in G$, $N(g) = 0 \iff g$ is the identity element of $G$
- For all $h, g \in G$, $N(gh) \leq N(g) + N(h)$
- For all $g \in G$, $N(g) = N(g^{-1})$

(a) Show that $N$ gives rise to a natural metric on $G$.

(b) Suppose that $G$ is generated by the set of elements $\Gamma = \{g_1, g_2, \ldots, g_k\}$. The **Cayley graph of $G$ with generating set $\Gamma$** is the graph $\mathcal{C}(G; \Gamma) = (G, E)$ whose vertices are the elements of $G$ and such that $\{x, y\} \in E$ if and only if $x = gy$ for some $g$ satisfying $g \in \Gamma$ or $g^{-1} \in \Gamma$.

For $x, y \in G$, define $d_\Gamma(x, y)$ to be the shortest path distance between $x$ and $y$ in the Cayley graph $\mathcal{C}(G; \Gamma)$ (with all edges having unit length). Define a group norm $N$ on $G$ that induces the metric $d_\Gamma$ (as in part (a)).

(c) Let $\Gamma'$ be another finite generating set of $G$. Show that the identity mapping $\text{Id} : (G, d_\Gamma) \to (G, d_{\Gamma'})$ has finite distortion.

3. **Dimension of finite subsets of $\ell_1$.**

Let $V = \{v_1, \ldots, v_n\}$ be an $n$-point set and let $N = \binom{n}{2}$. Every semi-metric $d : V \times V \to \mathbb{R}$ can be thought of as a vector $d \in \mathbb{R}^N$ (since $d$ specifies exactly $N$ real numbers). Recall that a semi-metric satisfies the axioms of a metric except possibly $d(x, y) = 0$ even when $x \neq y$.

Consider any mapping $f : V \to \ell^k_1$. This induces a semi-metric $d_f$ on $V$ defined by $d_f(v_i, v_j) = \|f(v_i) - f(v_j)\|_1$ for all $i, j \in [n]$. For $k \in \mathbb{N}$, let

$$M_{1,k}(V) = \{d_f : f : V \to \ell^k_1\}.$$ 

denote the set of all such induced semi-metrics, and set $M_1(V) = \bigcup_{k=1}^{\infty} M_{1,k}(V)$. Recall that we can think of $M_1(V) \subseteq \mathbb{R}^N$. 

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CSE 599I: Geometric embeddings (Winter 2007)  
Instructor: James R. Lee  
Homework #1  
Due: Tuesday, Jan. 30
(a) Show that $M_1(V)$ is equal to the convex hull of $M_{1,1}(V)$.
(b) Lookup the definition of Caratheodory’s theorem (about convex sets) on the internet.
(c) Show that for any $k \in \mathbb{N}$ and any $n$-point subset $X \subseteq \ell^k_1$, there is a map $f : X \rightarrow \ell^N_1$ with distortion 1.
(d) Show that for any $n$-point subset $X \subseteq \ell_1$, there is a map $f : X \rightarrow \ell^N_1$ with distortion 1.

4. Meditations on Bourgain’s embedding.

Let $(X, d)$ be an $n$-point metric space, and let $h = \lceil \log_2 n \rceil$. Recall that Bourgain’s embedding is a map $f : X \rightarrow L_2$ where

$$\left(\|f(x) - f(y)\|_2\right)^2 = \frac{1}{h + 1} \sum_{j=0}^{h} \mathbb{E}_{A_j} |d(x, A_j) - d(y, A_j)|^2,$$

and $A_j \subseteq X$ is a random subset which samples every point of $X$ independently with probability $2^{-j}$. In class, we showed that Bourgain’s embedding has distortion $O(\log n)$.

(a) Let $T = (V, E)$ be the complete binary tree of height $k$ so that $n = |V| = 2^k$, and let $d_T$ be the unweighted shortest-path metric on $T$. Argue that Bourgain’s embedding has distortion $\Omega(\log n)$ when applied to $T$.

(b) Let $P = \{1, 2, \ldots, n\}$ be equipped with the metric $d_P(i, j) = |i-j|$ (i.e. the path metric).

Show that Bourgain’s embedding has distortion $O(\sqrt{\log n})$ when applied to $(P, d_P)$.

(c) Consider the hypercube $X = \{0, 1\}^k$ equipped with the usual Hamming metric

$$d_H(x, y) = |\{i \in [k] : x_i \neq y_i\}|.$$

Give an informal argument (along the lines of part (a)) that Bourgain’s embedding applied to $(X, d_H)$ has $\Omega(k)$ distortion.

Let $Z$ be a (uniformly) random point of $\{0, 1\}^k$. Consider the mapping $f : \{0, 1\}^k \rightarrow L_2$ given by $f(x) = d(x, Z)$. Show that $\text{dist}(f) = O(\sqrt{k})$. 
